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A GENERALIZATION OF THE b_2-METRIC SPACE AND SOME FIXED POINT RESULTS

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Abstract. In this paper, we introduce a generalization of the b_2-metric space by weakening the rectangular inequality. Fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in the frame of the generalized b_2-metric type space.

Keywords: 2-metric; b_2 metric; Geraghty-type contraction.

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1. Introduction

Czerwik, gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem, [2]. In 1998, Czerwik, provided many fixed-point results in the generalized space, [3].

The notion of a 2-metric space was introduced by Gähler, in [4]. Several fixed-point results were obtained in [1, 6], as a generalization of the concept of a metric space. A 2-metric is not a continuous function of its variables, whereas an ordinary metric is. The basic philosophy is that

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since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line.

Z. Mustafa introduced a new type of generalized metric space called b_2-metric space, as a generalization of the 2-metric space, [8].

Recently, Kamran et al., have dealt with an extended b-metric space and obtained unique fixed-point results, [7].

Definition 1.1. [4, 9] *Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}_+$ be a map satisfying the following properties*

(i) $d(x,y,z) = 0$ if at least two of the three points are the same .

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) rectangle inequality:

$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$$

for $x,y,z,t \in X$.

Then d is a 2-metric and (X,d) is a 2-metric space.

Definition 1.2. [8] *Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}_+$ be a map satisfying the following properties*

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) s-rectangle inequality:there exists $s \geq 1$ such that

$$d(x,y,z) \leq s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$$

for $x,y,z,t \in X$.

Then d is a b_2-metric and (X,d) is a b_2-metric space.

If $s = 1$, the b_2-metric reduces to the 2-metric.

Example 1.3. *Let $X = [0,\infty)$ and define $d(x,y,z) = [xy + yz + zx]^p$ where $p > 1$. it suffices to only verify property (iv) of definition 1.2. For $x,y,z,t \in X$ we get by using the Jensen inequality,*

$$\begin{aligned} d(x,y,z) &= [xy + yz + zx]^p \\ &= 3^p \left(\frac{1}{3}xy + \frac{1}{3}yz + \frac{1}{3}zx \right)^p \\ &\leq 3^p \left(\tfrac{1}{3}[xy]^p + \tfrac{1}{3}[yz]^p + \tfrac{1}{3}[zx]^p \right) \\ &\leq 3^p \left(\tfrac{1}{3}[xy + yt + xt]^p + \tfrac{1}{3}[yz + zt + yt]^p + \tfrac{1}{3}[zx + xt + zt]^p \right) \\ &= 3^{p-1}[d(x,y,t) + d(y,z,t) + d(z,x,t)] \end{aligned}$$

It follows that d is a b_2-metric with $s = 3^{p-1}$.

2. Main Result

Definition 2.1. [10] *Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}_+$ be a map satisfying the following properties:*

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) symmetry property: for $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) modified rectangle inequality:there exists $\alpha, \beta, \gamma \geq 1$ such that

$$d(x,y,z) \leq \alpha d(x,y,t) + \beta d(y,z,t) + \gamma d(z,x,t)]$$

for $x,y,z,t \in X$.

Then d is a generalized b_2-metric and (X,d) is a generalized b_2- metric space.

If $\alpha = \beta = \gamma = s$ then a generalized b_2-metric is a b_2-metric. If $\alpha = \beta = \gamma = 1$ then the b_2-metric is a 2-metric. The example that follows provides a motivation for the generalization of the concept of a b_2-metric.

Example 2.2. *Let $X=(0,4)$ and define*

$$d(x,y,z)=\begin{cases} 0, & \text{if at least two of the three points are the same} \\ m(x,y,z)e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}}, & \text{otherwise} \end{cases}$$

where $\xi \geq 1$ and $m: X\times X\times X \to [0,\infty)$ is a continuous function such that $d(x,y,z)$ is symmetric with respect to x,y,z. It suffices to only verify property (iv) of definition 2.1:
For $x,y,z\in X$ and using Jensen's inequality, we get

$$\begin{aligned}
&d(x,y,z)\\
&=m(x,y,z)e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}}\\
&\leq m(x,y,z)\left[\frac{1}{2}e^{|x-y|^{\xi}}+\frac{1}{3}e^{|y-z|^{\xi}}+\frac{1}{6}e^{|z-x|^{\xi}}\right]\\
&\leq m(x,y,z)\left[\frac{1}{2}e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{2}|x-y|^{\xi}}+\frac{1}{3}e^{\frac{1}{2}|y-z|^{\xi}+\frac{1}{2}|y-z|^{\xi}}+\frac{1}{6}e^{\frac{1}{2}|z-x|^{\xi}+\frac{1}{2}|z-x|^{\xi}}\right]\\
&\leq m(x,y,z)\\
&\left[e^{2^{2\xi-1}}\frac{1}{2}e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-t|^{\xi}+\frac{1}{6}|t-x|^{\xi}}+e^{2^{2\xi-1}}\frac{1}{3}e^{\frac{1}{2}|z-y|^{\xi}+\frac{1}{3}|y-t|^{\xi}+\frac{1}{6}|t-z|^{\xi}}+e^{2^{2\xi-1}}\frac{1}{6}e^{|z-x|^{\xi}+|x-t|^{\xi}+|t-z|^{\xi}}\right]\\
&=\alpha d(x,y,t)+\beta d(z,y,t)+\gamma d(z,x,t)
\end{aligned}$$

where $\alpha=\frac{1}{2}e^{2^{2\xi-1}}\geq 1$, $\beta=\frac{1}{3}e^{2^{2\xi-1}}\geq 1$ and $\gamma=\frac{1}{6}e^{2^{2\xi-1}}\geq 1$ and some $t\in X$. An example of a function m such that d is symmetric in x,y,z maybe defined as follows: $m(x,y,z)=e^{\frac{1}{2}|x-y|^{\xi}+\frac{2}{3}|y-z|^{\xi}+\frac{5}{6}|z-x|^{\xi}}$.

It follows that d is a generalized b_2-metric and not a b_2-metric.

Definition 2.3. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a generalized b_2-metric space (X,d).*
a) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x\in X$ iff for all $\xi\in X$, $\lim_{n\to\infty}d(x_n,x,\xi)=0$.
b) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X iff for all $\xi\in X$, $\lim_{n,m\to\infty}d(x_n,x_m,\xi)=0$

Definition 2.4. *Let $\mathfrak{F}$ denote all functions $f:[0,\infty)\to[0,\frac{1}{\beta})$, where $\beta\geq 1$, satisfying the following condition:*
$f(t_n)\to\frac{1}{\beta}$ as $n\to\infty$ implies $t_n\to 0$ as $n\to\infty$.

In 1973, in an attempt to generalize the Banach contraction principle, Geraghty, proved a similar result in the theorem that follows for a complete metric space, [5].

Theorem 2.5. *Let (X,d) be a complete generalized b_2-metric space and $T : X \to X$ be a self mapping. Suppose that there exists $f \in \mathfrak{F}$ such that*

$$\begin{aligned} &\beta d(Tx,Ty,\xi) \\ &\leq f(d(x,y,\xi)) \max\left\{d(x,y,\xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\} \\ &+\mu \min\{d(x,Tx,\xi), d(x,Ty,\xi), d(y,Ty,\xi)\} \end{aligned} \tag{1}$$

for all $x,y,\xi \in X$. If T is continuous then T has a fixed point in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. We shall show that the sequence $\{d(x_n,x_{n+1},\xi)\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers. Using (1), we get

$$\begin{aligned} &\beta d(x_n,x_{n+1},\xi) \\ &= \beta d(Tx_{n-1},Tx_n,\xi) \\ &\leq f(d(x_{n-1},x_n,\xi)) \max\left\{d(x_{n-1},x_n,\xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(Tx_{n-1},Tx_n,\xi)},\right. \\ &\qquad\qquad \left.\tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(x_{n-1},x_n,\xi)}\right\} \\ &+\mu \min\{d(x_{n-1},Tx_n,\xi), d(x_n,Tx_n,\xi), d(x_{n-1},Tx_{n-1},\xi), d(x_n,Tx_{n-1},\xi)\}. \end{aligned} \tag{2}$$

It follows that

$$\begin{aligned} &\max\left\{d(x_{n-1},x_n,\xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(Tx_{n-1},Tx_n,\xi)}, \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(x_{n-1},x_n,\xi)}\right\} \\ &\leq \max\{d(x_{n-1},x_n,\xi), d(x_n,x_{n+1},\xi)\} \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\min\{d(x_{n-1},Tx_n,\xi), d(x_n,Tx_n,\xi), d(x_{n-1},Tx_{n-1},\xi), d(x_n,Tx_{n-1},\xi)\} \\ &= \min\{d(x_{n-1},x_{n+1},\xi), d(x_n,x_{n+1},\xi), d(x_{n-1},x_n,\xi), d(x_n,x_n,\xi)\} \\ &= 0. \end{aligned} \tag{4}$$

Using (3) and (4), inequality (2) reduces to

$$\beta d(x_n,x_{n+1},\xi) \le f(d(x_{n-1},x_n,\xi)) \max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\}. \tag{5}$$

If we assume that

$$\max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\} = d(x_n,x_{n+1},\xi),$$

then (5) reduces to

$$\begin{aligned}\beta d(x_n,x_{n+1},\xi) &\le f(d(x_{n-1},x_n,\xi))d(x_n,x_{n+1},\xi)\\ &\le \tfrac{1}{\beta} d(x_n,x_{n+1},\xi)\\ &< d(x_n,x_{n+1},\xi), \end{aligned} \tag{6}$$

which leads to a contradiction. Thus assume otherwise, $\max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\} = d(x_{n-1},x_n,\xi)$. Hence, we have

$$d(x_n,x_{n+1},\xi) \le \frac{1}{\beta} f(d(x_{n-1},x_n,\xi))d(x_{n-1},x_n,\xi) < \frac{1}{\beta^2} d(x_{n-1},x_n,\xi) < d(x_{n-1},x_n,\xi), \tag{7}$$

and it follows that $\{d(x_n,x_{n+1},\xi)\}_{n\in\mathbb{N}}$ is decreasing. We next shall show that $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = 0$. Suppose that $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = r$, where $r > 0$ then taking limit as $n \to \infty$, in inequality (7), we get

$$\tfrac{1}{\beta} r \le \beta r \le \lim_{n\to\infty} f(d(x_{n-1},x_n,\xi))r \tag{8}$$

which implies that

$$\tfrac{1}{\beta} \le \lim_{n\to\infty} f(d(x_{n-1},x_n,\xi) \tag{9}$$

but since $\lim_{n\to\infty} f(d(x_{n-1},x_n,\xi) \le \frac{1}{\beta}$ and $f \in \mathfrak{F}$, we obtain $\lim_{n\to\infty} f(d(x_{n-1},x_n,\xi) = \frac{1}{\beta}$ and hence, deduce that

$$\lim_{n\to\infty} d(x_{n-1},x_n,\xi) = 0$$

which is a contradiction, hence $r = 0$ ie., $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = 0$.

We next shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. From the rectangular inequality we obtain,

$$
\begin{aligned}
& d(x_n,x_m,\xi) \\
& \leq \alpha d(x_n,x_m,x_{n+1})+\beta d(x_m,\xi,x_{n+1})+\gamma d(\xi,x_n,x_{n+1}) \\
& \leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi)+\beta^2 d(x_{n+1},x_{m+1},\xi) \\
& +\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi) \\
& \leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi) \\
& +\beta^2 d(x_{n+1},x_{m+1},\xi)+\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi). \qquad (10)
\end{aligned}
$$

Using inequality (1) in (10) we get

$$
\begin{aligned}
& d(x_n,x_m,\xi) \\
& \leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi) \\
& +\beta f(d(x_n,x_m,\xi))\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(Tx_n,Tx_m,\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(x_n,x_m,\xi)}\right\} \\
& +\mu\min\{d(x_n,Tx_n,\xi),d(x_n,Tx_m,\xi),d(x_m,Tx_n,\xi),d(x_m,Tx_m,\xi)\} \\
& +\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi). \qquad (11)
\end{aligned}
$$

Taking $m,n\to\infty$ we get,

$$
\begin{aligned}
& \lim_{m,n\to\infty}\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(Tx_n,Tx_m,\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(x_n,x_m,\xi)}\right\} \\
& =\lim_{m,n\to\infty}\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,x_{n+1},\xi)d(x_m,x_{m+1},\xi)}{1+d(x_{n+1},x_{m+1},\xi)},\tfrac{d(x_n,x_{n+1},\xi)d(x_m,x_{m+1},\xi)}{1+d(x_n,x_m,\xi)}\right\} \\
& =\lim_{m,n\to\infty} d(x_n,x_m,\xi) \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
& \lim_{m,n\to\infty}\min\{d(x_n,Tx_n,\xi),d(x_n,Tx_m,\xi),d(x_m,Tx_n,\xi),d(x_m,Tx_m,\xi)\} \\
& =\lim_{m,n\to\infty}\min\{d(x_n,x_{n+1},\xi),d(x_n,x_{m+1},\xi),d(x_m,x_{n+1},\xi),d(x_m,x_{m+1},\xi)\} \\
& =0 \qquad (13)
\end{aligned}
$$

Taking $m,n\to\infty$ in (11), using (12) and (13), we get

$$\lim_{m.n\to\infty} d(x_n,x_m,\xi) \leq \beta \lim_{m,n\to\infty} f(d(x_n,x_m,\xi)) \lim_{m,n\to\infty} d(x_n,x_m,\xi). \tag{14}$$

We claim that $\lim_{m,n\to\infty} d(x_n,x_m,\xi) = 0$. On the contrary, if $\lim_{m,n\to\infty} d(x_n,x_m,\xi) \neq 0$, then we get

$$\tfrac{1}{\beta} \leq \lim_{m,n\to\infty} f(d(x_n,x_m,\xi)). \tag{15}$$

Since $\lim_{m,n\to\infty} f(d(x_n,x_m,\xi)) \leq \frac{1}{\beta}$ and $f \in \mathfrak{F}$, we deduce that $\lim_{m,n\to\infty} d(x_n,x_m,\xi) = 0$, which is a contradiction. Thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X,d) is complete there exists $x' \in X$ such that $\lim_{n\to\infty} d(x_n,x',\xi) = 0$.
Finally, we show that $x' \in X$ is a fixed point of T. From the rectangle inequality, we get

$$d(Tx',x',\xi) \leq \alpha d(Tx',x',Tx_n) + \beta d(x',\xi,Tx_n) + \gamma d(\xi,Tx',Tx_n). \tag{16}$$

Letting $n \to \infty$ and using the continuity of T, we get

$$d(Tx',x',\xi) \leq 0, \tag{17}$$

hence, we get $Tx' = x'$. Thus x' is a fixed point of T. □

Definition 2.6. [11] *Let (X,d) be a complete generalized b_2-metric space. Assume that $T : X \to X$ and $\nu : X \times X \times X :\to [0,\infty)$ are functions. The function T is an ν-admissible mapping if for all $\xi \in X$, $x,y \in X$ and if $\nu(x,y,\xi) \geq 1$ implies that $\nu(Tx,Ty,\xi) \geq 1$.*

In the theorem that follows, we prove existence of fixed points for mapping that are ν-admissible for the contraction used in theorem 2.5.

Theorem 2.7. *Let (X,d) be a complete generalized b_2-metric space, $T : X \to X$ and ν be functions such that T is an ν-admissible mapping. Suppose that*

$$\begin{aligned}
&\beta \nu(x,Tx,\xi)\nu(y,Ty,\xi)d(Tx,Ty,\xi) \\
&\leq f(d(x,y,\xi)) \max\left\{d(x,y,\xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\} \\
&+ \mu \min\{d(x,Tx,\xi), d(x,Ty,\xi), d(y,Ty,\xi)\}
\end{aligned} \tag{18}$$

for $f \in \mathfrak{F}$ and for all $x, y, \xi \in X$.

If there exists $x_0 \in X$ such that $\nu(x_0, Tx_0, \xi) \geq 1$, and if T is continuous then T has a fixed point.

Proof :

Let $x_0 \in X$ such that $\nu(x_0, Tx_0, \xi) \geq 1$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Since T is ν-admissible mapping and $\nu(x_0, Tx_0, \xi) \geq 1$, it follows that $\nu(x_1, Tx_1, \xi) = \nu(Tx_0, T^2x_0, \xi) \geq 1$. By continuing with the process, we get $\nu(x_n, Tx_n, \xi) \geq 1$ for all $n = 0, 1, 2, \cdots$. Then it follows that the product

$$\nu(x_n, Tx_n, \xi)\nu(x_{n-1}, Tx_{n-1}, \xi) \geq 1$$

for all $n = 1, 2, \cdots$. We shall now show that the sequence $\{d(x_n, x_{n+1}, \xi)\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers. Using (18) and that the product $\nu(x_n, Tx_n, \xi)\nu(x_{n-1}, Tx_{n-1}, \xi) \geq 1$, for the sequence $\{x_n\}_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned}
&\beta d(x_n, x_{n+1}, \xi)\\
&= \beta d(Tx_{n-1}, Tx_n, \xi)\\
&\leq \beta \nu(x_{n-1}, Tx_{n-1}, \xi)\nu(x_n, Tx_n, \xi) d(Tx_{n-1}, Tx_n, \xi)\\
&\leq f(d(x_{n-1}, x_n, \xi)) \max\left\{d(x_{n-1}, x_n, \xi), \tfrac{d(x_{n-1}, Tx_{n-1}, \xi)d(x_n, Tx_n, \xi)}{1+d(Tx_{n-1}, Tx_n, \xi)},\right.\\
&\qquad\qquad \left.\tfrac{d(x_{n-1}, Tx_{n-1}, \xi)d(x_n, Tx_n, \xi)}{1+d(x_{n-1}, x_n, \xi)}\right\}\\
&+ \mu \min\left\{d(x_{n-1}, Tx_n, \xi), d(x_n, Tx_n, \xi), d(x_{n-1}, Tx_{n-1}, \xi), d(x_n, Tx_{n-1}, \xi)\right\}.
\end{aligned} \tag{19}$$

It follows that

$$\begin{aligned}
&\max\left\{d(x_{n-1}, x_n, \xi), \tfrac{d(x_{n-1}, Tx_{n-1}, \xi)d(x_n, Tx_n, \xi)}{1+d(Tx_{n-1}, Tx_n, \xi)}, \tfrac{d(x_{n-1}, Tx_{n-1}, \xi)d(x_n, Tx_n, \xi)}{1+d(x_{n-1}, x_n, \xi)}\right\}\\
&\leq \max\left\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\right\}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\min\left\{d(x_{n-1}, Tx_n, \xi), d(x_n, Tx_n, \xi), d(x_{n-1}, Tx_{n-1}, \xi), d(x_n, Tx_{n-1}, \xi)\right\}\\
&= \min\left\{d(x_{n-1}, x_{n+1}, \xi), d(x_n, x_{n+1}, \xi), d(x_{n-1}, x_n, \xi), d(x_n, x_n, \xi)\right\}\\
&= 0.
\end{aligned} \tag{21}$$

Using (20) and (21), inequality (19) reduces to

$$\beta d(x_n,x_{n+1},\xi) \leq f(d(x_{n-1},x_n,\xi)) \max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\}. \tag{22}$$

Inequality (22) further reduces, if we assume that

$$\max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\} = d(x_{n-1},x_n,\xi)$$

for otherwise, we assume that

$$\max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\} = d(x_n,x_{n+1},\xi).$$

In the latter case, inequality (22), reduces to

$$\begin{aligned}\beta d(x_n,x_{n+1},\xi) &\leq f(d(x_{n-1},x_n,\xi))d(x_n,x_{n+1},\xi)\\ &\leq \tfrac{1}{\beta}d(x_n,x_{n+1},\xi)\\ &< d(x_n,x_{n+1},\xi) \qquad (23)\end{aligned}$$

which leads to a contradiction. Thus, we conclude that $\max\{d(x_{n-1},x_n,\xi),d(x_n,x_{n+1},\xi)\} = d(x_{n-1},x_n,\xi)$. Hence, we have

$$\beta d(x_n,x_{n+1},\xi) \leq f(d(x_{n-1},x_n,\xi))d(x_{n-1},x_n,\xi) \tag{24}$$

and it follows that $\{d(x_n,x_{n+1},\xi)\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers. Next, we shall show that $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = 0$. Suppose that $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = r$ where $r > 0$ then taking limit as $n \to \infty$ in inequality (24) we get

$$\tfrac{1}{\beta}r \leq \beta r \leq \lim_{n\to\infty} f(d(x_{n-1},x_n,\xi))r \tag{25}$$

Since $\lim_{n\to\infty} f(d(x_{n-1},x_n,\xi) \leq \frac{1}{\beta}$ and $f \in \mathfrak{F}$, we deduce that

$$\lim_{n\to\infty} d(x_{n-1},x_n,\xi) = 0,$$

which is a contradiction, hence $r = 0$, ie., $\lim_{n\to\infty} d(x_n,x_{n+1},\xi) = 0$.

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. From the rectangular inequality we obtain,

$$\begin{aligned}
&d(x_n,x_m,\xi)\\
&\leq \alpha d(x_n,x_m,x_{n+1})+\beta d(x_m,\xi,x_{n+1})+\gamma d(\xi,x_n,x_{n+1})\\
&\leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi)+\beta^2 d(x_{n+1},x_{m+1},\xi)\\
&+\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi)\\
&\leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi)\\
&+\beta^2\nu(x_n,Tx_n,\xi)\nu(x_m,Tx_m,\xi)d(x_{n+1},x_{m+1},\xi)\\
&+\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi) \qquad (26)
\end{aligned}$$

Using inequality (19) in (26) we get

$$\begin{aligned}
&d(x_n,x_m,\xi)\\
&\leq \alpha d(x_n,x_{n+1},x_m)+\beta\alpha d(x_m,x_{m+1},\xi)\\
&+\beta f(d(x_n,x_m,\xi))\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(Tx_n,Tx_m,\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(x_n,x_m,\xi)}\right\}\\
&+\mu\min\{d(x_n,Tx_n,\xi),d(x_n,Tx_m,\xi),d(x_m,Tx_n,\xi),d(x_m,Tx_m,\xi)\}\\
&+\beta\gamma d(x_m,x_{m+1},x_{n+1})+\gamma d(x_n,x_{n+1},\xi). \qquad (27)
\end{aligned}$$

Taking $m,n\to\infty$, we obtain,

$$\begin{aligned}
&\lim_{m,n\to\infty}\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(Tx_n,Tx_m,\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x_m,Tx_m,\xi)}{1+d(x_n,x_m,\xi)}\right\}\\
&=\lim_{m,n\to\infty}\max\left\{d(x_n,x_m,\xi),\tfrac{d(x_n,x_{n+1},\xi)d(x_m,x_{m+1},\xi)}{1+d(x_{n+1},x_{m+1},\xi)},\tfrac{d(x_n,x_{n+1},\xi)d(x_m,x_{m+1},\xi)}{1+d(x_n,x_m,\xi)}\right\}\\
&=\lim_{m,n\to\infty}d(x_n,x_m,\xi) \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{m,n\to\infty}\min\{d(x_n,Tx_n,\xi),d(x_n,Tx_m,\xi),d(x_m,Tx_n,\xi),d(x_m,Tx_m,\xi)\}\\
&=\lim_{m,n\to\infty}\min\{d(x_n,x_{n+1},\xi),d(x_n,x_{m+1},\xi),d(x_m,x_{n+1},\xi),d(x_m,x_{m+1},\xi)\}\\
&=0 \qquad (29)
\end{aligned}$$

Taking $m,n \to \infty$ in (27), using (28) and (29), we get

$$\lim_{m.n\to\infty} d(x_n,x_m,\xi) \leq \beta \lim_{m,n\to\infty} f(d(x_n,x_m,\xi)) \lim_{m,n\to\infty} d(x_n,x_m,\xi). \tag{30}$$

We claim that $\lim_{m,n\to\infty} d(x_n,x_m,\xi) = 0$. On the contrary, if $\lim_{m,n\to\infty} d(x_n,x_m,\xi) \neq 0$, then we get

$$\frac{1}{\beta} \leq \lim_{m,n\to\infty} f(d(x_n,x_m,\xi)). \tag{31}$$

Since $\lim_{m,n\to\infty} f(d(x_n,x_m,\xi)) \leq \frac{1}{\beta}$ and $f \in \mathfrak{F}$, we deduce that $\lim_{m,n\to\infty} d(x_n,x_m,\xi) = 0$ which is a contradiction. Thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X,d) is complete there exists $x' \in X$ such that $\lim_{m,n\to\infty} d(x_n,x',\xi) = 0$.

Finally, we show that $x' \in X$ is a fixed point of T. From the rectangle inequality, we get

$$d(Tx',x',\xi) \leq \alpha d(Tx',x',Tx_n) + \beta d(x',\xi,Tx_n) + \gamma d(\xi,Tx',Tx_n)$$

Letting $n \to \infty$ and using the continuity of T, we get

$$d(Tx',x',\xi) \leq 0 \tag{32}$$

hence, we get $Tx' = x'$. Thus x' is a fixed point of T.

3. Conclusion

In this paper, we demonstrated that a Geraghty type contraction has a fixed point in the new generalized b_2 metric space. The results can be extend for Geraghty contractions of type II and Type III. It should be noted that the continuity of the mapping can be dropped if one considers a partial ordering of the space.

Conflict of Interests

The authors declare that there is no conflict of interests.

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