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ISHIKAWA ITERATION CONVERGENCE TO FIXED POINTS OF A MULTI-VALUED MAPPING IN MODULAR FUNCTION SPACES

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Abstract. We prove the $\rho-$convergence of Ishikawa iterative algorithm to fixed points of a multi-valued mapping $T : C \to P_\rho(C)$, where C is a nonempty $\rho-$bounded $\rho-$closed subset of L_ρ, ρ is a convex function modular satisfying Δ_2-type condition, and $P_\rho(C)$ is the family of nonempty ρ-bounded ρ-proximinal subsets of C, such that the mapping P_T is ρ-nonexpansive. This is the modular version of approximating fixed points of multi-valued nonexpansive mapping in Banach spaces by Ishikawa iterative algorithm and it generalizes some results in the literature.

Keywords: Ishikawa iteration; fixed point; multivalued; ρ-convergence.

2020 AMS Subject Classification: 47H05, 54H25.

1. Introduction

The purpose of this paper is to prove $\rho-$convergence of Ishikawa-type iterative algorithm to fixed points of mappings defined on some subsets of modular function spaces[initiated by Nakano [15] which are natural generalizations of both function and sequence variants of many

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important, from applications perspective, spaces such as Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces,(see eg.[14] and the references therein) and many others.

The importance for applications of modular function spaces consists in the richness of structure of modular function spaces, that besides being Banach spaces (or F-spaces in a more general space)—are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural and modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces. Khamsi et al. [8] gave an example of a mapping which is $\rho-$nonexpansive but it is not norm nonexpansive. They demonstrated that for a mapping T to be norm nonexpansive in a modular function space L_ρ, a stronger than $\rho-$nonexpansiveness assumption is needed:

$$\rho(\lambda(Tx-Ty)) \leq \rho(\lambda(x-y)) \text{ for any } \lambda \geq 0.$$

It has been also shown (see eg. [1]) that there are fixed point free nonexpansive mppings on, even weakly compact, subsets of uniformly convex Banach spaces. But in modular function spaces the existence of fixed points of multivalued, more difficult than single valued mappings, nonexpansive mappings is guaranteed(see eg. [3], Corollary 3.5).

From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

Khamsi et al.[8] were the profoundest of the study of fixed point theory in the context of modular function spaces. Kozlowski [11] has contributed a lot towards the study of modular function spaces both on his own and with his collaborators and also Kuman[12] obtained some fixed point theorems for nonexpansive mappings in modular function spaces. In 2006, Dhompongsa et.al in [3] proved the existence of fixed points of multivalued $\rho - contarction$ and ρ-nonexpansive mappings in modular function spaces. Of course, most of the works we have discussed so far on fixed points in these spaces were of existential nature.

In the fixed point theory, approximating fixed points of nonlinear mappings is an important isue as the existence. So now a days, many well known analysts have shown their interests to the approximation processes of fixed points of nonexpansive mappings in modular function spaces. In 2012, Dehaish and Kozlowski [2] initiated the approximation of fixed points in modular function spaces by Mann iterative process for asymptotically pointwise nonexpansive mappings.

In 2014, Khan and Abbas [9] generalized the results of Dehaish and Kozlowski [2] to the multivalued mapping setting to approximate the fixed points of a ρ-nonexpansive mapping in modular function space by using the Mann iteration process [13]. Zegeye et al. [16] were able to extend the results of [9] to the common fixed points of finite family, which reduces to the results of Khan and Abbas [9] if considered for single mapping.

In this paper, we generalize the following important results of Khan and Abbas [9] to an Ishikawa iterative algorithm.

Theorem 1.1. [9] *Let ρ satisfy (UUC1) and C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset C$ be defined by the Mann iterative process*

$$f_{n+1} = (1-\alpha_n)f_n + \alpha_n u_n,$$

where $u_n \in P_\rho^T(f_n)$ and $\{\alpha_n\} \subset (0,1)$ is bounded away from both 0 and 1. Then,

$$\lim_{n\to\infty} \rho(f_n - c) \text{ exists for all } c \in F_\rho(T).$$

and

$$\lim_{n\to\infty} d_\rho(f_n, P_\rho^T(f_n)) = 0.$$

Theorem 1.2. [9] *Let ρ satisfy (UUC1) and C a nonempty ρ-compact, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is ρ-nonexpansive mapping. Suppose that $F_\rho(T) \neq \emptyset$. Let $\{f_n\}$ be defined as in the Theorem 1.1. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Theorem 1.3. [9] *Let ρ satisfy (UUC1) and C a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping with $F_\rho(T) \neq \emptyset$ and satisfying* *condition(I) such that P_ρ^T is ρ-nonexpansive mapping. Let $\{f_n\}$ be as defined in Theorem 1.1. Then, $\{f_n\}$ ρ-converges to a fixed point of T.*

2. PRELIMINARIES

Now, we recall some basic notions and facts about modular spaces as formulated by Kozlowski [10]. For more details the reader may consult [6, 11] and the references therein.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathscr{P}$ be a nontrivial δ-ring of subsets of Ω such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $\Omega = \cup_{n=1} K_n$. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from $\mathscr{P}$. By $\mathscr{M}_\infty$ we denote the space of all extended measurable functions, that is, all functions $f : \Omega \longrightarrow [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathscr{E}$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f(w)$ for all $w \in \Omega$. By χ_A we denote the characteristic function of the set A.

Definition 2.1. *Let $\rho : \mathscr{M}_\infty \longrightarrow [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudo-modular if it satisfies the following:*

a) $\rho(0) = 0$;

b) ρ is monotone; that is, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$ where $f, g \in \mathscr{M}_\infty$;

c) ρ is orthogonally subadditive; that is, $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, where $f \in \mathscr{M}_\infty$;

d) ρ has Fatou property; that is, $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies that $\rho(f_n) \uparrow \rho(f)$ where $f \in \mathscr{M}_\infty$ and

e) ρ is order continuous in $\mathscr{E}$; that is, $g_n \in \mathscr{E}$ and $|g_n| \downarrow 0$ implies that $\rho(g_n) \downarrow 0$.

We say that a set $A \in \Sigma$ is ρ-null if $\rho(g\chi_A) = 0$ for every $g \in \mathscr{E}$. We say that a property p holds ρ-almost every where if the exceptional set $\{w \in \Omega : p(w)$ does not hold$\}$ is ρ-null. As usual we identify any pair of measurable functions f and g differing only on ρ-null set by $f = g$ ρ-a.e. With this in mind we define

$$\mathscr{M} = \{f \in \mathscr{M}_\infty : |f(w)| < \infty\ \rho - a.e\},$$

where $f \in \mathscr{M}$ is actually an equivalence class of functions equal ρ-a.e rather than an individual function.

Definition 2.2. *Let ρ be a regular convex function pseudo-modular.*

a) We say that ρ is a regular convex function semi-modular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies that $f = 0$ ρ-a.e.

b) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies that $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\mathscr{R}$.

Remark 2.3. *Let us denote $\rho(f,E) = \rho(f\chi_E)$ for $f \in \mathscr{M}, E \in \Sigma$. Also by convention for $\alpha > 0$ we will write $\rho(\alpha,E)$ instead of $\rho(\alpha\chi_E)$. We will use these notations when convenient. It is easy to prove that $\rho(f,E)$ is a convex function modular in the sense of Definition 2.1.*

Remark 2.4. *Note that if ρ is a regular convex function modular, then to verify that a set E is ρ-null it suffices to prove that there exists $\alpha > 0$ such that $\rho(\alpha,E) = 0$.*

Definition 2.5. *Let ρ be a convex function modular.*

(1) A modular function space is the vector space $L_\rho(\Omega,\Sigma)$ or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathscr{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

(2) The following formula defines a norm in L_ρ frequently called the Luxemburg norm:

$$\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}.$$

Definition 2.6. [11] *Let $\rho \in \mathscr{R}$.*

a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f(\rho)$ if $\rho(f_n - f) \to 0$.

b) A sequence $\{f_n\}$ in L_ρ is called a ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $n,m \to \infty$.

c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $\{f_n\} \subset B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B.

d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter is finite; the ρ-diameter of B is defined as

$$\delta_\rho(B) = \sup\{\rho(f-g) : f \in B, g \in B\}.$$

e) A set $B \subset L_\rho$ is called ρ-compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ and an $f \in B$ such that $\rho(f_{n_k} - f) \to 0$.

f) A set $B \subset L_\rho$ is called ρ- a.e closed if if for any $\{f_n\}$ in B,which ρ-a.e converges to some f, we have $f \in B$.

g) A set $B \subset L_\rho$ is called ρ-a.e compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e converges to some $f \in B$.

h) Let $f \in L_\rho$ and $B \subset L_\rho$. The ρ-distance between f and B is defined as $d_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}$.

Theorem 2.7. [11] *Let $\rho \in \mathscr{R}$. L_ρ is complete with respect to ρ-convergence.*

The following definition plays very crucial role in modular function space and following this definition we get an important property that characterizes the convergence in function modular by the norm (*Luxemburg norm*)convergence(see the detail in [11]).

Definition 2.8. *Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$.*

Proposition 2.9. [11] *The following statements are equivalent:*

(i) ρ satisfies the Δ_2-condition.

(ii) $\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for all $\lambda > 0$ if and only if $\|f_n - f\|_\rho \to 0$.

Definition 2.10. [11] *Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - type condition if there exists a constant $0 < k < \infty$ such that for every $f \in L_\rho$, we have $\rho(2f) \le k\rho(f)$.*

Remark 2.11. *If ρ satisfies the Δ_2 - type condition, then it satisfies Δ_2 - property, and that the converse is not true (see, e.g.,*[11]*).*

Let $\rho \in \mathscr{R}$ and C be a nonempty subset of the modular space L_ρ. We denote a collection of all nonempty ρ-closed and ρ- bounded subsets of C by $\mathscr{C}_\rho(C)$ and a collection of all nonempty ρ-compact subsets of C by $\mathscr{K}_\rho(C)$.

Definition 2.12. [9] A set $C \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in C$ such that

$$\rho(f-g) = d_\rho(f,C) = \inf\{\rho(f-h) : h \in C\}.$$

We denote the family of nonempty ρ-bounded ρ-proximinal subsets of C by $P_\rho(C)$.

Definition 2.13. [9] We define a Hausdorff distance on $\mathscr{C}_\rho(C)$ by,

$$H_\rho(A,B) = max\{\sup_{f\in A} d_\rho(f,B),\ \sup_{g\in B} d_\rho(g,A)\},$$

$A,B \in \mathscr{C}_\rho(C)$.

Definition 2.14. [9] A multivalued mapping $T : C \to \mathscr{C}_\rho(C)$ is called ρ-Lipschitzian if there exists a number $k \geq 0$ such that

$$H_\rho(T(f),T(g)) \leq k\rho(f-g) \ for\ all\ f,g \in C.$$

If $k \leq 1$ then, T is called ρ-nonexpansive and if $k < 1$, T is called ρ-contractive.

We find the following uniform convexity type property definitions of the function modular ρ in [7] and [5].

Definition 2.15. Let $\rho \in \mathscr{R}$. Let $t \in (0,1)$, $r > 0$, $\varepsilon > 0$. Define,

$$D_1(r,\varepsilon) = \{(f,g) : f,g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f-g) \geq \varepsilon r\}.$$

Let

$$\delta_1^t(r,\varepsilon) = \inf\{1 - \frac{1}{r}\rho(tf + (1-t)g : (f,g) \in D_1(r,\varepsilon)\},\ if\ D_1(r,\varepsilon) \neq \emptyset$$

and $\delta_1^t(r,\varepsilon) = 1,\ if\ D_1(r,\varepsilon) = \emptyset$.

We will use the following notational convention: $\delta_1 = \delta_1^{\frac{1}{2}}$.

Definition 2.16. We say that ρ satisfies (UC1) if for every $r > 0,\ \varepsilon > 0,\ \delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. We say that ρ satisfies (UUC1) if for every $s \geq 0,\ \varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending only on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0,\ for\ any\ r > s.$$

Definition 2.17. A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $t_n \geq a$, for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $0 < b < 1$ such that $t_n \leq b$, for every $n \in \mathbb{N}$.

Lemma 2.18. [2] Let ρ satisfies (UUC1) and let $\{t_n\} \subset (0,1)$ be bounded away from both 0 and 1. If there exists $R > 0$ such that

$$\limsup_{n\to\infty} \rho(f_n) \leq R, \ \limsup_{n\to\infty} \rho(g_n) \leq R$$

and

$$\lim_{n\to\infty} \rho(t_n f_n + (1-t_n)g_n) = R, \ then \ \lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

Lemma 2.19. [9] *Let $T : C \to P_\rho(C)$ be a multi-valued mapping and*

$$P_\rho^T(f) = \{g \in Tf : \rho(f-g) = d_\rho(f, Tf)\}.$$

Then, the following are equivalent:

(i) $f \in F_\rho(T)$, *that is* $f \in Tf$;

(ii) $P_\rho^T(f) = \{f\}$, *that is,* $f = g$ *for each* $g \in P_\rho^T(f)$;

(iii) $f \in F_\rho(P_\rho^T(f))$, *that is,* $f \in P_\rho^T(f)$. *Further,* $F_\rho(T) = F(P_\rho^T)$ *where* $F(P_\rho^T)$ *denotes the set of fixed points of* P_ρ^T.

Definition 2.20. [9] A multivalued mapping $T : C \to \mathscr{C}_\rho(C)$ is said to satisfy condition(I) if there exists a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for all $r \in (0,\infty)$ such that

$$d_\rho(f, Tf) \geq \varphi(d_\rho(f, F_\rho(T))),$$

for all $f \in C$.

The following lemma will be used in what follows, so we state it here.

Lemma 2.21. [16] *Let $\rho \in \mathscr{R}$. Let $A, B \in P_\rho(L_\rho)$. For every $f \in A$, there exists $g \in B$ such that* $\rho(f-g) \leq H_\rho(A,B)$.

3. Main Results

We now prove our main result that gives a major support to our ρ-convergence result for approximating fixed points of multivalued ρ-nonexpansive mappings in modular function spaces using an Ishikawa iterative scheme formulated below.

Let $C \subset L_\rho$ be nonempty set and $T : C \to P_\rho(C)$ a multi-valued mapping. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$\begin{cases} f_1 \in C, \\ g_n = \beta_n u_n + (1-\beta_n) f_n, \\ f_{n+1} = (1-\alpha_n) v_n + \alpha_n f_n \end{cases} \tag{3.1}$$

where $u_n \in P_\rho^T(f_n)$, $v_n \in P_\rho^T(g_n)$, and $\{\alpha_n\}$ & $\{\beta_n\}$ are sequences in $(0,1)$ such that both are bounded away from 0 and 1.

Theorem 3.1. *Let ρ satisfy (UUC1) and C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset C$ be defined by the Ishikawa iteration process in (3.1). Then,*

$$\lim_{n\to\infty} \rho(f_n - p) \; exists \; for \; all \; p \in F_\rho(T),$$

and

$$\lim_{n\to\infty} d_\rho(f_n, T(f_n)) = 0.$$

Proof. Let $p \in F_\rho(T)$. By Lemma 2.19, $P_\rho^T(p) = \{p\}$. Moreover, by the same lemma, $F_\rho(T) = F(P_\rho^T)$. Then,

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho((1-\alpha_n) v_n + \alpha_n f_n - p) \\ &= \rho((1-\alpha_n)(v_n - p) + \alpha_n (f_n - p)) \\ &\leq \alpha_n \rho(v_n - p) + (1-\alpha_n)\rho(f_n - p) \\ &\leq \alpha_n H_\rho(P_\rho^T(g_n), P_\rho^T(p)) + (1-\alpha_n)\rho(f_n - p) \\ &\leq \alpha_n \rho(g_n - p) + (1-\alpha_n)\rho(f_n - p). \qquad (3.2) \end{aligned}$$

Again from (3.1), we have

$$\begin{aligned}
\rho(g_n - p) &= \rho(\beta_n u_n + (1-\beta_n) f_n - p) \\
&\leq \beta_n \rho(u_n - p) + (1-\beta_n)\rho(f_n) \\
&\leq \beta_n H_\rho(P_\rho^T(f_n), P_\rho^T(p)) + (1-\beta_n)\rho(f_n - p) \\
&\leq \beta_n \rho(f_n - p) + (1-\beta_n)\rho(f_n - p) \\
&= \rho(f_n - p). \qquad (3.3)
\end{aligned}$$

From (3.2) and (3.3), we get that

$$\rho(f_{n+1} - p) \leq \rho(f_n - p). \qquad (3.4)$$

Thus, $\{\rho(f_n - p)\}$ is a decreasing and nonnegative sequence of reals for all $p \in F_\rho(T)$. Hence,

$$\lim_{n\to\infty} \rho(f_n - p) \text{ exists for all } p \in F_\rho(T).$$

Next we show that

$$\lim_{n\to\infty} d_\rho(f_n, T(f_n)) = 0.$$

Let

$$\lim_{n\to\infty} \rho(f_n - p) = L \geq 0. \qquad (3.5)$$

From (3.3), we have

$$\rho(g_n - p) \leq \rho(f_n - p). \qquad (3.6)$$

Therefore from (3.5) and (3.6), we have

$$\limsup_{n\to\infty} \rho(g_n - p) \leq L. \qquad (3.7)$$

By the similar argument, we have

$$\limsup_{n\to\infty} \rho(u_n - p) \leq L. \qquad (3.8)$$

and

$$\limsup_{n\to\infty} \rho(v_n - p) \leq L. \qquad (3.9)$$

Since $\alpha_n \in (0,1)$ is bounded away from 0 and 1, there exists $\alpha \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = \alpha$ (passing to subsequence if necessary).

Now,

$$\begin{aligned} \rho(f_{n+1}-p) &= \rho(\alpha_n v_n + (1-\alpha_n)f_n - p) \\ &\leq \alpha_n \rho(v_n - p) + (1-\alpha_n)\rho(f_n - p). \end{aligned} \tag{3.10}$$

Applying $liminf$ to both sides of (3.10) and using the fact that C is ρ-bounded, we get that

$$L \leq \liminf_{n\to\infty} \rho(v_n - p). \tag{3.11}$$

From (3.9) and (3.11), we have

$$\lim_{n\to\infty} \rho(v_n - p) = L. \tag{3.12}$$

Using the fact that $\rho(v_n - p) \leq \rho(g_n - p)$ and (3.12), we have

$$L \leq \liminf_{n\to\infty} \rho(g_n - p). \tag{3.13}$$

Thus from (3.7) and (3.13), we get that

$$\lim_{n\to\infty} \rho(g_n - p) = L. \tag{3.14}$$

Now,

$$\rho(g_n - p) = \rho(\beta_n(u_n - p) + (1-\beta_n)(f_n - p)).$$

Then by (3.14), we have

$$\lim_{n\to\infty} \rho(\beta_n(u_n - p) + (1-\beta_n)(f_n - p)) = L. \tag{3.15}$$

Now by (3.5),(3.8), (3.15) and Lemma 2.18, we have

$$\lim_{n\to\infty} \rho(u_n - f_n) = 0. \tag{3.16}$$

Since $u_n \in P_\rho^T(f_n)$, $d_\rho(f_n, P_\rho^T(f_n)) \leq \rho(u_n - f_n)$, so that

$$\lim_{n\to\infty} d_\rho(f_n, P_\rho^T(f_n)) = 0.$$

Which in turn implies that

$$\lim_{n\to\infty} d_\rho(f_n, T(f_n)) = 0.$$

This completes the proof.

Theorem 3.2. *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and $C \subset L_\rho$ be nonempty ρ-compact, ρ-bounded and convex set. Suppose $T : C \to P_\rho(C)$ is a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset C$ be defined by the Ishikawa iteration process in (3.1). Then, $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. By the ρ-compactness of C, there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in C$ such that $\rho(f_{n_k} - f) \to 0$, as $k \to \infty$.

We next prove that f is a fixed point of T, that is, $f \in T(f)$. Let $g \in P_\rho^T(f)$ be arbitrary. Then by Lemma 2.21 there is $h_k \in P_\rho^T(f_{n_k})$ such that $\rho(g - h_k) \leq H_\rho(P_\rho^T(f), P_\rho^T(f_{n_k})$. Now by convexity of ρ we have,

$$\begin{aligned}
\rho(\frac{f-g}{3}) &= \rho(\frac{f-f_{n_k}}{3} + \frac{f_{n_k}-h_k}{3} + \frac{h_k-g}{3}) \\
&\leq \frac{1}{3}\rho(f-f_{n_k}) + \frac{1}{3}\rho(f_{n_k}-h_k) + \frac{1}{3}\rho(h_k-g) \\
&\leq \rho(f-f_{n_k}) + d_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f)) \\
&\leq \rho(f-f_{n_k}) + d_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + \rho(f-f_{n_k}) \to 0,\ k\ \to \infty.
\end{aligned}$$

Since by Theorem 3.1, we have $\lim_{n\to\infty} d_\rho(f_{n_k}, P_\rho^T(f_{n_k}) = 0$. Hence, $f = g$ ρ-a.e. Since $g \in P_\rho^T(f)$ was arbitrary we have $P_\rho^T(f) = \{f\}$. Thus, by Lemma 2.19, $f \in T(f)$.

Theorem 3.3. *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-proprty and $C \subset L_\rho$ be nonempty ρ-closed, ρ-bounded and convex set. Suppose $T : C \to P_\rho(C)$ is a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$ and T satisfies condition(I). Let $\{f_n\} \subset C$ be defined by the Ishikawa iteration process in (3.1). Then, $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. By Theorem3.1 $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$. If $\lim_{n\to\infty} \rho(f_n - p) = 0$, then we are through.

Assume that $\lim_{n\to\infty} \rho(f_n - p) = L > 0$. Again by the same theorem, using (3.4, we have

$$d_\rho(f_{n+1}, F_\rho(T)) \leq d_\rho(f_n, F_\rho(T)).$$

Hence,

$$\lim_{n\to\infty} d_\rho(f_n, F_\rho(T)) \text{ exists.}$$

We now prove that $d_\rho(f_n, F_\rho(T)) \to 0$ as $n \to \infty$. By Theorem 3.1 and the fact that T satisfies condition(I) we have

$$\limsup_{n\to\infty} \varphi(d_\rho(f_n, F_\rho(T))) = 0.$$

Therefore,

$$\lim_{n\to\infty} d_\rho(f_n, F_\rho(T)) \quad = \quad 0 \tag{3.17}$$

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} d_\rho(f_n, F_\rho(T)) = 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$d_\rho(f_n, F_\rho(T)) < \frac{\varepsilon}{2},\ \forall n \geq n_0.$$

In particular,

$$inf\{\rho(f_{n_0} - p) : p \in F_\rho(T)\} < \frac{\varepsilon}{2}.$$

Thus there must exist a $p_0 \in F_\rho(T)$ such that

$$\rho(f_{n_0} - p_0) \quad < \quad \varepsilon. \tag{3.18}$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned}
\rho(\frac{f_m - f_n}{2}) \quad &\leq \quad \frac{1}{2}\rho(f_m - p_0) + \frac{1}{2}\rho(f_n - p_0) \\
&\leq \quad \frac{\rho(f_{n_0} - p_0)}{2} + \frac{\rho(f_{n_0} - p_0)}{2} (\text{ since } \{\rho(f_n - p)\} \text{ is decreasing}) \\
&< \quad \varepsilon.
\end{aligned}$$

Since ρ satisfies Δ-2 condition, by Proposition 2.9, $\{f_n\}$ is a ρ-Cauchy sequence in C. Since L_ρ is complete with respect to ρ-convergence and C is ρ-closed, there exists $h \in C$ such that $\rho(f_n - h) \to 0$. Now, h is a fixed point of T follows immediately from Theorem 3.2.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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