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CONVERGENCE ANALYSIS OF PICARD-ABBAS HYBRID ITERATIVE PROCESS

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Abstract. Iterative methods play a crucial role in numerical analysis and optimization problems. This paper introduces a hybrid fixed-point iterative process and compares its convergence rate with the established Abbas iterative approach, using Berinde's criteria. Additionally, the stability and data dependency of the improved iterative process are established. This study's purpose is to demonstrate the superiority, in terms of convergence speed, of the hybrid approach and provide further insights into its stability and data dependency. Furthermore, we apply the iterative process to find solutions of delay differential equations.

Keywords: new iterative process; convergence; stability; data dependency; hybrid iterative process.

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1. INTRODUCTION

One of the several applications of the Banach contraction theorem is to ascertain that a unique fixed point exists. Consequently, it is commonly employed in the development of numerous iterative processes. Several well-known iterative processes include the Picard process [1], the Mann process [2], the Ishikawa process [3], the Noor process [4], the Abbas process [5], the Agarwal process [6], the SP process [7], the S* process [8] and the CR process [9]. Several hybrid iterative processes were introduced, including the Picard-Mann [10], Picard-Ishikawa

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[11] and Picard-Noor [12] hybrid iterative processes. In Berinde's sense [13], it was observed that these procedures exhibited faster convergence compared to some well-known iterative processes. This study presents a novel approach that combines the Picard and Abbas iterative processes to create a hybrid. We establish the superiority of this hybrid over the Abbas iteration, in terms of convergence rate, in Berinde's sense [13]. We shall call this hybrid process the *Picard-Abbas hybrid iterative process* or simply *P-A hybrid iterative process*. Furthermore, the data dependency and stability results of the process are established. We also present an application of the process.

2. PRELIMINARIES

Let C be a non-empty closed convex subset of a Banach space X. A point $p \in C$ is a fixed point of $T : C \to C$ if $Tp = p$. Let $Fix(T) = \{p \in C | Tp = p\}$.
A mapping $T : C \to C$ is a contraction if $\exists \theta \in (0,1)$, such that

$$\|Tx - Ty\| \leq \theta \|x - y\| \text{ for all } x, y \in C \tag{2.1}$$

Definition 2.1. [13] *If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are real sequences with $x_n \to x$ and $y_n \to y$, and $\lim_{n\to\infty} \frac{|x_n - x|}{|y_n - y|} = 0$, then $\{x_n\}_{n=1}^{\infty}$ converges to x faster than $\{y_n\}_{n=1}^{\infty}$ does to y.*

Definition 2.2. [13] *Let the fixed point iterative processes $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be such that $x_n \to p$ and $t_n \to p$. Suppose $\|x_n - p\| \leq a_n$ and $\|t_n - p\| \leq b_n$ for all $n \in \mathbb{N}$, where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences with $a_n \to 0, b_n \to 0$. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{t_n\}_{n=1}^{\infty}$ to p.*

Lemma 2.1. *Let a real sequence $\{k_n\}_{n=1}^{\infty}$ ($k_n \geq 0$) satisfy $k_{n+1} \leq (1-\mu_n)k_n$. If $\{\mu_n\}_{n=1}^{\infty} \subset (0,1)$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} k_n = 0$.*

Lemma 2.2. *Let the real sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ ($a_n \geq 0, b_n \geq 0$) satisfy $a_{n+1} \leq (1-\lambda_n)a_n + b_n$, where $\lambda_n \in (0,1)$ $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\frac{b_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3. *Let θ be such that $0 \leq \theta \leq 1$, and let $\{\varepsilon_n\}_{n=1}^{\infty}$ ($\varepsilon_n > 0$) be a sequence, such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then for any sequence $\{\rho_n\}_{n=1}^{\infty}$, satisfying $\rho_{n+1} \leq \theta\rho_n + \varepsilon_n$, $n = 1, 2, 3, ...$, we have $\lim_{n\to\infty} \rho_n = 0$.*

Lemma 2.4. *Let $\{\xi_n\}_{n=1}^{\infty}$ be non-negative real sequence. Suppose there exists $n_1 \in \mathbb{N}$, such that $\forall n \geq n_1$, the inequality $\xi_{n+1} \leq (1-\zeta_n)\xi_n + \zeta_n \xi_n$ is satisfied, where $\zeta_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \zeta_n = \infty$ and $\lambda_n \geq 0$, $\forall n \in \mathbb{N}$. Then the inequality below holds*

$$0 \leq \limsup_{n\to\infty} \xi_n \leq \limsup_{n\to\infty} \lambda_n \tag{2.2}$$

Picard-Mann hybrid iterative process

It is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} p_{n+1} &= Tq_n \\ q_n &= (1-\alpha_n)p_n + \alpha_n T p_n \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Picard-Ishikawa hybrid iterative process

For any fixed p_1 in C, this process is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} p_{n+1} &= Tq_n \\ q_n &= (1-\alpha_n)p_n + \alpha_n T r_n \\ r_n &= (1-\beta_n)p_n + \beta_n T p_n \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$.

Picard-Noor hybrid iterative process

For any fixed p_1 in C, this process is defined by $\{p_n\}_{n=1}^{\infty}$ as

$$\begin{aligned} p_{n+1} &= Tq_n \\ q_n &= (1-\alpha_n)p_n + \alpha_n T r_n \\ r_n &= (1-\beta_n)p_n + \beta_n T s_n \\ s_n &= (1-\gamma_n)p_n + \gamma_n T p_n \end{aligned} \tag{2.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

Abbas iterative process

For any fixed t_1 in C, this process is defined by $\{t_n\}_{n=1}^{\infty}$ as

$$t_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n$$

$$u_n = (1-\beta_n)Tt_n + \beta_n Tv_n$$

$$v_n = (1-\gamma_n)t_n + \gamma_n Tt_n \tag{2.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

Picard-Abbas hybrid iterative process

We introduce the Picard-Abbas hybrid iterative process below.

For any fixed x_1 in C, the Picard-Abbas hybrid iterative process is defined by the sequence $\{x_n\}_{n=1}^{\infty}$ as

$$x_{n+1} = Ty_n$$

$$y_n = (1-\alpha_n)Tz_n + \alpha_n Tw_n$$

$$z_n = (1-\beta_n)Tx_n + \beta_n Tw_n$$

$$w_n = (1-\gamma_n)x_n + \gamma_n Tx_n \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

3. Convergence and Stability Analysis

Theorem 3.1. *Let $T : C \to C$ be a self-mapping on C and let T satisfy (2.1). If the process (2.7) generates the iterative sequence $\{x_n\}_{n=1}^{\infty}$ with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ being real sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p of T.*

Proof. The Banach contraction principle assures that a unique $p \in Fix(T)$ exists. We will show that $x_n \to p$ as $n \to \infty$. Using (2.7) we have

$$\begin{aligned}\|w_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|Tx_n - Tp\|\end{aligned}$$

$$\leq (1-\gamma_n)\|x_n-p\|+\gamma_n\theta\|x_n-p\|$$

$$= (1-\gamma_n(1-\theta))\|x_n-p\| \tag{3.1}$$

$$\begin{aligned}
\|z_n-p\| &= \|(1-\beta_n)Tx_n+\beta_nTw_n-p\| \\
&\leq (1-\beta_n)\|Tx_n-Tp\|+\beta_n\|Tw_n-Tp\| \\
&\leq (1-\beta_n)\theta\|x_n-p\|+\beta_n\theta\|w_n-p\| \\
&\leq (1-\beta_n)\theta\|x_n-p\|+\beta_n\theta\big(1-\gamma_n(1-\theta)\big)\|x_n-p\| \\
&= \theta[1-\beta_n\gamma_n(1-\theta)]\|x_n-p\|
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\|y_n-p\| &= \|(1-\alpha_n)Tx_n+\alpha_nTz_n-p\| \\
&\leq (1-\alpha_n)\|Tx_n-Tp\|+\alpha_n\|Tz_n-Tp\| \\
&\leq (1-\alpha_n)\theta\|x_n-p\|+\alpha_n\theta\|z_n-p\| \\
&\leq (1-\alpha_n)\theta\|x_n-p\|+\alpha_n\theta^2\big[1-\beta_n\gamma_n(1-\theta)\|x_n-p\|\big] \\
&\leq (1-\alpha_n)\theta\|x_n-p\|+\alpha_n\theta\big(1-\beta_n\gamma_n(1-\theta)\big)\|x_n-p\| \\
&= \theta[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-p\|
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\|x_{n+1}-p\| &= \|Ty_n-Tp\| \\
&\leq \theta\|y_n-p\| \\
&= \theta^2[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-p\|
\end{aligned} \tag{3.4}$$

Repeating the above process, we get

$$\begin{cases}
\|x_{n+1}-p\| \leq \theta^2[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-p\| \\
\|x_n-p\| \leq \theta^2[1-\alpha_{n-1}\beta_{n-1}\gamma_{n-1}(1-\theta)]\|x_{n-1}-p\| \\
\|x_{n-1}-p\| \leq \theta^2[1-\alpha_{n-2}\beta_{n-2}\gamma_{n-2}(1-\theta)]\|x_{n-2}-p\| \\
\cdots \\
\|x_2-p\| \leq \theta^2[1-\alpha_1\beta_1\gamma_1(1-\theta)]\|x_1-p\|
\end{cases} \tag{3.5}$$

From (3.5) we get

$$\|x_{n+1}-p\| \leq \|x_1-p\|\theta^{2n}\prod_{k=1}^{n}[1-\alpha_k\beta_k\gamma_k(1-\theta)] \tag{3.6}$$

Since $\theta, \alpha_n, \beta_n, \gamma_n \in (0,1)$, we have

$$1-\alpha_n\beta_n\gamma_n(1-\theta) < 1 \tag{3.7}$$

We know that $\forall x \in (0,1)$, $1-x < e^{-x}$. Using these facts and (3.6), we get

$$\|x_{n+1}-p\| \leq \|x_1-p\|\theta^{2n}e^{-(1-\theta)\sum_{n=1}^{\infty}\alpha_k\beta_k\gamma_k} \tag{3.8}$$

Taking limit as $n \to \infty$ on both sides of (3.8), we get $\lim_{n\to\infty}\|x_n-p\| = 0$.
Therefore, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p of T. □

Theorem 3.2. *Let $T : C \to C$ be a self-mapping on C and let T satisfy (2.1). If the process (2.7) generates the iterative sequence $\{x_n\}_{n=1}^{\infty}$ with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ being real sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$, then (2.7) is T-stable.*

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be any sequence in C. Let (2.7) generate the sequence $x_{n+1} = F(T,x_n)$ which converge to a unique $x^* \in Fix(T)$ (by Theorem 3.1) and $\varepsilon_n = \|t_{n+1}-F(T,x_n)\|$.
We will show that $\lim_{n\to\infty}\varepsilon_n = 0 \iff \lim_{n\to\infty}t_n = x^*$.
Let $\lim_{n\to\infty}\varepsilon_n = 0$. We have

$$\begin{aligned}\|t_{n+1}-x^*\| &\leq \|t_{n+1}-F(T,x_n)\| + \|F(T,x_n)-x^*\| \\ &\leq \varepsilon_n + \theta^2\big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\| \\ &\leq \varepsilon_n + \big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\|\end{aligned}$$

Now, $\theta \in (0,1)$, $\alpha_n, \beta_n, \gamma_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\varepsilon_n = 0$.
Using Lemma (2.2), we get $\lim_{n\to\infty}t_n = x^*$.
Conversely, let $\lim_{n\to\infty}t_n = x^*$.

$$\begin{aligned}\varepsilon_n &= \|t_{n+1}-F(T,x_n)\| \\ &\leq \|t_{n+1}-x^*\| + \|F(T,x_n)-x^*\| \\ &\leq \|t_{n+1}-x^*\| + \theta^2\big(1-\alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n-x^*\|\end{aligned}$$

$$\leq \|t_{n+1} - x^*\| + \big(1 - \alpha_n\beta_n\gamma_n(1-\theta)\big)\|t_n - x^*\|$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$.
Hence, (2.7) is T-stable. □

We now prove that (2.7) converges faster than (2.6) in Berinde's sense.

Theorem 3.3. *Let $T : C \to C$ be a contraction satisfying (2.1) with a unique $p \in Fix(T)$. For $t_1 = x_1 \in C$, let (2.6) and (2.7) generate the iterative sequences $\{t_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ respectively, with $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ being real sequences in $(0,1)$ satisfying*

(i) $\alpha \leq \alpha_n < 1, \beta \leq \beta_n < 1, \gamma \leq \gamma_n < 1$ for some $\alpha, \beta, \gamma > 0$ and $\forall n \in \mathbb{N}$.

Then $\{x_n\}_{n=1}^{\infty}$ converges to p faster than $\{t_n\}_{n=1}^{\infty}$.

Proof. From inequality (3.6) of Theorem 3.1, we have

$$\|x_{n+1} - p\| \leq \|x_1 - p\|\theta^{2n} \prod_{k=1}^{n}[1 - \alpha_k\beta_k\gamma_k(1-\theta)] \tag{3.9}$$

Using assumption (i) to (3.9), we get

$$\|x_{n+1} - p\| \leq \theta^{2n}[1 - \alpha\beta\gamma(1-\theta)]^n\|x_1 - p\| \tag{3.10}$$

Let

$$a_n = \theta^{2n}[1 - \alpha\beta\gamma(1-\theta)]^n\|x_1 - p\| \tag{3.11}$$

Using (2.6), we have

$$\begin{aligned}
\|v_n - p\| &= \|(1-\gamma_n)t_n + \gamma_n Tt_n - (1 - \gamma_n + \gamma_n)p\| \\
&\leq (1-\gamma_n)\|t_n - p\| + \gamma_n\|Tt_n - Tp\| \\
&\leq (1-\gamma_n)\|t_n - p\| + \gamma_n\theta\|t_n - p\| \\
&= [1 - \gamma_n(1-\theta)]\|t_n - p\| \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
\|u_n - p\| &= \|(1-\beta_n)Tt_n + \beta_n Tv_n - p\| \\
&\leq (1-\beta_n)\|Tt_n - Tp\| + \beta_n\|Tv_n - Tp\| \\
&\leq (1-\beta_n)\theta\|t_n - p\| + \beta_n\theta\|v_n - p\|
\end{aligned}$$

$$\leq (1-\beta_n)\theta\|t_n - p\| + \beta_n\theta\big(1-\gamma_n(1-\theta)\big)\|t_n - p\|$$

$$= \theta[1-\beta_n\gamma_n(1-\theta)]\|t_n - p\| \tag{3.13}$$

$$\begin{aligned}
\|t_{n+1}-p\| &= \|(1-\alpha_n)Tu_n + \alpha_n Tv_n - p\| \\
&\leq (1-\alpha_n)\|Tu_n - Tp\| + \alpha_n\|Tv_n - Tp\| \\
&\leq (1-\alpha_n)\theta\|u_n - p\| + \alpha_n\theta\|v_n - p\| \\
&\leq (1-\alpha_n)\theta^2[1-\beta_n\gamma_n(1-\theta)]\|t_n - p\| + \alpha_n\theta[1-\gamma_n(1-\theta)]\|t_n - p\| \\
&< \theta\{1-\alpha_n-(1-\theta)(1-\alpha_n)\beta_n\gamma_n + \alpha_n - (1-\theta)\alpha_n\gamma_n\}\|t_n - p\| \\
&\leq \theta\{1-(1-\theta)\alpha_n\beta_n\gamma_n + (1-\theta)\alpha_n\beta_n\gamma_n - (1-\theta)\alpha_n\beta_n\gamma_n\}\|t_n - p\| \\
&= \theta[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|t_n - p\| \qquad (3.14)
\end{aligned}$$

Repeating the above process, we get

$$\begin{cases}
\|t_{n+1}-p\| \leq \theta[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|t_n - p\| \\
\|t_n - p\| \leq \theta[1-\alpha_{n-1}\beta_{n-1}\gamma_{n-1}(1-\theta)]\|t_{n-1}-p\| \\
\|t_{n-1}-p\| \leq \theta[1-\alpha_{n-2}\beta_{n-2}\gamma_{n-2}(1-\theta)]\|t_{n-2}-p\| \\
\qquad \cdots \\
\|t_2 - p\| \leq \theta[1-\alpha_1\beta_1\gamma_1(1-\theta)]\|t_1 - p\|
\end{cases} \tag{3.15}$$

From (3.15) we get

$$\|t_{n+1}-p\| \leq \|t_1 - p\|\theta^n \prod_{k=1}^{n}[1-\alpha_k\beta_k\gamma_k(1-\theta)] \tag{3.16}$$

Using assumption (i) to (3.16), we get

$$\|t_{n+1}-p\| \leq \theta^n[1-\alpha\beta\gamma(1-\theta)]^n\|t_1 - p\| \tag{3.17}$$

Let

$$b_n = \theta^n[1-\alpha\beta\gamma(1-\theta)]^n\|t_1 - p\| \tag{3.18}$$

$$\frac{a_n}{b_n} = \frac{\theta^{2n}[1-\alpha\beta\gamma(1-\theta)]^n\|x_1 - p\|}{\theta^n[1-\alpha\beta\gamma(1-\theta)]^n\|t_1 - p\|} = \frac{\theta^n\|x_1 - p\|}{\|t_1 - p\|} \to 0 \text{ as } n \to \infty$$

Therefore, the process (2.7) converges faster than the process (2.6). □

4. Data Dependency

Theorem 4.1. *Let $\tilde{T}$ be an approximate operator of a contraction mapping T. Let the process (2.7) generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$ for T . We define $\{\tilde{x}_n\}_{n=1}^{\infty}$ as follows*

$$
\begin{aligned}
\tilde{x}_{n+1} &= \tilde{T}\tilde{y}_n \\
\tilde{y}_n &= (1-\alpha_n)\tilde{T}\tilde{z}_n + \alpha_n\tilde{T}\tilde{w}_n \\
\tilde{z}_n &= (1-\beta_n)\tilde{T}\tilde{x}_n + \beta_n\tilde{T}\tilde{w}_n \\
\tilde{w}_n &= (1-\gamma_n)\tilde{x}_n + \gamma_n\tilde{T}\tilde{x}_n
\end{aligned} \tag{4.1}
$$

where α_n, β_n, γ_n in $(0,1)$ satisfying

(i) $\frac{1}{2} \leq \alpha_n\beta_n\gamma_n, \forall n \in \mathbb{N}$

(ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$

If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ be such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$, then $\|p-\tilde{p}\| \leq \frac{8\varepsilon}{1-\theta}$, where $\varepsilon > 0$ is a fixed number.

Proof.

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &= \|Ty_n - \tilde{T}\tilde{y}_n\| \\
&= \|Ty_n - T\tilde{y}_n + T\tilde{y}_n - \tilde{T}\tilde{y}_n\| \\
&\leq \|Ty_n - T\tilde{y}_n\| + \|T\tilde{y}_n - \tilde{T}\tilde{y}_n\| \\
&\leq \theta\|y_n - \tilde{y}_n\| + \varepsilon
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\|w_n - \tilde{w}_n\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n)\tilde{x}_n - \gamma_n\tilde{T}\tilde{x}_n\| \\
&\leq (1-\gamma_n)\|x_n-\tilde{x}_n\| + \gamma_n\|Tx_n - \tilde{T}\tilde{x}_n\| \\
&\leq (1-\gamma_n)\|x_n-\tilde{x}_n\| + \gamma_n\left\{\|Tx_n - T\tilde{x}_n\| + \|T\tilde{x}_n - \tilde{T}\tilde{x}_n\|\right\} \\
&\leq (1-\gamma_n)\|x_n-\tilde{x}_n\| + \gamma_n\theta\|x_n-\tilde{x}_n\| + \gamma_n\varepsilon \\
&= \big(1-\gamma_n(1-\theta)\big)\|x_n-\tilde{x}_n\| + \gamma_n\varepsilon
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\|z_n - \tilde{z}_n\| &= \|(1-\beta_n)Tx_n + \beta_n Tw_n - (1-\beta_n)\tilde{T}\tilde{x}_n - \beta_n \tilde{T}\tilde{w}_n\| \\
&\le (1-\beta_n)\|Tx_n - \tilde{T}\tilde{x}_n\| + \beta_n\|Tw_n - \tilde{T}\tilde{w}_n\| \\
&\le (1-\beta_n)\left\{\|Tx_n - T\tilde{x}_n\| + \|T\tilde{x}_n - \tilde{T}\tilde{x}_n\|\right\} + \beta_n\left\{\|Tw_n - T\tilde{w}_n\| + \|T\tilde{w}_n - \tilde{T}\tilde{w}_n\|\right\} \\
&\le (1-\beta_n)\theta\|x_n - \tilde{x}_n\| + (1-\beta_n)\varepsilon + \beta_n\theta\|w_n - \tilde{w}_n\| + \beta_n\varepsilon \\
&= (1-\beta_n)\theta\|x_n - \tilde{x}_n\| + \beta_n\theta\|w_n - \tilde{w}_n\| + \varepsilon \\
&\le (1-\beta_n)\theta\|x_n - \tilde{x}_n\| + \beta_n\theta\left\{\big(1-\gamma_n(1-\theta)\big)\|x_n - \tilde{x}_n\| + \gamma_n\varepsilon\right\} + \varepsilon \\
&\le \theta[1-\beta_n\gamma_n(1-\theta)]\|x_n - \tilde{x}_n\| + \varepsilon(1+\beta_n\gamma_n\theta) \\
&\le [1-\beta_n\gamma_n(1-\theta)]\|x_n - \tilde{x}_n\| + \varepsilon(1+\beta_n\gamma_n\theta) \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
\|y_n - \tilde{y}_n\| &= \|(1-\alpha_n)Tz_n + \alpha_n Tw_n - (1-\alpha_n)\tilde{T}\tilde{z}_n - \alpha_n \tilde{T}\tilde{w}_n\| \\
&\le (1-\alpha_n)\|Tz_n - \tilde{T}\tilde{z}_n\| + \alpha_n\|Tw_n - \tilde{T}\tilde{w}_n\| \\
&\le (1-\alpha_n)\|Tz_n - T\tilde{z}_n + T\tilde{z}_n - \tilde{T}\tilde{z}_n\| + \alpha_n\|Tw_n - T\tilde{w}_n + T\tilde{w}_n - \tilde{T}\tilde{w}_n\| \\
&\le (1-\alpha_n)\left\{\|Tz_n - T\tilde{z}_n\| + \|T\tilde{z}_n - \tilde{T}\tilde{z}_n\|\right\} + \alpha_n\left\{\|Tw_n - T\tilde{w}_n\| + \|T\tilde{w}_n - \tilde{T}\tilde{w}_n\|\right\} \\
&\le (1-\alpha_n)\theta\|z_n - \tilde{z}_n\| + (1-\alpha_n)\varepsilon + \alpha_n\theta\|w_n - \tilde{w}_n\| + \alpha_n\varepsilon \\
&= (1-\alpha_n)\theta\|z_n - \tilde{z}_n\| + \alpha_n\theta\|w_n - \tilde{w}_n\| + \varepsilon \\
&\le (1-\alpha_n)\theta\left\{[1-\beta_n\gamma_n(1-\theta)]\|x_n - \tilde{x}_n\| + \varepsilon(1+\beta_n\gamma_n\theta)\right\} \\
&\quad + \alpha_n\theta\left\{\big(1-\gamma_n(1-\theta)\big)\|x_n - \tilde{x}_n\| + \gamma_n\varepsilon\right\} + \varepsilon \\
&\le \theta\left\{(1-\alpha_n)(1-\beta_n\gamma_n(1-\theta)) + \alpha_n(1-\gamma_n(1-\theta))\right\}\|x_n - \tilde{x}_n\| \\
&\quad + \theta\varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) + \varepsilon(1+\alpha_n\gamma_n\theta) \\
&\le \left\{1 + \alpha_n\beta_n\gamma_n(1-\theta) - \beta_n\gamma_n(1-\theta) - \alpha_n\gamma_n(1-\theta)\right\}\|x_n - \tilde{x}_n\| \\
&\quad + \theta\varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) + \varepsilon(1+\alpha_n\gamma_n\theta) \\
&\le \left\{1 + \alpha_n\beta_n\gamma_n(1-\theta) - \alpha_n\beta_n\gamma_n(1-\theta) - \alpha_n\beta_n\gamma_n(1-\theta)\right\}\|x_n - \tilde{x}_n\| \\
&\quad + \theta\varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) + \varepsilon(1+\alpha_n\gamma_n\theta) \\
&= \left\{1 - \alpha_n\beta_n\gamma_n(1-\theta)\right\}\|x_n - \tilde{x}_n\| + \theta\varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) + \varepsilon(1+\alpha_n\gamma_n\theta) \qquad (4.5)
\end{aligned}
$$

From (4.2) and (4.5), we have

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq \theta[1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + \theta^2\varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) \\
&\quad + \theta\varepsilon(1+\alpha_n\gamma_n\theta)+\varepsilon \\
&\leq [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + \varepsilon(1-\alpha_n)(1+\beta_n\gamma_n\theta) \\
&\quad + \varepsilon(1+\alpha_n\gamma_n\theta)+\varepsilon \\
&\leq [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + \varepsilon(1-\alpha_n)(1+1)+\varepsilon(1+1)+\varepsilon \\
&= [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + 2\varepsilon(1-\alpha_n)+3\varepsilon \\
&\leq [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + 2\varepsilon(1-\alpha_n\beta_n\gamma_n) \\
&\quad + 3(1-\alpha_n\beta_n\gamma_n+\alpha_n\beta_n\gamma_n)\varepsilon \qquad (4.6)
\end{aligned}
$$

From assumption (i) we have $1-\alpha_n\beta_n\gamma_n \leq \alpha_n\beta_n\gamma_n$. Using this in (4.6) we get

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + 8\alpha_n\beta_n\gamma_n\varepsilon \\
&= [1-\alpha_n\beta_n\gamma_n(1-\theta)]\|x_n-\tilde{x}_n\| + \alpha_n\beta_n\gamma_n(1-\theta)\frac{8\varepsilon}{1-\theta} \qquad (4.7)
\end{aligned}
$$

Let $\xi := \|x_n-\tilde{x}_n\|$, $\zeta_n := \alpha_n\beta_n\gamma_n(1-\theta) \in (0,1)$, $\lambda_n := \dfrac{8\varepsilon}{1-\theta}$.
By Lemma (2.4), we have

$$
0 \leq \limsup_{n\to\infty}\|x_n-\tilde{x}_n\| \leq \limsup_{n\to\infty}\frac{8\varepsilon}{1-\theta} \qquad (4.8)
$$

We know that $\lim_{n\to\infty} x_n = p$ (by Theorem 3.1). Also by assumption, $\lim_{n\to\infty}\tilde{x}_n = \tilde{p}$.
Therefore, we have $\|p-\tilde{p}\| \leq \dfrac{8\varepsilon}{1-\theta}$.
This completes the proof. □

5. Numerical Illustration

We now compare the P-A hybrid iterative process with Abbas, Picard-Mann, Picard-Ishikawa and Picard-Noor iterative processes by an illustration.

Example 5.1. *Let $C=[1,6]\subseteq X=\mathbb{R}$ and $T: C\to C$ be defined in C by $Tx=\sqrt{x+2}$ for all $x\in C$. Choose $\alpha_n=\beta_n=\gamma_n=\frac{1}{2}$ for each $n\in\mathbb{N}$ with initial value $x_1=5$. Clearly, T is a contraction and $Fix(T)=2$. The comparison is shown in table 1.*

Step	Picard-Abbas	Abbas	Picard-Mann	Picard-Ishikawa	Picard-Noor
1	5.00000000000	5.00000000000	5.00000000000	5.00000000000	5.00000000000
2	2.06655558777	2.27065199733	2.41306354154	2.38883481446	2.38629784546
3	2.00171524137	2.02760217242	2.06322476109	2.05528840652	2.05441369100
4	2.00004438379	2.00285282760	2.00984688238	2.00797195435	2.00776739377
5	2.00000114860	2.00029526479	2.00153779502	2.00115180092	2.00111089764
6	2.00000002972	2.00003056401	2.00024026142	2.00016646296	2.00015892486
7	2.00000000077	2.00000316385	2.00003754038	2.00002405893	2.00002273666
8	2.00000000002	2.00000032751	2.00000586567	2.00000347726	2.00000325285
9	**2.00000000000**	2.00000003390	2.00000091651	2.00000050257	2.00000046537
10	2.00000000000	2.00000000351	2.00000014320	2.00000007264	2.00000006658
11	2.00000000000	2.00000000036	2.00000002238	2.00000001050	2.00000000953
12	2.00000000000	2.00000000004	2.00000000350	2.00000000152	2.00000000136
13	2.00000000000	**2.00000000000**	2.00000000055	2.00000000022	2.00000000019
14	2.00000000000	2.00000000000	2.00000000009	2.00000000003	2.00000000003
15	2.00000000000	2.00000000000	2.00000000001	**2.00000000000**	**2.00000000000**
16	2.00000000000	2.00000000000	**2.00000000000**	2.00000000000	2.00000000000
17	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000
18	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000
19	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000
20	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000

TABLE 1. Comparison of convergence rate of iterative processes (2.3), (2.4), (2.5), (2.6) and (2.7).

6. APPLICATION TO DELAY DIFFERENTIAL EQUATIONS

Let us endow the space $C\big([a,b]\big)$ with the norm $\|x-y\|_\infty = \max\limits_{t\in[a,b]} |x(t)-y(t)|$ with $C\big([a,b]\big)$ being the space of real-valued functions which are continuous on the interval $[a,b]$. The space $\big(C\big([a,b]\big), \|\cdot\|_\infty\big)$ is known to be a Banach space.

Let the following delay differential equation be considered:

$$x'(t) = f(t, x(t), x(t-\tau)),\ t \in [t_0, b] \tag{6.1}$$

with initial condition

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0] \tag{6.2}$$

We assume the following conditions are satisfied:

- (C_1) $t_0, b \in \mathbb{R}$, $\tau > 0$;
- (C_2) $f \in C\big([t_0, b] \times \mathbb{R}^2, \mathbb{R}\big)$;
- (C_3) $\varphi \in C\big([t_0 - \tau, b], \mathbb{R}\big)$;
- (C_4) there exists $L_f > 0$ such that $\forall u_i, v_i \in \mathbb{R}$, $i = 1, 2$, $t \in [t_0, b]$

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \leq L_f \sum_{i=1}^{2} |u_i - v_i| \tag{6.3}$$

- (C_5) $2L_f(b - t_0) < 1$

A solution x of problem (6.1) – (6.2) is a function $x \in C\big([t_0 - \tau, b], \mathbb{R}\big) \bigcap C^1\big([t_0, b], \mathbb{R}\big)$.

We can reformulate the given problem (6.1) – (6.2) as an integral equation below:

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b] \end{cases} \tag{6.4}$$

We first state the following result established by Coman et al. [14]

Theorem 6.1. *If (C_1) – (C_5) are satisfied, then a unique solution, say x^*, to the problem (6.1) – (6.2) exists in $C\big([t_0 - \tau, b], \mathbb{R}\big) \bigcap C^1\big([t_0, b], \mathbb{R}\big)$ and*

$$x^* = \lim_{n \to \infty} T^n(x) \text{ for any } x \in C\big([t_0 - \tau, b], \mathbb{R}\big). \tag{6.5}$$

Theorem 6.2. *If (C_1) – (C_5) are satisfied, then a unique solution, say x^*, to the problem (6.1) – (6.2) exists in $C\big([t_0 - \tau, b], \mathbb{R}\big) \bigcap C^1\big([t_0, b], \mathbb{R}\big)$ and the Picard-Abbas iterative process (2.7) with real sequences $\alpha_n, \beta_n, \gamma_n$ in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$, converges to x^*.*

Proof. Let the iteration (2.7) generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$ for the operator

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b] \end{cases} \tag{6.6}$$

Let $x^* \in Fix(T)$. We will prove that $x_n \to x^*$ as $n \to \infty$.

We can see that $x_n \to x^*$ for each $t \in [t_0 - \tau, t_0]$.

Now, for each $t \in [t_0, b]$ we have

$$\begin{aligned}
\|w_n - x^*\|_\infty &= \|(1-\gamma_n)x_n + \gamma_n T x_n - x^*\|_\infty \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \|Tx_n - Tx^*\|_\infty \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \big|Tx_n(t) - Tx^*(t)\big| \\
&= (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \left| \begin{aligned} &\varphi(t_0) + \int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))ds \\ &- \varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{aligned} \right| \\
&= (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \left| \begin{aligned} &\int_{t_0}^{t} f(s, x_n(s), x_n(s-\tau))ds \\ &- \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{aligned} \right| \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{aligned} &f(s, x_n(s), x_n(s-\tau))ds \\ &- f(s, x^*(s), x^*(s-\tau))ds \end{aligned} \right| \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty \\
&\quad + \gamma_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\big(|x_n(s) - x^*(s)| + |x_n(s-\tau) - x^*(s-\tau)|\big)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f \left(\begin{aligned} &\max_{t\in[t_0-\tau,b]} |x_n(s) - x^*(s)| \\ &+ \max_{t\in[t_0-\tau,b]} |x_n(s-\tau) - x^*(s-\tau)| \end{aligned} \right) ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + \gamma_n \int_{t_0}^{t} L_f\big(\|x_n - x^*\|_\infty + \|x_n - x^*\|_\infty\big)ds \\
&\leq (1-\gamma_n)\|x_n - x^*\|_\infty + 2\gamma_n L_f (t - t_0)\|x_n - x^*\|_\infty \\
&\leq [1 - \gamma_n\big(1 - 2L_f(b - t_0)\big)]\|x_n - x^*\|_\infty \qquad (6.7)
\end{aligned}$$

$$\begin{aligned}
\|z_n - x^*\|_\infty &= \|(1-\beta_n)x_n + \beta_n T w_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \|Tw_n - Tx^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \left| Tw_n(t) - Tx^*(t)\right| \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \left| \begin{array}{l} \varphi(t_0) + \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ -\varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{array}\right| \\
&= (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \left| \begin{array}{l} \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{array}\right| \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{array}{l} f(s, w_n(s), w_n(s-\tau))ds \\ - f(s, x^*(s), x^*(s-\tau))ds \end{array}\right| \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty \\
&\quad + \beta_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f \big(|w_n(s) - x^*(s)| + |w_n(s-\tau) - x^*(s-\tau)|\big) ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f \left(\begin{array}{l} \max_{t\in[t_0-\tau,b]} |w_n(s) - x^*(s)| \\ + \max_{t\in[t_0-\tau,b]} |w_n(s-\tau) - x^*(s-\tau)| \end{array}\right) ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n \int_{t_0}^{t} L_f \big(\|w_n - x^*\|_\infty + \|w_n - x^*\|_\infty\big) ds \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f (t-t_0)\|w_n - x^*\|_\infty \\
&\leq (1-\beta_n)\|x_n - x^*\|_\infty + 2\beta_n L_f (b-t_0)[1-\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \qquad (6.8)
\end{aligned}$$

Using condition (C_5), that is $2L_f(b-t_0) < 1$ in (6.8), we have

$$\begin{aligned}
\|z_n - x^*\|_\infty &\leq (1-\beta_n)\|x_n - x^*\|_\infty + \beta_n [1-\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&= [1-\beta_n\gamma_n\big(1-2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \qquad (6.9)
\end{aligned}$$

$$\begin{aligned}
\|y_n - x^*\|_\infty &= \|(1-\alpha_n)z_n + \alpha_n T w_n - x^*\|_\infty \\
&\leq (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \|Tw_n - Tx^*\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \big|Tw_n(t) - Tx^*(t)\big| \\
&= (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \varphi(t_0) + \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ -\varphi(t_0) - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&= (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \left| \begin{matrix} \int_{t_0}^{t} f(s, w_n(s), w_n(s-\tau))ds \\ - \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\le (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left| \begin{matrix} f(s, w_n(s), w_n(s-\tau))ds \\ - f(s, x^*(s), x^*(s-\tau))ds \end{matrix} \right| \\
&\le (1-\alpha_n)\|z_n - x^*\|_\infty \\
&\quad + \alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\big(|w_n(s) - x^*(s)| + |w_n(s-\tau) - x^*(s-\tau)|\big)ds \\
&\le (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t} L_f \left(\begin{matrix} \max_{t\in[t_0-\tau,b]} |w_n(s) - x^*(s)| \\ + \max_{t\in[t_0-\tau,b]} |w_n(s-\tau) - x^*(s-\tau)| \end{matrix} \right) ds \\
&\le (1-\alpha_n)\|z_n - x^*\|_\infty + \alpha_n \int_{t_0}^{t} L_f\big(\|w_n - x^*\|_\infty + \|w_n - x^*\|_\infty\big)ds \\
&\le (1-\alpha_n)\|z_n - x^*\|_\infty + 2\alpha_n L_f(t-t_0)\|w_n - x^*\|_\infty \\
&\le (1-\alpha_n)[1 - \beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&\quad + 2\alpha_n L_f(b-t_0)[1 - \gamma_n\big(1 - 2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&\le (1-\alpha_n)[1 - \beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&\quad + \alpha_n[1 - \gamma_n\big(1 - 2L_f(b-t_0)\big)]\|x_n - x^*\|_\infty \\
&= \left[\begin{matrix} 1 - \beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big) + \alpha_n\beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big) \\ - \alpha_n\gamma_n\big(1 - 2L_f(b-t_0)\big) \end{matrix} \right] \|x_n - x^*\|_\infty \\
&\le \left[\begin{matrix} 1 - \alpha_n\beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big) + \alpha_n\beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big) \\ - \alpha_n\beta_n\gamma_n\big(1 - 2L_f(b-t_0)\big) \end{matrix} \right] \|x_n - x^*\|_\infty
\end{aligned}
$$

$$= \left[1 - \alpha_n \beta_n \gamma_n \left(1 - 2L_f(b - t_0)\right)\right] \|x_n - x^*\|_\infty \tag{6.10}$$

$$\begin{aligned}
\|x_{n+1} - x^*\|_\infty &= \|Ty_n - Tx^*\|_\infty \\
&= \max_{t \in [t_0 - \tau, b]} \left| \int_{t_0}^{t} f(s, y_n(s), y_n(s - \tau))ds - \int_{t_0}^{t} f(s, x^*(s), x^*(s - \tau))ds \right| \\
&\leq \max_{t \in [t_0 - \tau, b]} \int_{t_0}^{t} \left| f(s, y_n(s), y_n(s - \tau))ds - f(s, x^*(s), x^*(s - \tau)) \right| ds \\
&\leq \max_{t \in [t_0 - \tau, b]} \int_{t_0}^{t} L_f \left(|y_n(s) - x^*(s)| + |y_n(s - \tau) - x^*(s - \tau)| \right) ds \\
&\leq 2L_f(b - t_0) \|y_n - x^*\|_\infty
\end{aligned}$$

$$\leq 2L_f(b - t_0) \left[1 - \alpha_n \beta_n \gamma_n \left(1 - 2L_f(b - t_0)\right)\right] \|x_n - x^*\|_\infty \tag{6.11}$$

Using condition (C_5), that is $2L_f(b - t_0) < 1$ in (6.11), we have

$$\|x_{n+1} - x^*\|_\infty \leq \left[1 - \alpha_n \beta_n \gamma_n \left(1 - 2L_f(b - t_0)\right)\right] \|x_n - x^*\|_\infty \tag{6.12}$$

Now, take $\mu_n = \alpha_n \beta_n \gamma_n \left(1 - 2L_f(b - t_0)\right) < 1$ and $k_n = \|x_n - x^*\|_\infty$.
By Lemma 2.1, we get $\lim_{n \to \infty} \|x_n - x^*\|_\infty = 0$.
Hecne the proof. □

7. Conclusion

We are able to demonstrate from the results above that the P-A iteration converges more quickly than the Abbas iteration. Example (5.1) demonstrates our point. Additionally, the P-A iteration is stable, and a data dependence result is obtained for the P-A iteration. On the application side, we were able to employ the P-A iteration to find solution of delay differential equations. It's interesting to observe that the convergence rate appears to improve when well-known iterative techniques are combined to create a hybrid. This actually creates a pathway for more study in this area.

Conflict of Interests

The authors declare that there is no conflict of interests.

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