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BIFURCATION AND CHAOTIC BEHAVIOR OF TWO PARAMETER FAMILY OF GENERALIZED LOGISTIC MAPS

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Abstract. The goal in the article is to study the bifurcation and chaotic behavior of the maps $\eta x(1-x)^n$ over the real domain in the real parameter space, considering η is a positive real parameter which is continuous and n is positive integer. The dynamic properties of the proposed family are not only theoretically analyzed, but also they are analyzed graphically and numerically. The fixed points (real) are simulated theoretically and the periodic points are computed numerically. Furthermore, we discussed the stability of the fixed points as well as periodic points. The plot of the bifurcation of the maps are given by altering the parameters. The presence of chaos in the dynamics of this family is investigated by studying period-doubling phenomena in the bifurcation diagram, and chaotic behavior is been quantified by finding positive Lyapunov exponents.

Keywords: bifurcation; chaos; fixed points; periodic points; attractor; Fatou sets; Julia sets.

2020 AMS Subject Classification: 37F10, 30D05.

1. Introduction

For last four decades, advancement in computer graphics uses of technology that helped to derive new formulation and develop non-linear technique for convoluted systems [4, 3]. As many physical, socioeconomic, and natural systems are inherently non-linear in nature, hence this systems shows a large number of various characteristics. At the same time, chaos is not

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only been a center of focus in many various fields of sciences and engineering [5, 6, 11] but also redesigns many researches; the analysis of chaotic behavior in the dynamical systems is an interesting field of study of many engineers, scientists, and mathematicians. Multiple disciplines, such as stoke market [7], fashion cycle model [8], modeling [9, 10], optimization [16], photovoltaic plant [12] and many other uses it extensively. The comprehensive dynamical analysis is described for an H-bridge parallel resonant converter within a zero current switching control in [19]. The dynamics of DC-AC resonant self-oscillating LC series inverter is explored in [15]. The analysis has been done in the sense of piecewise smooth dynamical system and the bifurcation analysis has been done for the parameter space that are one dimensional. A discrete dynamics approach has been used to see different ruses on using sporadic computing methods on ontology learning and determine its effectiveness through tests. The recrudescence in non-autonomous discrete dynamical systems is explored in [22]. Bifurcation analysis of fertility and gender equality is explained in [18]. In [23] the efficiency-wage competition model introduced by Hahn (1987) is illustrated, and the chaotic behavior is described in the parameter space as well.

The word chaos in a system is united appearance of non-linear alliance, determinism, order and sensitive dependence. Chaos is a wonderful phenomenon in mathematics, physics and many other fields of sciences. The research in chaos indicate that an elementary system can show a complicated, uncertain pattern. A chaos can be described in various shapes, as per circumstances, observations, or application to the object. The chaos, as a mathematical concept was emerged around 1980.

The idea of chaos is usually connected to the field called dynamical systems in Mathematics. This can also be classified in the dynamics as the tactful reliance on the preliminary conditions. The dynamical system is a discussion of the process that vary over the time. The primodial goal to elaborate a dynamical system and to see the asymptotic behavior of the trajectories which is related to long-term monitoring. Moreover, one can say that the initial condition has gone identical long term behavior. A primordial focus in dynamical systems to find out the behavior of the orbits near fixed points. Fixed points are called stable if the initial disturbance is set to be stable over time where is it is unstable if the primary inconsistency tends to grow over long-term. G. Julia, a French mathematician, looked into the dynamics of quadratic polynomials in the complex plane and used the Julia set to disclose a well-known fractal example. Benoit B. Mandelbrot, a French-American mathematician, first introduced the Mandelbrot set concept in 1975 [24]. To study the dynamics of meromorphic maps reader can see ([13], [14]). For local connectivity of Julia set see ([20], [26], [27]).

The article is arranged as follows. We have defined all the basic terminologies that is been used in the article in section 2. In section 3, we have discussed the stable and unstable domains of the function and have used graphs to elaborate the attracting domain in the real line when the parameter value $n = 1$. Section 4 gives a more generalized dynamics of the maps for the parameter value $n > 1$. Furthermore, we use numerical simulation to dig the details of the periodic fixed points of higher order in this section. In section 5, we showed how the period bifurcation occurs for a perturbation of the parameter values and also use the positive Lyapunov exponents to analyze the chaos in the parameter region. The last section is reserved for giving details of application of this phenomenon in different fields and it talks about the future scope of extending this work in a broder sense for higher dimension.

2. Preliminaries

The proposed map in this article is the generalized logistic maps over the real domain with real parameters defined by $g_\eta(x) = \eta x(1-x)^n$ where n is defined as the set of positive discrete values. This family is denoted by $\mathscr{D}$ and we will denote any map $g_\eta(x)$ in $\mathscr{D}$ by g until unless we need more detail description of the map for convenience of the reader. A sequence of points $\{x, g(x), g^{\circ 2}(x), g^{\circ 3}(x), \dots\}$ for $x \in \mathbb{R}$ is called an *orbit* of x under the iteration of g. A point x_0 is said to be a fixed point of the function g if $g(x_0) = x_0$, where $x_0 \in \mathbb{R}$. The following theorem classifies the points in the *Fatou* set. We will use the theorem to define the points in Fatou set (Stable set).

A periodic point p is called *attractor* if $|(g^m)'(p)| < 1$ and the point p is called *repellor* if $|(g^m)'(p)| > 1$. Suppose x_0 is a periodic fixed point of a function g of period p, that is $g^p(x_0) = x_0$. We define $\lambda = |(g^p)'(x_0)|$, as the multiplier of the function g at x_0. We classify the fixed point as follows:

(i) If $\lambda = 0$, x_0 is called a *super-attracting* fixed point.

(ii) If $|\lambda| < 1$, x_0 is called an *attracting* fixed point.

(iii) If $|\lambda| > 1$, x_0 is said to be a *repelling* fixed point.

(iv) If $|\lambda| = 1$, x_0 is called a *neutral* fixed point.

If a point q is a m-periodic attracting fixed point of g, then there is a neighborhood V of q so that all points of V are forward asymptotic to q under g^k; that is, we have $g^{mk}(s) \to q$ as $k \to \infty$ for all s in V. Such a set V is called a *local stable set* of q and is denoted by $W^t_{loc}(q)$. The *stable set* of q, denoted by $W^t(q)$, as the set of points forward asymptotic to q, or equivalently as the set of points mapped to $W^t_{loc}(q)$ under iteration of g^{hm}

$$W^t(q) = \bigcup_{k=1}^{\infty} \{x | g^{hm}(x) \in U\}$$

The *Fatou set* is given as the union of all stable sets. Complement of stable set is said to be the *Julia set* or *unstable set*.

3. Stable and Unstable Components for $n = 1$

We study the dynamics of the family of maps when $n = 1$ through the following results. We are interested to study the real fixed points and real periodic points for a certain range of the parameter values η as well as study about the connectivity of Julia sets.

Lemma 3.1. Let $g_\eta \in \mathscr{D}$. If $-1 < \eta < 1$, $x = 0$ is a super-attracting fixed point whereas $x = \frac{\eta - 1}{\eta}$ is an attracting fixed point for $1 < \eta < 3$.

Proof. $g'_\eta(x) = \eta(1 - 2x)$. The fixed points of the equation is given by, $x_1 = 0$, $x_2 = \frac{\eta - 1}{\eta}$. So $|g'_\eta(0)| = |\eta| < 1$ when $|\eta| < 1$. Therefore, $x = 0$ is attracting fixed point for $-1 < \eta < 1$ and repelling fixed point for $|\eta| > 1$. In other ways, we have $|g'_\eta(\frac{\eta-1}{\eta})| = |\eta(1 - \frac{2(\eta-1)}{\eta})| = |\eta(\frac{\eta - 2\eta + 2)}{\eta})| = |2 - \eta| < 1$. Thus, $-1 < 2 - \eta < 1$ and $\frac{\eta-1}{\eta}$ is an attracting fixed point for $1 < \eta < 3$. □

Lemma 3.2. The function $g_\eta(x) \in \mathscr{D}$ has an attracting periodic two cycle for $3 < \eta < 3.45$.

Proof. If x is an attracting periodic point in a period two cycle of $g_\eta(x)$, then it must satisfy $g_\eta^2(x) = x$. That implies, $\eta^2 x^2 - \eta x(\eta+1) + (\eta+1) = 0$. Thus we get the two periodic points as follows.

$x_1 = \frac{1}{2\eta}\{(\eta+1) + \sqrt{(\eta-3)(\eta+1)}\}$ and $x_2 = \frac{1}{2\eta}\{(\eta+1) - \sqrt{(\eta-3)(\eta+1)}\}$

The multiplier is given by, $|(g^2)^{'}(x_1)| = |(g^2)^{'}(x_2)| = |g^{'}(x_1)g^{'}(x_2)| = |1-(\eta-3)(\eta+1)| = |\eta^2 - 2\eta - 4|$. For x_1 and x_2 to be an attracting periodic cycle we need to have $|\eta^2 - 2\eta - 4| < 1$ implies, $3 < \eta < 3.45$. □

Lemma 3.3. For each $\eta \in \mathbb{R}$, there is $k_\eta \in \mathbb{R}$ such that $|g_\eta^n(x)| \to \infty, \forall\ |x| > k_\eta$ as $n \to \infty$.

Proof. A function $g_\eta(x)$ is a parabola concave upward or downward according to $\eta > 0$ or $\eta < 0$ and $g_\eta(x)$ is monotonic over $(-\infty, 1/2) \cup (1/2, \infty,)$. $x_\eta = \frac{1-\eta}{\eta}$ is an attracting fixed point of g_η for $2 < \eta < 3$ by (3.1). Take $k_\eta = \frac{1-\eta}{\eta}$ for $\eta \in (-\infty, 2] \cup [3, \infty)$. Since $|g_\eta'(k_\eta)| > 1$, $\forall \eta \in (-\infty, 2] \cup [3, \infty)$, k_η is a repelling fixed point. Therefore the orbit $\{g_\eta^n(x)\}$ of x is divergent and $|g_\eta^n(x)| \to \infty\ \forall\ |x| > k_\eta$ as $n \to \infty$.

If $2 < \eta < 3$, x_η is an attracting periodic fixed point. Let U_{x_η} be the set of points attracted by the attracting periodic fixed points. We claim that U_{x_η} is bounded.

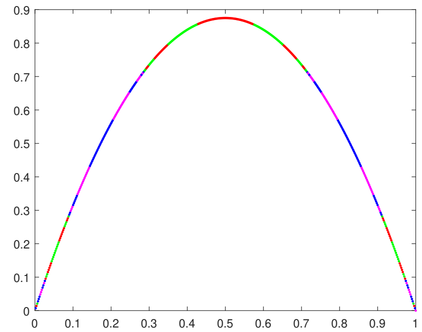
If not, there is a sequence of points $x_n = g^{2n}(x)$ such that $|x_n| \to \infty$ as $n \to \infty$. Therefore the orbit of x has no uniformly convergent sub-sequence. Therefore x is not in Fatou set. Thus U_{x_η} must be bounded. So there is an M such that $|x| < M\ \forall\ x \in U_{x_\eta}$. Let $k_\eta = M_\eta \forall\ \eta \in (2,3)$. Then $|g^n(x)| \to \infty$ as $n \to \infty\ \forall\ |x| > k_\eta$ for $2 < \eta < 3$. □

Theorem 3.4. The Julia set has no connected components.

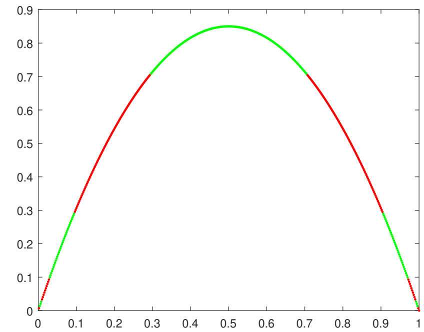
Proof. By Lemma 3.3 we have that for each $\eta \in \mathbb{R}$, there exists a k_η so that $|x| < k_\eta\ \forall\ x \in \mathscr{J}$. If there is a connected component U of $\mathscr{J}$, then $|x| < k_\eta\ \forall\ x \in U$. So $g_\eta^n(x)$ is uniformly convergent and forms a normal family. Contradiction. □

Lemma 3.5. The Fatou set is the union of connected components.

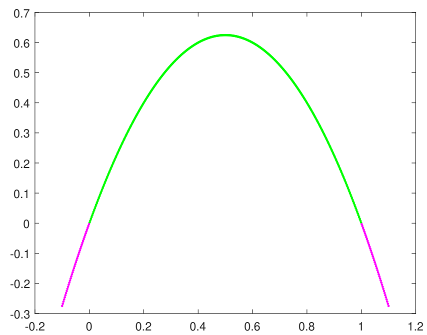
Proof. We have $\mathscr{F} = \mathbb{R} \setminus \mathscr{J}$. By theorem 3.4, $\mathscr{J}$ as no connected components in $\mathbb{R}$. Hence $\mathscr{F}$ must be the union of connected components in $\mathbb{R}$. □



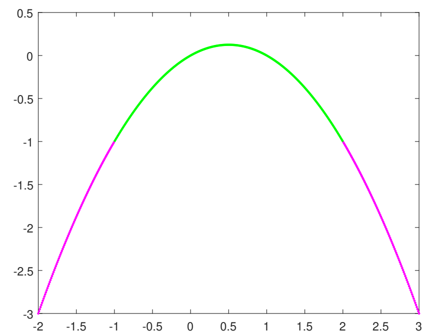
(A) Bounded Fatou domain $\eta = 3.5$



(B) Bounded Fatou domain $\eta = 3.4$



(C) Bounded and unbounded Fatou domain $\eta = 2.5$



(D) Bounded and unbounded Fatou domain $\eta = 0.5$

FIGURE 1. Fatou and Julia set for various parameter values

Lemma 3.6. The Fatou set is non-empty.

Proof. Using Lemma 3.4 one can find that for every $\eta \in \mathbb{R}$, $\exists\ k_\eta \in \mathbb{R}$ such that $|x| < k_\eta$ and $\mathscr{J} \subset [-k_\eta, k_\eta]$. Therefore $x \in \mathscr{F}\ \forall\ |x| > k_\eta$. Hence $\mathscr{F}$ is non-empty. □

Theorem 3.7. A function $g_\eta(x)$ has finite attracting (or super-attracting) periodic cycle for $-2 \leq \eta \leq 4$.

Proof. **Case 1:** $\eta > 4$. In this case the immediate attracting domain of an attracting periodic point should contain a critical value. Then it is enough to trace the orbit of the critical value.

$g_\eta(1/2) = \frac{\eta}{2^2}$,

Thus we have the $g_\eta^n(1/2)$ as

$g_\eta^n(1/2) = \eta^n \frac{(2-1)(2^2-1)(2^4-(2^2-1))(2^8-(2^4-(2^2-1)))(2^{2^{n-1}}-(2^{2^{n-2}}-(\ldots-(2^2-1)))}{(2^2)^n}$ and

$g_\eta^n(1/2) = \eta^n \frac{(2-1)(2^2-1)(2^4-(2^2-1))(2^8-(2^4-(2^2-1)))(2^{2^{n-1}}-(2^{2^{n-2}}-(...-(2^2-1)))}{(4)^n} > (\frac{\eta}{4})^n, \ \forall\, n \geq 2$

Therefore $(\frac{\eta}{4})^n \to \infty$ as $n \to \infty$ for $\eta > 4$, implies $g_\eta^n(1/2) \to \infty$ as $n \to \infty$.

Case 2: $\eta < 2$. $g_\eta(1/2) = \eta.\frac{1}{2}.(1-\frac{1}{2}) = \eta\frac{(2-1)}{2^2}$

$g_\eta^2(1/2) = \eta^2 \frac{(2-1)(2^2-(2-1))}{2^{2^2}}$

$\vdots$

$g_\eta^n(1/2) = (\frac{\eta}{2})^n \frac{(2-1)(2^2-1)(2^4-(2^2-1))(2^8-(2^4-(2^2-1)))(2^{2^{n-1}}-(2^{2^{n-2}}-(...-(2^2-1)))}{2^n}$

Therefore we have $|g_\eta^n(1/2)| > \frac{(2-1)(2^2-1)(2^4-(2^2-1))(2^8-(2^4-(2^2-1)))(2^{2^{n-1}}-(2^{2^{n-2}}-(...-(2^2-1)))}{2^n}$ as $|\eta| > 2$. Furthermore, $\frac{(2-1)(2^2-1)(2^4-(2^2-1))(2^8-(2^4-(2^2-1)))(2^{2^{n-1}}-(2^{2^{n-2}}-(...-(2^2-1)))}{2^n} > n!$

Thus $|g_\eta^n(1/2)| \to \infty$ as $n \to \infty$. This concludes that the function $g_\eta^n(x)$ has attracting periodic fixed points for $-2 < \eta < 4$. □

Remark 3.1. $x = 3/2$ is a repelling periodic point for $\eta = 2$ and at $\eta = 4$, $x = 1$ is a repelling periodic point.

Lemma 3.8. The Julia set is finite.

Proof. By theorem 3.7, a function $g_\eta(x)$ has bounded Fatou components for $2 < \eta < 4$. So for $\eta \in (-\infty, 2] \cup [4, \infty)$, the Fatou set is given by $(-\infty, k_\eta) \cup (k_\eta, \infty)$ for some k_η satisfying $|x| > k_\eta$. Clearly the Julia set is finite and a singleton set $\{k_\eta\}$. On the other hand when Fatou set has bounded components, the set will be given by the union of countably many open intervals. So the Julia set is given by the boundary points of these intervals and are the union of countably many boundary points. Therefore the points in the Julia set is given by an infinite sequence of points in $\mathbb{R}$. Because the sequence is bounded, it has a convergent subsequence in $\mathbb{R}$ and the limit point will be in the Julia set. Let p be the limit point. Then we can have a small neighborhood U around p such that U contains at-least one point of the Julia set except p. In other words there is no open interval around p belonging to the Fatou set. That is a contradiction! Thus Julia set must be finite. □

Theorem 3.9. Let $g_{\eta,n}(x) \in \mathscr{D}$. The map $g_{\eta,n}(x)$ has a fixed point at 0, for all η, and one non-zero real fixed point. Furthermore, the non-zero fixed point is negative for $\eta < 1$ and positive for $\eta \in (-\infty, 0) \cup (1, \infty)$.

Proof. For real fixed point of $g_{\eta,n}(x)$ we set $g_{\eta,n}(x) = x \implies \eta x(1-x)^n = x \implies x = 0, \eta(1-x)^n = 1$. Therefore $x = 0$ and $x_{\eta,n} = 1 - \sqrt[n]{\frac{1}{\eta}}$ are the solution of the equation and gives the fixed point of $g_{\eta,n}(x)$. Further it is easy to see that $x_{\eta,n} < 0$ when $\eta < 1$ and $x_{\eta,n} > 0$ for $\eta > 1$. □

Remark 3.2. The fixed point $x_{\eta,n}$ is always less than 1.

4. Dynamics of the Family for $n > 1$

This family shows very interesting dynamical properties for $n > 1$. The following results pledges with the study of dynamical charecteristic of the proposed map but this study is been extended into numerical stimulation to get the higher order attracting periodic points.

Proposition 4.1. Suppose that $k(x) = \frac{1-nx}{1-x},\ n > 1,\ x \in \mathbb{R}$. Then function does not attain any maximum or minimum in $\mathbb{R}$. Moreover, $k(x) \to \infty$, as $x \to 1^+$ and $k(x) \to -\infty$ as $x \to 1^-$ and $k(x) \to \pm n$ as $x \to \pm\infty$.

Theorem 4.2. Suppose $g_{\eta,n}(x) \in \mathscr{D}$. Then the following hold,

a) The real fixed point $x = 0$ is attracting if $|\lambda| < 1$, rationally indiffernt if $|\lambda| = 1$, and repelling when $|\lambda| > 1$.

b) The real fixed point $x_{\eta,n}$ is attracting for $x_{\eta,n} \in (0, \frac{2}{1+n})$, rationally indifferent for $x_{\eta,n} = 0, \frac{2}{1+n}$, and repelling otherwise.

Proof. We have that $g'_{\eta,n}(x) = \eta(1-x)^{n-1}(1-nx)$.

a) For $x = 0, |g'_{\eta,n}(x)| = |\eta| < 1, = 1$, or > 1 conclusion holds accordingly.

b) For the real fixed point $x_{\eta,n}$, when $\eta > 0,\ g'_{\eta,n}(x_{\eta,n}) = \frac{(1-nx_{\eta,n})}{(1-x_{\eta,n})}$. Therefore, $|g'_{\eta,n}(x_{\eta,n})| < 1 \implies |\frac{(1-nx_{\eta,n})}{(1-x_{\eta,n})}| < 1 \implies 0 < x_{\eta,n} < \frac{2}{1+n}$. For $|g'_{\eta,n}(x_{\eta,n})| = 1$, we have $x = \frac{2}{1+n}$. Furthermore $x_{\eta,n} = 0$ implies $\eta = 1$ which is already considered in (a). Since $g'_{\eta,n}(x_{\eta,n})$ is decreasing for $x < 1$, and $g'_{\eta,n}(x = \frac{2}{1+n}) = -1$, therefore $|g'_{\eta,n}(x_{\eta,n})| > 1$ for $1 > x > \frac{2}{1+n}$. This completes the proof. □

4.1. Numerical iteration of real periodic points of period > 1.

To see the higher periodic points of period larger than one is quite cumbersome. We need to use mathematical simulation to calculate the periodic points greater than one. When $0 < \eta < 1,\ x = 0$ is the only attracting

fixed point and $x_{\eta,n} < 0$, so it is not an attracting fixed point.
For a fixed n, $\exists\ \eta_n > 1$ such that $x_{\eta,n}$ is attracting periodic point of period one.

The numerical stimulation is been done for $n = 2,3,4,8$ and choose $\eta > \eta^*$ such that $g_{\eta,n}$ has periodic points of period greater than 1. For $n = 2$, the periodic points of period two starts from $\eta \approx 4.1$. Clearly these periodic points will be the roots of the equation $g^k_{\eta,n}(x) = \eta g^{k-1}_{\eta,n}(x)(1 - g^{k-1}_{\eta,n}(x))^n = x$.

4.2. Periodic doubling of logistic maps.

- We calculate the periodic cycle of period $2,4,8$ for values of $n = 2,3,4$, and 5, where the other parameter value η is taken as $\eta = 4.1, 5.1, 5.25$. For $n = 2$, the periodic points of period $2,4,8$ are calculated for $\eta = 4.1, 4.9$ and 5.2.
- For $\eta = 4.1$ the 2-periodic cycle points p_1 and p_2 of $g_{\eta,n}$ are as following,
  $p_1 \approx 0.5781$ and $p_2 \approx 0.4219$. $g_{\eta,n}(p_1) = -0.2702$ and $g_{\eta,n}(p_2) = 0.3702$.
  Then $|g'_{\eta,n}(p_1).g'_{\eta,n}(p_2)| \approx 0.1000 < 1$ and p_1 and p_2 are 2-cycle attracting periodic points.
- If $\eta = 5.1$, the $4-$ periodic cycle points p_1, p_2, p_3, p_4 are as following,
  $p_1 \approx 0.6961$, $p_2 \approx 0.3279$, $p_3 \approx 0.7554$, $p_4 \approx 0.2305$, Then $g'_{\eta,n}(p_1) = -0.6079$, $g'_{\eta,n}(p_2) = -0.6079$, $g'_{\eta,n}(p_3) = -0.6372$, $g'_{\eta,n}(p_4) = 2.1153$.
  $m_{\eta,n} = \prod_{i=1}^{4} |g'_{\eta,n}(p_i)| = 0.9666 < 1$. Therefore p_1, p_2, p_3 & p_4 are $4-$ cycle attracting periodic points.
- For $\eta = 5.25$, the $8-$ periodic cycle points $p_1, p_2, \dots p_8$ of $g_{\eta,n}(x)$ are as follows:
  $p_1 \approx 0.2019$, $p_2 =\approx 0.6752$, $p_3 \approx 0.3739$, $p_4 \approx 0.7695$, $p_5 \approx 0.2147$, $p_6 \approx 0.6951$, $p_7 \approx 0.3393$, $p_8 \approx 0.7776$. Thus, $g'_{\eta,n}(p_1) = 0.4267$, $g'_{\eta,n}(p_2) = -0.5804$, $g'_{\eta,n}(p_3) = 0.8053$, $g'_{\eta,n}(p_4) = -0.6336$, $g'_{\eta,n}(p_5) = 0.2853$, $g'_{\eta,n}(p_6) = -0.6058$, $g'_{\eta,n}(p_7) = 1.0830$, $g'_{\eta,n}(p_8) = -0.6297$.
  $m_{\eta,n} = \prod_{i=1}^{8} |g'_{\eta,n}(p_i)| = 0.6797 < 1$. Hence $\{p_i\}_{i=1}^{8}$ are $8-$ cycle attracting periodic points.
- When $n = 3$, we calculate the periodic points of period 2, for $\eta = 4.8$. If $\eta = 4.8$, the $2-$ periodic cycle points $p_1, \& p_2$ of $g'_{\eta,n}(x)$ are as follows.

$p_1 \approx 0.4819,\ p_2 \approx 0.0771$ and $m_{\eta,n} = \prod_{i=1}^{2} |g'_{\eta,n}(p_i)| = 0.0443 < 1$. So, $p_1 \& p_2$ are 2-cycle attracting periodic points.

- When $n = 4$, we calculate the periodic point of period 2, for $\eta = 5.5$, of $g_{\eta,n}(x)$ as follows:

  $p_1 = 0.4427,\ p_2 = 0.2348$ and $g_{\eta,n}(p_1) = -0.7338,\ g_{\eta,n}(p_2) = 0.1498$. Thus, $m_{\eta,n} = \prod_{i=1}^{2} |g'_{\eta,n}(p_i)| = 0.1099 < 1$ and $\{p_i\}_{i=1}^{2}$ are $2-$ cycle attracting periodic points.

  For $n = 5$, we conclude the periodic points of period 2 for $\eta = 5.8$. The periodic points are as below,

  $p_1 = 0.3781$ & $p_2 = 0.2040$ and $g'_{\eta,n}(p_1) \approx -1.0123,\ g'_{\eta,n}(p_2) \approx -0.0618$ implies $m_{\eta,n} = \prod_{i=1}^{2} |g'_{\eta,n}(p_i)| = 0.0618 < 1$. Hence, $\{p_i\}_{i=1}^{2}$ are $2-$ cycle attracting periodic points.

- If $n = 3$, and $\eta = 6.5$, the periodic points are as following: $p_1 \approx 0.1309,\ p_2 \approx 0.5768,\ p_3 \approx 0.2842,\ p_4 \approx 0.6775,\ p_5 \approx 0.1477,\ p_6 \approx 0.5944,\ p_7 \approx 0.2578,\ p_8 \approx 0.6851$. Furthermore, $g'_{\eta,n}(p_1) \approx 2.1395,\ g'_{\eta,n}(p_2) \approx -1.5218,\ g'_{\eta,n}(p_3) \approx -0.4556,\ g'_{\eta,n}(p_4) \approx -1.1560,\ g'_{\eta,n}(p_5) \approx 1.9321,\ g'_{\eta,n}(p_6) \approx -1.4731,\ g'_{\eta,n}(p_7) \approx -0.1117,\ g'_{\eta,n}(p_8) \approx -1.1218$.

  The multiplier $m_{\eta,n} = \prod_{i=1}^{8} |g'_{\eta,n}(p_i)| \approx 0.6116 < 1$. Therefore there is a $8-$ periodic attracting cycle.

  For $\eta = 6.4$, the periodic points are as given, $p_1 \approx 0.1495,\ p_2 \approx 0.5887,\ p_3 \approx 0.2622,\ p_4 \approx 0.6739,\ g'_{\eta,n}(p_1) \approx 1.8610,\ g'_{\eta,n}(p_2) \approx -1.4668,\ g'_{\eta,n}(p_3) \approx -0.1700,\ g'_{\eta,n}(p_4) \approx -1.540$.

  $m_{\eta,n} \approx \prod_{i=1}^{4} |g'_{\eta,n}(p_i)| = 0.5356 < 1$. Hence, $g_{\eta,n}(x)$ has an attracting periodic $4-$cycle.

- If $n = 4$, and $\eta = 7.1$, the periodic points are given below;

  $p_1 \approx 0.1906,\ p_2 \approx 0.5808,\ p_3 \approx 0.1273,\ p_4 \approx 0.5243$ and $g'_{\eta,n}(p_1) \approx 0.1769,\ g'_{\eta,n}(p_2) \approx -0.9958,\ g'_{\eta,n}(p_3) \approx 1.7154,\ g'_{\eta,n}(p_4) \approx -1.2393$. Hence, $m_{\eta,n} \approx \prod_{i=1}^{4} |g'_{\eta,n}(p_i)| = 0.3746 < 1$. Therefore, $g_{\eta,n}(x)$ has an attracting periodic $4-$ cycle.

For $\eta = 7.5$, the periodic points are given as:

$p_1 \approx 0.1989$, $p_2 \approx$, 0.6144, $p_3 \approx 0.1019$, $p_4 \approx 0.4972$, $p_5 \approx 0.2384$, $p_6 \approx 0.6016$, $p_7 \approx 0.1137$, $p_8 \approx 0.5262$. Furthermore, we have $g'_{\eta,n}(p_1) \approx 0.0212$, $g'_{\eta,n}(p_2) \approx -0.8910$, $g'_{\eta,n}(p_3) \approx 2.6649$, $g'_{\eta,n}(p_4) \approx -1.4167$, $g'_{\eta,n}(p_5) \approx -0.6361$, $g'_{\eta,n}(p_6) \approx -0.9523$, $g'_{\eta,n}(p_7) \approx 2.2531$, $g'_{\eta,n}(p_8) \approx -1.3011$. The multiplier value is given by, $m_{\eta,n} \approx \prod_{i=1}^{8} |g'_{\eta,n}(p_i)| \approx 0.1267 < 1$. Hence, $g_{\eta,n}(x)$ has an attracting periodic cycle of period 8.

- For $n = 5$, $\eta = 8.2$ the periodic points are as follows:

  $p_1 \approx 0.1913$, $p_2 \approx 0.5426$, $p_3 \approx 0.0891$, $p_4 \approx 0.4582$, $p_5 \approx 0.1754$, $p_6 \approx 0.5484$, $p_7 \approx 0.0845$, $p_8 \approx 0.4456$, and the value of the corresponding multiplier map is given by $g'_{\eta,n}(p_1) \approx -0.5184$, $g'_{\eta,n}(p_2) \approx -0.8096$, $g'_{\eta,n}(p_3) \approx 2.6274$, $g'_{\eta,n}(p_4) \approx -1.2360$, $g'_{\eta,n}(p_5) \approx 0.1987$, $g'_{\eta,n}(p_6) \approx -0.7812$, $g'_{\eta,n}(p_7) \approx 2.8398$, $g'_{\eta,n}(p_8) \approx -1.2965$.

  $m_{\eta,n} \approx \prod_{i=1}^{8} |g'_{\eta,n}(p_i)| \approx 0.7787 < 1$. Therefore, $g_{\eta,n}(x)$ has an attracting $8-$ periodic cycle.

  For $\eta = 7.58$, the periodic points are as following:

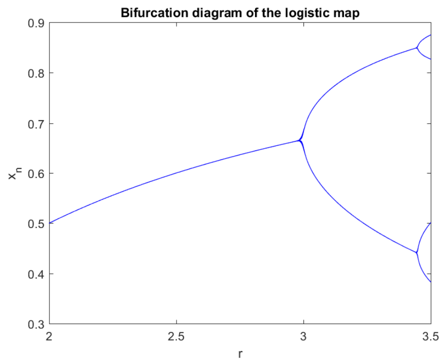
  $p_1 \approx 0.9432$, $p_2 \approx -0.9873$, $p_3 \approx 1.0604$, $p_4 \approx -1.0057$, the value of the first order derivative of the map at those points are given as $g'_{\eta,n}(p_1) \approx 0.9432$, $g'_{\eta,n}(p_2) \approx -0.9873$, $g'_{\eta,n}(p_3) \approx 1.0604$, $g'_{\eta,n}(p_4) \approx -1.0057$.

  $m_{\eta,n} \approx \prod_{i=1}^{4} |g'_{\eta,n}(p_i)| \approx 0.9930 < 1$. Hence, $g_{\eta,n}(x)$ has an attracting $4-$ periodic cycle.

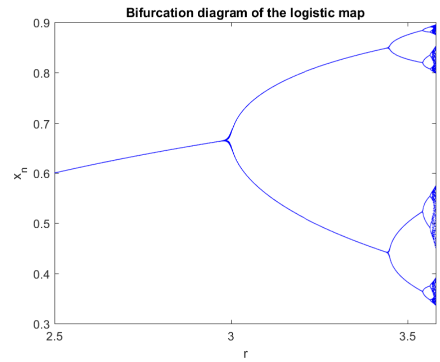
5. PERIOD DOUBLING BIFURCATION

We studied the bifurcation properties of this family of maps based on our earlier analysis. We emphasized the period doubling bifurcation of the maps in the family in Figure 3. Since the critical value of the map is always attracted by an attracting periodic cycle wherever attracting cycle exists, we used this phenomena to find the period doubling bifurcation. The critical value of a map g_η is given by $x = \eta/4$.

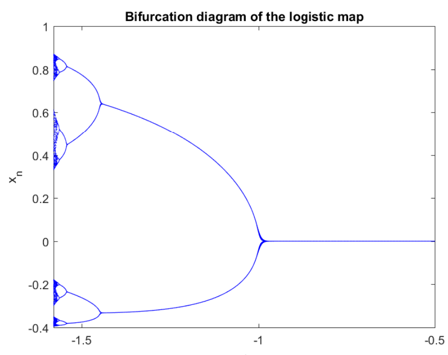
Here we discuss the bifurcation of the periodic points when η and n varies. We have plotted the bifurcation diagram for $n = 2, 3, 4$ & 5. It is clear from the Figure 2 that as η increases the periodic cycle bifurcates into a higher order cycle. However, the bifurcation does not maintain the attracting behavior of the cycle.



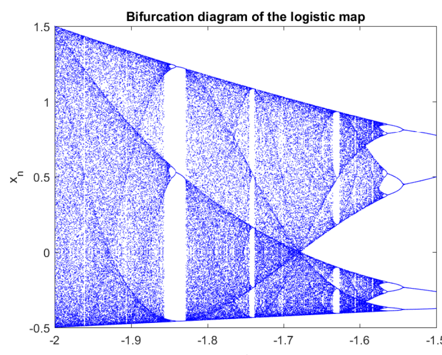
(A) Period doubling bifurcation for $\eta \in (2, 3.5)$



(B) Period doubling bifurcation for $\eta \in (2.5, 3.58)$



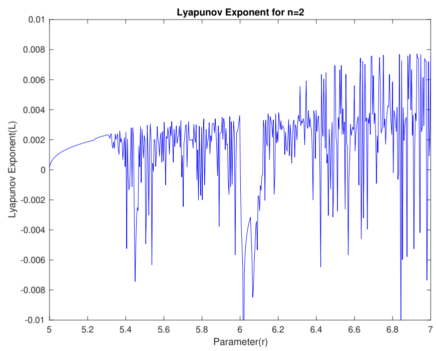
(C) Period doubling bifurcation for $\eta \in (-1.58, -0.5)$



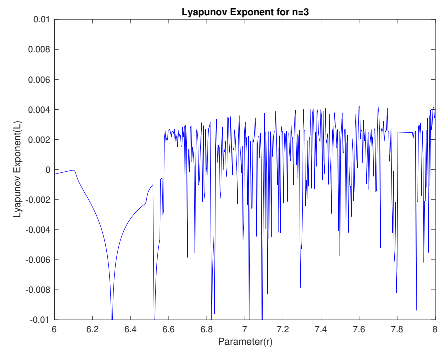
(D) Period doubling bifurcation for $\eta \in (-2, -1.5)$

FIGURE 2. Period doubling bifurcation

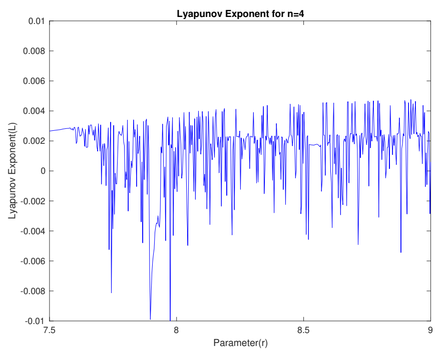
In other words, for $n = 3, 4, 5$, the attracting $2-$ cycle bifurcates into an attracting $4-$ cycle, an attracting $8-$ cycle and so on, however for $n = 2$, an attracting $2-$ cycle bifurcates into a $4-$ cycle, $8-$ cycle and so on. The period-doubling phenomenon in the bifurcation diagrams steers the way to chaos of the $g_{\eta,n} \in \mathscr{F}$ in real dynamics.



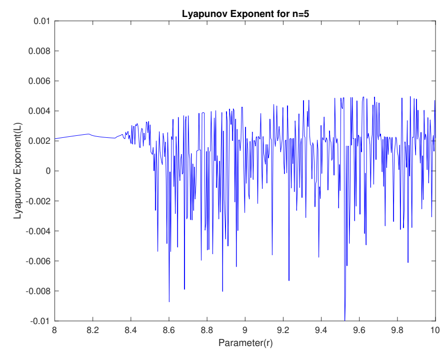
(A) $n = 2,\ 5 < \eta < 3.5$



(B) $n = 3,\ 6 < \eta < 8$



(C) $n = 4,\ 7.5 < \eta < 9$



(D) $n = 5,\ 8 < \eta < 10$

FIGURE 3. Value of Lyapunov Exponent for various parameter values

5.1. Lyapunov exponents. The Lyapunov exponent of the map $g_{\eta,n} \in \mathscr{F}$ is to be calculated. This is another key to know the chaotic systems for those η, which are responsible for the period doubling phenomenon in the bifurcation diagrams.

Using the formula, we have the Lyapunov exponent for the family of maps as

$$L = \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln |\eta(1-(k+1)x(i))(1-x(i))^{k-1}|,\ n = 2,3$$

For our calculation, we choose $x_0 = 0.33,\ x_0 = 0.20,\ (n = 4,5)$ and $k = 500$. The calculated value of the Lyapunov exponents are explained in the following figure for $n = 2,3,4,5$ with $5 \le \eta \le 7,\ 6 \le \eta \le 8,\ 7.5 \le \eta \le 9,\ \&\ 8 \le \eta \le 10$ respectively. This is clearly seen from these figures that, the Lyapunov Exponents are negative for same values of the parameter η, and it shows tactful dependence on the preliminary conditions. Therefore, chaotic behavior exists in the real dynamics of $g_{\eta,n} \in \mathscr{F}$.

The bifurcation diagrams (in bifurcation diagram figure) and the corresponding Lyapunov exponents are discussed here in the last diagram for intervals of parameters η. For positive Lyapunov exponents, the bifurcation diagram has dense blue region which shows chaotic behavior in the real dynamics of $g_{\eta,n}(x)$ for some fixed ranges of parameter values. Furthermore, for other values the Lyapunov exponents that are negative, the bifurcation diagrams have white regions; it indicates that the chaotic regions breaks up into non-chaotic initially and then goes back to being chaotic.

We have iterated the critical value when the parameter value is varying over the real line. The length of the attracting periodic cycle will be determined by the attracting periodic points where the critical value lands after a large number of iterations complete. The following diagrams are given when parameter values range over the positive and negative real number in the range of $(-2,3.58)$.

6. Conclusive Remarks

The article discusses the dynamics of a freshly introduced family of real maps with two parameters including the logistic function in one-dimensional settings. The fixed points (real) with their properties are theoretically investigated however, numerical simulation is being applied to see the periodic points. The period doubling along with period-three window is being investigated through bifurcation diagrams, the existence of chaotic behavior is studied. The chaos is being determined by calculating the positive Lyapunov exponents. The fractal structures that are generated for the stable and unstable components are topologically rich enough to investigate. Moreover, there are some questions in terms of the combinatorial structures of the fractals, that are not addressed in this article but would be something interesting to investigate in some future work. One can be interested to study the dynamics along the negative real axis and analyze the stable and unstable domains. What will be the dynamics of the maps when the multiplier is one. In other words, one can see the dynamics of the map when the function has neutral fixed point. However, such results can be elaborated with the family of maps with three or more parameters values, furthermore, for a family of functions whose dimension are equal to two.

It will be fascinating to extend these work for complex polynomials generated fractals which are often used to encode information of various objects and are very useful in various branches of science and engineering. Moreover, it is being noticed that as we enlarge the edges of the petals of the Mandelbrot set, we come across the Julia set for complex polynomial generated fractals. In other word, each point of the Mandelbrot set carries a large scale of image data of a Julia set. We plan to provide some fascinating features to compare with the existing one and elaborate the significance of the outcomes such as some of the fractals resemble the traditional Kachhi Thread Works found in the Kutch district of Gujarat (India) which are useful in the textile industry.

Conflict of Interests

The author declares that there is no conflict of interests.

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