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SOME NEW FIXED POINT RESULTS FOR MONOTONE PARTIALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we study monotone partially nonexpansive mappings and present some new existence and convergence theorems for these mappings in the setting of ordered Banach spaces. Furthermore, we extend the results reported by M. R. Alfuraidan and M. A. Khamsi (Carpathian J. Math., 36(2): 199–204, 2020) and E. Llorens-Fuster (Adv. Theory Nonlinear Anal. Appl., 6(4): 565–573, 2022).

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1. Introduction

Let $(\mathscr{B}, \|.\|)$ denote a Banach space, and $\mathscr{Y}$ a nonempty subset of $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is termed nonexpansive if it adheres to the condition $\|\Phi(\xi) - \Phi(\upsilon)\| \leq \|\xi - \upsilon\|$ for all $\xi, \upsilon \in \mathscr{Y}$. A point $z \in \mathscr{Y}$ is a fixed point of Φ if $\Phi(z) = z$. It is well-known that within a general Banach space, a nonexpansive mapping might not possess a fixed point. However, in 1965, Browder [3], Göhde [11], and Kirk [13] independently established fixed point theorems for nonexpansive mappings meeting certain geometric prerequisites, such as uniform convexity or normal structure.

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Nonexpansive mappings have played a crucial role in nonlinear functional analysis, particularly in their connections to variational inequalities and the theory of monotone and accretive operators. They represent a variation of traditional Banach contractions. The exploration of fixed point existence for nonexpansive mappings and their dynamic behavior has evolved since the mid-1960s. Primarily investigated within closed convex subsets of Banach spaces, this area has now developed into a specialized domain within metric fixed point theory. Over time, numerous researchers have contributed to advancements in understanding nonexpansive mappings, resulting in various generalizations and extensions of their properties. A number of significant expansions and generalizations of nonexpansive mappings appeared in literature, see [7, 9, 10, 14, 18, 21].

In 2008, Suzuki [21] introduced a class of mappings characterized by condition (C), belonging to the broader category of nonexpansive type mappings. Suzuki's exploration of these mappings notably contributed to the formulation of significant fixed point theorems.

Definition 1.1. [21]. Let $\mathscr{B}$ be a Banach space and $\mathscr{Y}$ a nonempty subset of $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|\xi - \Phi(\xi)\| \leq \|\xi - \upsilon\| \text{ implies } \|\Phi(\xi) - \Phi(\upsilon)\| \leq \|\xi - \upsilon\| \ \forall\, \xi, \upsilon \in \mathscr{Y}.$$

This condition occupies a position between nonexpansiveness and quasinonexpansiveness in terms of its strength. The author demonstrated the existence of fixed points for mappings that satisfy condition (C) and presented various fixed point theorems and convergence theorems for such mappings. Moreover, the author established a robust convergence theorem linked to Ishikawa's theorem [12] and a more subtle weak convergence theorem associated with the contributions of Edelstein and O'Brien [6].

García-Falset *et al.* [7] further generalized condition (C) into the following two class of mappings.

Definition 1.2. [7]. For $\lambda \in (0,1)$, a mapping $\Phi : \mathscr{Y} \to \mathscr{B}$ is said to satisfy condition (C_λ) if

$$\lambda\|\xi - \Phi(\xi)\| \leq \|\xi - \upsilon\| \text{ implies } \|\Phi(\xi) - \Phi(\upsilon)\| \leq \|\xi - \upsilon\| \ \forall\, \xi, \upsilon \in \mathscr{Y}.$$

Definition 1.3. [7]. Let $\mathscr{Y}$ be a nonempty subset of a Banach space $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to fulfill condition (E_μ) if there exists $\mu \geq 1$ such that

$$\|\xi - \Phi(\upsilon)\| \leq \mu\|\xi - \Phi(\xi)\| + \|\xi - \upsilon\| \; \forall \, \xi, \upsilon \in \mathscr{Y}.$$

We say that Φ satisfies condition (E) if it satisfies (E_μ) for some $\mu \geq 1$.

The category of mappings adhering to condition (E) encompasses various significant classes of generalized nonexpansive mappings. Several crucial findings concerning nonexpansive mappings have been established within this class, see also [17].

In a recent development, Llorens-Fuster [14] introduced the class of partially nonexpansive mappings (PNE), which encompasses the class of Suzuki-nonexpansive mappings. Despite being defined on compact convex sets, PNE mappings might not possess fixed points. However, Llorens-Fuster demonstrated that when combining both properties, PNE and condition (E), fixed points can indeed be guaranteed in Banach spaces with appropriate geometric properties in their norm structure.

On the other hand, the surge of interest in monotone Lipschitzian mappings gained momentum following the publication of an extension of the Banach Contraction Principle to partially ordered metric spaces by Ran and Reurings [15] (also discussed in Turinici [22]). Subsequently, many authors tried to develop a metric fixed point theory for monotone Lipschitzian mappings, see for example [16, 19, 20]. However, mappings falling outside the realm of Lipschitzianity in the traditional sense received less attention. A valuable resource showcasing applications of fixed point theory for monotone mappings is the comprehensive book by Carl and Heikkilä [4]. For further exploration of metric fixed point theory and the geometric aspects of Banach spaces, readers may refer to the book by Goebel and Kirk [8]. In this vein, Alfuraidan and Khamsi [1] broadened the scope of condition (C) in ordered Banach spaces and derived existence and convergence theorems.

Motivated by Llorens-Fuster [14], and others, we extend the class of partially nonexpansive mappings within the framework of ordered Banach spaces. Through this extension, we establish various existence and convergence results for partially nonexpansive mappings under specific assumptions. Consequently, we extend, generalize, and complement results presented in [1, 7, 14, 21].

2. Preliminaries

Let $\mathscr{B}$ be a Banach space and $\mathscr{Y}$ a nonempty subset of $\mathscr{B}$ such that $\mathscr{Y} \neq \emptyset$. We denote $F(\Phi)$ the set of all fixed points of mapping Φ, i.e., $F(\Phi) = \{z \in \mathscr{Y} : \ \Phi(z) = z\}$. Let $\mathscr{B}$ be a Banach space with a partial order $\preceq$ compatible with the linear structure of $\mathscr{B}$, that is,

$$\xi \preceq \upsilon \text{ implies } \xi + z \preceq \upsilon + z,$$

$$\xi \preceq \upsilon \text{ implies } \lambda \xi \preceq \lambda \upsilon$$

for every $\xi, \upsilon, z \in \mathscr{B}$ and $\lambda \geq 0$. It follows that all order intervals $[\xi, \rightarrow] = \{z \in \mathscr{B} : \xi \preceq z\}$ and $[\leftarrow, \upsilon] = \{z \in \mathscr{B} : z \preceq \upsilon\}$ are convex. Moreover, we will assume that each $[\xi, \rightarrow]$ and $[\leftarrow, \upsilon]$ is closed. We will say that $(\mathscr{B}, \|\cdot\|, \preceq)$ is an ordered Banach space.

Definition 2.1. [8]. Let $\mathscr{Y}$ be a nonempty subset of a Banach space $\mathscr{B}$. A sequence $\{\xi_n\}$ in $\mathscr{Y}$ is said to be approximate fixed point sequence (in short, a.f.p.s.) for a mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ if $\lim_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$.

Lemma 2.2. [8]. *Let $(\mathscr{B}, \|.\|)$ be a Banach space. Let $\{\xi_n\}$ and $\{\upsilon_n\}$ be two bounded sequences in $\mathscr{B}$ and $\lambda \in (0,1)$. Assume that $\xi_{n+1} = \lambda \upsilon_n + (1-\lambda)\xi_n$ and $\|\upsilon_{n+1} - \upsilon_n\| \leq \|\xi_{n+1} - \xi_n\|$, for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|\xi_n - \upsilon_n\| = 0$ holds.*

Definition 2.3. [5]. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to be a quasi-nonexpansive (QNE) if

$$\|\Phi(\xi) - z\| \leq \|\xi - z\| \ \forall \xi \in \mathscr{Y} \text{ and } z \in F(\Phi).$$

A mapping satisfying condition (E) with a fixed point is QNE.

Definition 2.4. We say that $\mathscr{B}$ satisfies the monotone weak-Opial property if for any monotone sequence $\{\xi_n\}$ in $\mathscr{B}$ which converges weakly to ξ, we have

$$\liminf_{n\to\infty} \|\xi_n - \xi\| < \liminf_{n\to\infty} \|\xi_n - y\|,$$

for any $y \neq \xi$ and y is greater or less than all the elements of the sequence $\{\xi_n\}$.

Typically, when attempting to relax the compactness requirement, we often turn to the Opial property. It is commonly acknowledged that ℓ_p spaces for $p > 1$, as well as any Hilbert space, exhibit this property under the weak topology. However, Opial pointed out that Banach spaces like $L_p([0,1])$ for $p > 1$ lack the Opial property under the weak topology, despite their uniform convexity—a notable drawback. Consequently, in [2] the concept of the monotone weak-Opial property was introduced. Remarkably, they demonstrated that $L_p([0,1])$ spaces for $p > 1$ indeed satisfy the monotone weak-Opial property.

Definition 2.5. Let $\mathscr{B}$ be a Banach space and $\mathscr{Y}$ a closed convex subset of $\mathscr{B}$ such that $\mathscr{Y} \neq \emptyset$. Let $\Phi : \mathscr{Y} \to \mathscr{Y}$ be a mapping:

- The mapping Φ is said to be compact if $\Phi(\mathscr{Y})$ has a compact closure.
- The mapping Φ is said to be weakly compact if $\Phi(\mathscr{Y})$ has a weakly compact closure.

If a monotone sequence $\{\xi_n\}$ possesses a subsequence that converges weakly to some z, then the entire sequence $\{\xi_n\}$ converges weakly to z. Furthermore, if $\{\xi_n\}$ is monotone increasing (resp. decreasing), then $\xi_n \preceq z$ (resp. $z \preceq \xi_n$).

3. Monotone Partially Nonexpansive Mapping

Llorens-Fuster [14] introduced the following class of mappings:

Definition 3.1. Let $\Phi : \mathscr{Y} \to \mathscr{Y}$ be a mapping. A mapping Φ is called as partially nonexpansive, (in short, PNE), if

$$\left\| \Phi\left(\frac{1}{2}(\xi + \Phi(\xi))\right) - \Phi(\xi) \right\| \leq \frac{1}{2}\|\xi - \Phi(\xi)\|$$

for all $\xi \in \mathscr{Y}$.

Remark 3.2. Having fixed points for a mapping is not necessarily implied by either condition PNE or condition (E).

Proposition 3.3. [14] *If $\Phi : \mathscr{Y} \to \mathscr{Y}$ satisfies condition (C), then Φ is partially nonexpansive mapping.*

Thus, every nonexpansive mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is partially nonexpansive. However, it should be noted that the converse of above Proposition is not necessarily true.

Example 3.4. [14] Suppose that $\Phi : \mathscr{B}[0_{\mathscr{B}}, 1] \to \mathscr{B}[0_{\mathscr{B}}, 1]$ is the mapping given by the following definition:

$$\Phi(\xi) = \begin{cases} \frac{1}{2}\frac{\xi}{\|\xi\|} & \xi \in \mathscr{B}[0_{\mathscr{B}}, 1] \setminus \mathscr{B}\left[0_{\mathscr{B}}, \frac{1}{2}\right] \\ 0_{\mathscr{B}} & \xi \in \mathscr{B}\left[0_{\mathscr{B}}, \frac{1}{2}\right] \end{cases}$$

It is shown in [14] that the mapping Φ is PNE. The mapping Φ does not satisfy condition (C).

The class of partially nonexpansive mappings with fixed point and the class of quasi-nonexpansive mappings are independent in nature. The following two illustrate this facts:

Example 3.5. [14] Let $\left(\mathbb{R}^2, \|\cdot\|_\infty\right)$ be a Banach space. Consider $\Phi : \mathscr{B}_\infty[0_{\mathscr{B}}, 2] \to \mathscr{B}_\infty[0_{\mathscr{B}}, 2]$ is the mapping

$$\Phi(\xi) = \begin{cases} \frac{\xi}{\|\xi\|_\infty} & \xi \in \mathscr{B}_\infty[0_{\mathscr{B}}, 2] \setminus \mathscr{B}_\infty[0_{\mathscr{B}}, 1] \\ \xi & \xi \in \mathscr{B}_\infty[0_{\mathscr{B}}, 1]. \end{cases}$$

It is shown in [14] that mapping Φ is PNE. Now, let $s \in (0, 1)$, take $\xi := (1, 1)$ and $\upsilon_s := (1-s, 1+s)$. It can be noted that ξ is a fixed point of Φ. Hence Φ fails to be QNE (w.r.t. the norm $\|\cdot\|_\infty$).

Example 3.6. [14] Let $\mathscr{Y} = \left[0, \frac{2}{3}\right]$ be a subset of $\mathbb{R}$. Let $\Phi : \mathscr{Y} \to \mathscr{Y}$ be defined as

$$\Phi(x) = x^2, \text{ for all } x \in \left[0, \frac{2}{3}\right].$$

Then Φ is QNE. On the other hand, at $x = \dfrac{2}{3}$, Φ fails to be PNE.

Definition 3.7. Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathscr{Y}$ be a nonempty subset of $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to be monotone if

$$\xi \preceq \upsilon \text{ implies } \Phi(\xi) \preceq \Phi(\upsilon)$$

for all $\xi, \upsilon \in \mathscr{Y}$.

Definition 3.8. Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathscr{Y}$ be a nonempty convex subset of $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to be monotone partially nonexpansive if Φ is monotone and if

$$\left\|\Phi\left(\frac{1}{2}(\xi + \Phi(\xi))\right) - \Phi(\xi)\right\| \leq \frac{1}{2}\|\xi - \Phi(\xi)\|$$

for all $\xi \in \mathscr{Y}$ with $\xi \preceq \Phi(\xi)$.

Definition 3.9. Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space and $\mathscr{Y}$ be a nonempty subset of $\mathscr{B}$. A mapping $\Phi : \mathscr{Y} \to \mathscr{Y}$ is said to be monotone mapping satisfying condition (E) if Φ is monotone and there exists $\mu \geq 1$ such that

$$\|\xi - \Phi(\upsilon)\| \leq \mu \|\xi - \Phi(\xi)\| + \|\xi - \upsilon\|$$

for all $\xi, \upsilon \in \mathscr{Y}$ with $\xi \preceq \upsilon$.

Partially nonexpansive mappings enjoy an approximate fixed point property. We have a similar conclusion for monotone partially nonexpansive mappings.

Lemma 3.10. *Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space. Let $\mathscr{Y}$ be a nonempty bounded convex subset of $\mathscr{B}$ and $\Phi : \mathscr{Y} \to \mathscr{Y}$ be a monotone PNE mapping. Let $\xi_0 \in \mathscr{Y}$ such that ξ_0 and $\Phi(\xi_0)$ are comparable. Define the sequence $\{\xi_n\}$ by the successive iteration*

$$\xi_{n+1} = \frac{\xi_n + \Phi(\xi_n)}{2}$$

for any $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$ holds, i.e. $\{\xi_n\}$ is an approximate fixed point sequence of Φ.

Proof. Note that since order intervals are convex, if $\xi_0 \preceq \Phi(\xi_0)$ (resp. $\Phi(\xi_0) \preceq \xi_0$), then $\{\xi_n\}$ is monotone increasing (resp. monotone decreasing). In fact, one can easily prove by induction that if $\xi_0 \preceq \Phi(\xi_0)$, then we have

$$\xi_n \preceq \xi_{n+1} \preceq \Phi(\xi_n) \preceq \Phi(\xi_{n+1}),$$

for any $n \in \mathbb{N}$. Since

$$\frac{1}{2}(\xi_n - \Phi(\xi_n)) = (\xi_n - \xi_{n+1}),$$

and ξ_n is comparable to $\Phi(\xi_n)$. Using the fact that, Φ is monotone PNE mapping implies that

$$\begin{aligned} \|\Phi(\xi_{n+1}) - \Phi(\xi_n)\| &= \left\|\Phi\left(\frac{\Phi(\xi_n) + \xi_n}{2}\right) - \Phi(\xi_n)\right\| \\ &\leq \frac{1}{2}\|\xi_n - \Phi(\xi_n)\| \\ &= \|\xi_n - \xi_{n+1}\| \end{aligned}$$

for any $n \in \mathbb{N}$. Using Lemma 2.2, we conclude that $\lim_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$. □

By incorporating compactness into the assumptions of Lemma 3.10 along with condition (E), we achieve the initial convergence outcome for an iteration linked to a monotone partially nonexpansive (PNE) mapping.

Theorem 3.11. *Let $(\mathscr{B}, \|\cdot\|, \preceq)$ be an ordered Banach space. Let $\mathscr{Y}$ be a nonempty bounded closed convex subset of $\mathscr{B}$. Assume that $\Phi : \mathscr{Y} \to \mathscr{Y}$ is a compact monotone PNE mapping satisfying condition (E). Let $\xi_0 \in \mathscr{Y}$ such that ξ_0 and $\Phi(\xi_0)$ are comparable. Define the sequence $\{\xi_n\}$ by the successive iteration*

$$\xi_{n+1} = \frac{\xi_n + \Phi(\xi_n)}{2}$$

for any $n \in \mathbb{N}$. Then $\{\xi_n\}$ converges to a point $z \in \mathscr{Y}$ which is a fixed point of Φ, i.e., $z \in F(\Phi)$.

Proof. Without loss of generality, we may assume $\xi_0 \preceq \Phi(\xi_0)$. From Lemma 3.10, we know that $\lim\limits_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$. Since Φ is compact, there exists a point z and a subsequence $\left\{\Phi\left(\xi_{\phi(n)}\right)\right\}$ which converges to z. Note that since $\mathscr{Y}$ is closed, we have $z \in \mathscr{Y}$. Clearly, the subsequence $\left\{\xi_{\phi(n)}\right\}$ also converges to z. Since

$$\xi_n \preceq \xi_{n+1} \preceq \Phi(\xi_n) \preceq \Phi(\xi_{n+1}),$$

and order intervals are closed, we conclude that $\xi_n \preceq \Phi(\xi_n) \preceq z$, for any $n \in \mathbb{N}$. Since Φ satisfies condition (E), we have

$$\left\|\xi_{\phi(n)} - \Phi(z)\right\| \leq \mu \left\|\Phi\left(\xi_{\phi(n)}\right) - \xi_{\phi(n)}\right\| + \left\|\xi_{\phi(n)} - z\right\|$$

for any $n \in \mathbb{N}$. Therefore $\left\{\xi_{\phi(n)}\right\}$ also converges to $\Phi(z)$. Hence $\Phi(z) = z$, i.e., z is a fixed point of Φ. The quasi-nonexpansiveness of Φ and z is comparable to the sequence $\{\xi_n\}$ imply

$$\begin{aligned}
\|\xi_{n+1} - z\| &\leq \frac{1}{2}\|\Phi(\xi_n) - z\| + \frac{1}{2}\|\xi_n - z\| \\
&\leq \frac{1}{2}\|\xi_n - z\| + \frac{1}{2}\|\xi_n - z\| \\
&= \|\xi_n - z\|,
\end{aligned}$$

for any $n \in \mathbb{N}$. In other words, the sequence $\{\|\xi_n - z\|\}$ is a decreasing sequence of positive numbers of which a subsequence goes to 0 . Hence $\lim\limits_{n\to\infty} \|\xi_n - z\| = 0$, i.e., $\{\xi_n\}$ converges to z. □

Theorem 3.12. *Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space which satisfies the monotone weak-Opial property. Let $\mathscr{Y}$ be a nonempty bounded closed convex subset of $\mathscr{B}$ and $\Phi : \mathscr{Y} \to \mathscr{Y}$ be a weakly compact monotone PNE mapping satisfying condition (E). Let $\xi_0 \in \mathscr{Y}$ such that ξ_0 and $\Phi(\xi_0)$ are comparable. Define the sequence $\{\xi_n\}$ by the successive iteration*

$$\xi_{n+1} = \frac{\xi_n + \Phi(\xi_n)}{2}$$

for any $n \in \mathbb{N}$. Then $\{\xi_n\}$ converges weakly to a point $z \in \mathscr{Y}$ which is a fixed point of Φ.

Proof. Without loss of generality, we may assume $\xi_0 \preceq \Phi(\xi_0)$. From Lemma 3.10, we know that $\lim_{n\to\infty} \|\xi_n - \Phi(\xi_n)\| = 0$. Since Φ is weakly compact, there exists a point z and a subsequence $\left\{\Phi\left(\xi_{\phi(n)}\right)\right\}$ which converges weakly to z. Note that since $\mathscr{Y}$ is closed and convex, we have $z \in \mathscr{Y}$. Note that $\left\{\xi_{\phi(n)}\right\}$ also weakly converges to z as well. Since $\{\xi_n\}$ is monotone increasing, then $\{\xi_n\}$ weakly converges to z as well as $\{\Phi(\xi_n)\}$. Note that we have $\xi_n \preceq \Phi(\xi_n) \preceq z$, which implies by monotonicity of Φ that $\xi_n \preceq \Phi(z)$ for any $n \in \mathbb{N}$. Assume that $\Phi(z) \neq z$, then by the monotone weak-Opial property, we have

$$\liminf_{n\to\infty} \|\xi_n - z\| < \liminf_{n\to\infty} \|\xi_n - \Phi(z)\|.$$

On the other hand, mapping Φ satisfies condition (E), it implies that

$$\|\xi_n - \Phi(z)\| \leq \mu \|\Phi(\xi_n) - \xi_n\| + \|\xi_n - z\|$$

for any $n \in \mathbb{N}$. Hence

$$\liminf_{n\to\infty} \|\xi_n - \Phi(z)\| \leq \liminf_{n\to\infty} \|\xi_n - z\|.$$

This contradiction forces $\Phi(z) = z$, i.e., z is a fixed point of Φ. □

If we combine the two above theorems, we get an existence fixed point theorem for monotone PNE mapping satisfying condition (E).

Theorem 3.13. *Let $(\mathscr{B}, \|.\|, \preceq)$ be an ordered Banach space. Let $\mathscr{Y}$ be a nonempty convex subset of $\mathscr{B}$ and $\Phi : \mathscr{Y} \to \mathscr{Y}$ be a monotone PNE mapping satisfying condition (E). Let $\xi_0 \in \mathscr{Y}$ such that ξ_0 and $\Phi(\xi_0)$ are comparable. Assume either of the following assumptions holds*

(a) *$\mathscr{Y}$ is compact;*

(b) *$\mathscr{Y}$ is weakly compact and $\mathscr{B}$ satisfies the monotone weak-Opial property.*

Then Φ has a fixed point.

Authors' Contributions

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Conflict of Interests

The author declares that there is no conflict of interests.

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