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CERTAIN APPLICATIONS OF *JS*-QUASI-CONTRACTION FIXED POINT THEOREMS IN BIPOLAR METRIC SPACES

CH. SRINIVASA RAO[1], KURADA CHINNI NAGA TARANI[1,*], N.V.E.S. MURTHY[1], B. SRINUVASA RAO[2]

[1]Department of Mathematics, Andhra University, Visakhapatnam-530003, India

[2]Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam- 532410, Andhra Pradesh, India

Abstract: The concept of *JS*-quasi-contraction is introduced in this work as a step in the construction of a bipolar metric space, along with frequently certain fixed point theorems for these mappings in complete bipolar metric spaces under the presumption that the involved function is nondecreasing and continuous. In addition, we offer applications to homotopy and integral equations and offer an explanation that shows the significance of the discovered results.

Keywords: complete bipolar metric space; compatible mappings; *JS*-quasi contraction; common fixed point.

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1. INTRODUCTION

The study of non-linear phenomena benefits greatly from the use of fixed point theory. It is an interdisciplinary area of mathematics that has applications in many different areas of mathematics as well as in other disciplines, such as biology, chemistry, physics, engineering, game theory, mathematical economics, optimisation issues, approximation theory, initial and boundary value issues in ordinary and partial differential equations, and variational inequalities. The most important finding in fixed point theory, attributed to the Polish mathematician Stefan Banach in

*Corresponding author
E-mail address: kuradatarani627@gmail.com
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1922 and cited as [1], had a significant impact on a number of studies. In fact, he established a theory that ensures that any contraction mapping in all of metric space has a singular fixed point. The Banach fixed point theorem or Banach contraction principle are two names for this conclusion. Additionally, an intriguing constructive proof of the Banach fixed point theorem is one that results in an iterative approach for determining a fixed point. Scholars in mathematics are constantly interested in learning about new discoveries in space and their characteristics, therefore many scholars have made generalisations in different directions (see [2]-[16]). For example, the concepts of *Ćirić* contraction [2], quasi-contraction [3], JS-contraction [4], JS-*Ćirić* contraction [5], and JS-quasi contraction [6] have been introduced, and many interesting generalizations of the Banach contraction principle are obtained.

By modifying the domain of the function so that they took into account the distance between points of two separate sets instead of only one, Mutlu et al. [7] recently generalised the metric space structure. The theory is known as a bipolar metric space, and it extends a number of fixed point theorems, such as the Banach contraction principle, to the situations in which it is used (see [7]-[20] and references therein). Furthermore, Mutlu et al. ([7], [8]) demonstrated the coupled fixed point results and principle of locally and weakly contractive mappings in bipolar metric spaces, while Kishore et al. [9] proved certain common fixed point theorems in a bipolar metric space with significant applications. Hence, fixed point theory of bipolar metric space is an active research area and it is capturing a lot of attention for further work.

This article's goal is to put forth a general fixed point theorem for covariant JS-quasi contraction mappings in regard to bipolar metric spaces. Additionally, applications to homotopy and integral equations are given with suitable and pertinent examples.

What follows is in our subsequent conversations, we compile a few suitable definitions.

2. PRELIMINARIES

Definition 2.1:([7]) The mapping $d : \mathscr{S} \times \mathscr{T} \to [0,\infty)$ is said to be a Bipolar-metric on pair of non empty sets $(\mathscr{S},\mathscr{T})$.If

(B_1) $d(\mu,\nu) = 0$ implies that $\mu = \nu$;

(B_2) $\mu = \nu$ implies that $d(\mu,\nu) = 0$;

(B_3) if $\mu,\nu\in\mathscr{S}\cap\mathscr{T}$, then $d(\mu,\nu)=d(\nu,\mu)$;

(B_4) $d(\mu_1,\nu_2)\leq d(\mu_1,\nu_1)+d(\mu_2,\nu_1)+d(\mu_2,\nu_2)$,

for all $\mu,\mu_1,\mu_2\in\mathscr{S}$ and $\nu,\nu_1,\nu_2\in\mathscr{T}$, and the triple $(\mathscr{S},\mathscr{T},d)$ is called a Bipolar-metric space.

Definition 2.2:([7]) Let $\Omega:\mathscr{S}_1\cup\mathscr{T}_1\to\mathscr{S}_2\cup\mathscr{T}_2$ be a function defined on two pairs of sets $(\mathscr{S}_1,\mathscr{T}_1)$ and $(\mathscr{S}_2,\mathscr{T}_2)$ is said to be

(i) covariant if $\Omega(\mathscr{S}_1)\subseteq\mathscr{S}_2$ and $\Omega(\mathscr{T}_1)\subseteq\mathscr{T}_2$. This is denoted as $\Omega:(\mathscr{S}_1,\mathscr{T}_1)\rightrightarrows(\mathscr{S}_2,\mathscr{T}_2)$;

(ii) contravariant if $\Omega(\mathscr{S}_1)\subseteq\mathscr{T}_2$ and $\Omega(\mathscr{T}_1)\subseteq\mathscr{S}_2$. It is denoted as $\Omega:(\mathscr{S}_1,\mathscr{T}_1)\leftrightharpoons(\mathscr{S}_2,\mathscr{T}_2)$.

Particularly, if d_1 is bipolar metrics on $(\mathscr{S}_1,\mathscr{T}_1)$ and d_2 is bipolar metrics on $(\mathscr{S}_2,\mathscr{T}_2)$, we often write $\Omega:(\mathscr{S}_1,\mathscr{T}_1,d_1)\rightrightarrows(\mathscr{S}_2,\mathscr{T}_2,d_2)$ and $\Omega:(\mathscr{S}_1,\mathscr{T}_1,d_1)\leftrightharpoons(\mathscr{S}_2,\mathscr{T}_2,d_2)$ respectively.

Definition 2.3:([7]) In a bipolar metric space $(\mathscr{S},\mathscr{T},d)$ for any $\xi\in\mathscr{S}\cup\mathscr{T}$ is left point if $\xi\in\mathscr{S}$, is right point if $\xi\in\mathscr{T}$ and is central point if $\xi\in\mathscr{S}\cap\mathscr{T}$. Also, $\{\alpha_i\}$ in $\mathscr{S}$ and $\{\beta_i\}$ in $\mathscr{T}$ are left and right sequence respectively. In a bipolar metric space, we call a sequence, a left or a right one. A sequence $\{\xi_i\}$ is said to be convergent to ξ iff either $\{\xi_i\}$ is a left sequence, ξ is a right point and $\lim_{i\to\infty}d(\xi_i,\xi)=0$, or $\{\xi_i\}$ is a right sequence, ξ is a left point and $\lim_{i\to\infty}d(\xi,\xi_i)=0$. The bi-sequence $(\{\alpha_i\},\{\beta_i\})$ on $(\mathscr{S},\mathscr{T},d)$ is a sequence on $\mathscr{S}\times\mathscr{T}$. In the case where $\{\alpha_i\}$ and $\{\beta_i\}$ are both convergent, then $(\{\alpha_i\},\{\beta_i\})$ is convergent.
The bi-sequence $(\{\alpha_i\},\{\beta_i\})$ is a Cauchy bi-sequence if $\lim_{i,j\to\infty}d(\alpha_i,\beta_j)=0$.

Note that every convergent Cauchy bi-sequence is bi-convergent. The bipolar metric space is complete, if each Cauchy bi-sequence is convergent (and so it is biconvergent).

Definition 2.4:([9]) Let $(\mathscr{S}_1,\mathscr{T}_1,d_1)$ and $(\mathscr{S}_2,\mathscr{T}_2,d_2)$ be bipolar metric spaces.

(a) A map $\Gamma:(\mathscr{S}_1,\mathscr{T}_1,d_1)\rightrightarrows(\mathscr{S}_2,\mathscr{T}_2,d_2)$ is called left-continuous at a point $\xi_0\in\mathscr{S}_1$, if for every $\varepsilon>0$, there exists a $\delta>0$ such that $d_1(\xi_0,\wp)<\delta$ implies $d_2(\Gamma(\xi_0),\Gamma(\wp))<\varepsilon$ for all $\wp\in\mathscr{T}_1$.

(b) A map $\Gamma:(\mathscr{S}_1,\mathscr{T}_1,d_1)\rightrightarrows(\mathscr{S}_2,\mathscr{T}_2,d_2)$ is called right-continuous at a point $\wp_0\in\mathscr{T}_1$, if for every $\varepsilon>0$, $\exists\ \delta>0$ such that $d_1(\xi,\wp_0)<\delta$ implies $d_2(\Gamma(\xi),\Gamma(\wp_0))<\varepsilon$ for all $\xi\in\mathscr{S}_1$.

(c) A map Γ is called continuous, if it is left-continuous at each point $\xi \in \mathscr{S}_1$ and right-continuous at each point $\wp \in \mathscr{T}_1$.

(c) A contravariant map $\Gamma : (\mathscr{S}_1, \mathscr{T}_1, d_1) \leftrightharpoons (\mathscr{S}_2, \mathscr{T}_2, d_2)$ is continuous if and only if it is continuous as a covariant map $\Gamma : (\mathscr{S}_1, \mathscr{T}_1, d_1) \rightrightarrows (\mathscr{S}_2, \mathscr{T}_2, d_2)$

It can be seen from the Definition 2.4 that a covariant or a contravariant map Γ from $(\mathscr{S}_1, \mathscr{T}_1, d_1)$ to $(\mathscr{S}_2, \mathscr{T}_2, d_2)$ is continuous if and only if $(u_n) \to v$ on $(\mathscr{S}_1, \mathscr{T}_1, d_1)$ implies $(\Gamma(u_n)) \to \Gamma(v)$ on $(\mathscr{S}_2, \mathscr{T}_2, d_2)$.

Definition 2.5:([9]) Let $(\mathscr{S}, \mathscr{T}, d)$ be a bipolar metric space and $\Gamma, \Lambda : (\mathscr{S}, \mathscr{T}) \rightrightarrows (\mathscr{S}, \mathscr{T})$ be two covariant mappings. A pair (Γ, Λ) is called a compatible if and only if $\lim_{i\to\infty} d(\Gamma\Lambda\alpha_i, \Lambda\Gamma\beta_i) = \lim_{i\to\infty} d(\Lambda\Gamma\alpha_i, \Gamma\Lambda\beta_i) = 0$ whenever, $(\{\alpha_i\}, \{\beta_i\})$ is a sequence in $(\mathscr{S}, \mathscr{T})$ such that $\lim_{i\to\infty} \Gamma\alpha_i = \lim_{i\to\infty} \Gamma\beta_i = \lim_{i\to\infty} \Lambda\alpha_i = \lim_{i\to\infty} \Lambda\beta_i = \wp$ for some $\wp \in \mathscr{S} \cap \mathscr{T}$.

Now we prove our main result.

3. Main Results

In this section, two covariant mappings that meet new type contractive criteria in bipolar metric spaces are given some common fixed point theorems.

Definition 3.1: Let $(\mathscr{S}, \mathscr{T}, d)$ be a bipolar metric space. Suppose $\Gamma, \Lambda : (\mathscr{S}, \mathscr{T}) \rightrightarrows (\mathscr{S}, \mathscr{T})$ are called a *JS*-quasi contraction covariant mappings if there exist a mapping $\psi_\star : (0, +\infty) \to (1, +\infty)$ and $\ell \in (0,1)$ such that $\forall\, u \in \mathscr{S}$, $p \in \mathscr{T}$ with $\Gamma u \neq \Gamma p$

$$\psi_\star(d(\Gamma u, \Gamma p) \leq \psi_\star(d(\Lambda u, \Lambda p))^{\ell} \tag{1}$$

For convenience, we set: $\Omega = \{\psi_\star / \psi_\star : (0, +\infty) \to (1, +\infty)\}$ be a family of functions that satisfy the following properties;

(i) $\psi_\star$ is a continuously nondecreasing map;

(ii) $\psi_\star(t)$ is subadditive, $\psi_\star(p+q) \leq \psi_\star(p) + \psi_\star(q)$.

Theorem 3.2: Let $(\mathscr{S}, \mathscr{T}, d)$ be a complete bipolar metric space. Suppose that $\Gamma, \Lambda : (\mathscr{S}, \mathscr{T}) \rightrightarrows (\mathscr{S}, \mathscr{T})$ be two covariant mappings satisfies *JS*-quasi contraction with $\psi_\star \in \Omega$

(i_0) $\Gamma(\mathscr{S} \cup \mathscr{T}) \subseteq \Lambda(\mathscr{S} \cup \mathscr{T})$,

(i_1) pair (Γ, Λ) is compatible,

(i_2) Λ is continuous.

Then there is a unique common fixed point of Γ and Λ in $\mathscr{S} \cup \mathscr{T}$.

Proof Let $u_0 \in \mathscr{S}$ and $p_0 \in \mathscr{T}$ be arbitrary, and from (i_0), we construct the bisequences $(\{\alpha_\kappa\}, \{\zeta_\kappa\})$ in $(\mathscr{S}, \mathscr{T})$ as

$$\Gamma u_\kappa = \Lambda u_{\kappa+1} = \alpha_\kappa, \quad \Gamma p_\kappa = \Lambda p_{\kappa+1} = \zeta_\kappa \quad \text{where } \kappa = 0, 1, 2, \ldots.$$

Then from (1), we can get

$$\begin{aligned} \psi_\star(d(\alpha_\kappa, \zeta_{\kappa+1})) &= \psi_\star(d(\Gamma u_\kappa, \Gamma p_{\kappa+1}) \\ &\leq \psi_\star(d(\Lambda u_\kappa, \Lambda p_{\kappa+1})^\ell \\ &\leq \psi_\star(d(\alpha_{\kappa-1}, \zeta_\kappa))^\ell \\ &< \psi_\star(d(\alpha_{\kappa-1}, \zeta_\kappa)). \end{aligned}$$

By the property of $\psi_\star$, we get that

$$d(\alpha_\kappa, \zeta_{\kappa+1}) < d(\alpha_{\kappa-1}, \zeta_\kappa). \tag{2}$$

On the other hand, we have

$$\begin{aligned} \psi_\star(d(\alpha_{\kappa+1}, \zeta_\kappa)) &= \psi_\star(d(\Gamma u_{\kappa+1}, \Gamma p_\kappa) \\ &\leq \psi_\star(d(\Lambda u_{\kappa+1}, \Lambda p_\kappa)^\ell \\ &\leq \psi_\star(d(\alpha_\kappa, \zeta_{\kappa-1}))^\ell \\ &< \psi_\star(d(\alpha_\kappa, \zeta_{\kappa-1})). \end{aligned}$$

By the property of $\psi_\star$, we get that

$$d(\alpha_{\kappa+1}, \zeta_\kappa) < d(\alpha_\kappa, \zeta_{\kappa-1}). \tag{3}$$

Moreover,

$$\begin{aligned} \psi_\star(d(\alpha_\kappa, \zeta_\kappa)) &= \psi_\star(d(\Gamma u_\kappa, \Gamma p_\kappa) \\ &\leq \psi_\star(d(\Lambda u_\kappa, \Lambda p_\kappa)^\ell \\ &\leq \psi_\star(d(\alpha_{\kappa-1}, \zeta_{\kappa-1}))^\ell \end{aligned}$$

$$< \quad \psi_\star\left(d(\alpha_{\kappa-1},\zeta_{\kappa-1})\right).$$

By the property of $\psi_\star$, we get that

$$d(\alpha_\kappa,\zeta_\kappa) \quad < \quad d(\alpha_{\kappa-1},\zeta_{\kappa-1}). \tag{4}$$

Thus, from (2), (3) and (4) one shows that the bisequence $\{d(\alpha_\kappa,\zeta_\kappa)\}$ are nonincreasing bisequences of non-negative real numbers. So there exist $\iota \geq 0$ such that $\lim_{\kappa\to\infty} d(\alpha_\kappa,\zeta_\kappa) = \iota$ and

$$d(\alpha_\kappa,\zeta_\kappa) \geq \iota. \tag{5}$$

Suppose that $\iota > 0$. By (2), (3) and (4) and (5), since $\psi_\star$ is nondecreasing, we get

$$1 < \psi_\star(\iota) \leq \psi_\star(d(\alpha_\kappa,\zeta_\kappa)) \leq \psi_\star(d(\alpha_{\kappa-1},\zeta_{\kappa-1}))^{\ell} \leq \cdots \leq \psi_\star(d(\alpha_0,\zeta_0))^{\ell^\kappa}, \forall\ \kappa$$

Letting $\kappa\to\infty$ in this inequality, we get $\psi_\star(\iota) = 1$ which contradicts the assumption that $\psi_\star(s) > 1$ for each $s > 0$. Consequently, we have $\iota = 0$, that is,

$$\lim_{\kappa\to\infty} d(\alpha_\kappa,\zeta_\kappa) = 0$$

$$\text{Similarly, we have } \lim_{\kappa\to\infty} d(\alpha_{\kappa+1},\zeta_\kappa) = 0 \text{ and } \lim_{\kappa\to\infty} d(\alpha_\kappa,\zeta_{\kappa+1}) = 0. \tag{6}$$

Now we show that $\lim_{\kappa,\lambda\to\infty} d(\alpha_\kappa,\zeta_\lambda) = 0$ and $\lim_{\kappa,\lambda\to\infty} d(\alpha_\lambda,\zeta_\kappa) = 0$.
Otherwise, there exist $\varepsilon > 0$ and two bi-subsequences $(\{\alpha_{\kappa_p}\},\left\{\zeta_{\lambda_p}\right\})$ and $(\left\{\alpha_{\lambda_p}\right\},\{\zeta_{\kappa_p}\})$ of $(\{\alpha_\kappa\},\{\zeta_\kappa\})$ such that λ_p is the smallest index with $\lambda_p > \kappa_p > p$ for which

$$d\left(\alpha_{\kappa_p},\zeta_{\lambda_p}\right) \geq \varepsilon, \quad d\left(\alpha_{\lambda_p},\zeta_{\kappa_p}\right) \geq \varepsilon \tag{7}$$

which indicates that

$$d\left(\alpha_{\kappa_p},\zeta_{\lambda_p-1}\right) < \varepsilon, \quad d\left(\alpha_{\lambda_p},\zeta_{\kappa_p-1}\right) < \varepsilon \tag{8}$$

By (7), (8), and the triangle inequality we get

$$\begin{aligned}\varepsilon &\leq d\left(\alpha_{\kappa_p},\zeta_{\lambda_p}\right) \leq d\left(\alpha_{\kappa_p},\zeta_{\lambda_p-1}\right) + d\left(\alpha_{\lambda_p},\zeta_{\lambda_p-1}\right) + d\left(\alpha_{\lambda_p},\zeta_{\lambda_p}\right) \\ &< \varepsilon + d\left(\alpha_{\lambda_p},\zeta_{\lambda_p-1}\right) + d\left(\alpha_{\lambda_p},\zeta_{\lambda_p}\right), \forall\ \lambda_p > \kappa_p > p.\end{aligned}$$

Letting $p \to \infty$ in this inequality, by (6) we obtain

$$\lim_{p\to\infty} d\left(\alpha_{\kappa_p}, \zeta_{\lambda_p}\right) = \varepsilon$$

(9) $$\text{Similarly, we have} \quad \lim_{p\to\infty} d\left(\alpha_{\lambda_p}, \zeta_{\kappa_p}\right) = \varepsilon.$$

Also by the triangle inequality we get

$$\begin{aligned} \varepsilon &\le d\left(\alpha_{\kappa_p}, \zeta_{\lambda_p}\right) - d\left(\alpha_{\kappa_p}, \zeta_{\kappa_p+1}\right) - d\left(\alpha_{\kappa_p+1}, \zeta_{\kappa_p+1}\right) - d\left(\alpha_{\lambda_p}, \zeta_{\lambda_p+1}\right) - d\left(\alpha_{\lambda_p}, \zeta_{\lambda_p}\right) \\ &\le d\left(\alpha_{\kappa_p+1}, \zeta_{\lambda_p+1}\right) \\ &\le d\left(\alpha_{\kappa_p+1}, \zeta_{\kappa_p}\right) + d\left(\alpha_{\lambda_p}, \zeta_{\kappa_p}\right) + d\left(\alpha_{\lambda_p}, \zeta_{\lambda_p+1}\right). \end{aligned}$$

Letting $p \to \infty$ in this inequality, by (6) and (9) we obtain

$$\lim_{p\to\infty} d\left(\alpha_{\kappa_p+1}, \zeta_{\lambda_p+1}\right) = \varepsilon$$

(10) $$\text{Similarly, we have} \quad \lim_{p\to\infty} d\left(\alpha_{\lambda_p+1}, \zeta_{\kappa_p+1}\right) = \varepsilon.$$

In analogy to (10), by (6) and (9) we can prove that

$$\lim_{p\to\infty} d\left(\alpha_{\lambda_p+1}, \zeta_{\kappa_p}\right) = \lim_{p\to\infty} d\left(\alpha_{\lambda_p}, \zeta_{\kappa_p+1}\right) = \varepsilon.$$

Note that (10) and (9) implies that there exists a positive integer p_0 such that

$$d\left(\alpha_{\kappa_p+1}, \zeta_{\lambda_p+1}\right) > 0, d\left(\alpha_{\lambda_p+1}, \zeta_{\kappa_p+1}\right) > 0 \text{ and } d\left(\alpha_{\kappa_p}, \zeta_{\lambda_p}\right) > 0, d\left(\alpha_{\lambda_p}, \zeta_{\kappa_p}\right) > 0$$

$$\text{also } d\left(\alpha_{\lambda_p+1}, \zeta_{\kappa_p}\right) > 0, d\left(\alpha_{\lambda_p}, \zeta_{\kappa_p+1}\right) > 0 \ \forall\ p \ge p_0.$$

Thus, by (1) we get

$$\begin{aligned} \psi_\star\left(d\left(\alpha_{\kappa_p+1}, \zeta_{\lambda_p+1}\right)\right) &= \psi_\star\left(d\left(\Gamma u_{\kappa_p+1}, \Gamma p_{\lambda_p+1}\right)\right) \\ &\le \psi_\star\left(d\left(\Lambda u_{\kappa_p+1}, \Lambda p_{\lambda_p+1}\right)\right)^\ell \\ &\le \psi_\star\left(d\left(\alpha_{\kappa_p}, \zeta_{\lambda_p}\right)\right)^\ell, \ \forall\ \lambda_p > \kappa_p > p \ge p_0. \end{aligned}$$

Letting $p \to \infty$ in this inequality, by (9), (10), and the continuity of $\psi_\star$ we obtain

$$\psi_\star(\varepsilon) = \lim_{p\to\infty} \psi_\star\left(d\left(\alpha_{\kappa_p+1}, \zeta_{\lambda_p+1}\right)\right) \le \lim_{p\to\infty} \psi_\star\left(d\left(\alpha_{\kappa_p}, \zeta_{\lambda_p}\right)\right)^\ell = \psi_\star(\varepsilon)^\ell < \psi_\star(\varepsilon)$$

a contradiction. Consequently, $d(\alpha_\kappa,\zeta_\lambda)\to 0$ as $\kappa,\lambda\to\infty$ holds. Similarly, we can prove $d(\alpha_\lambda,\zeta_\kappa)\to 0$ as $\kappa,\lambda\to\infty$, that is,the bisequences $(\{\alpha_\kappa\},\{\zeta_\kappa\})$ is a Cauchy sequence in $(\mathscr{S},\mathscr{T})$. Since $(\mathscr{S},\mathscr{T},d)$ is complete, $(\{\alpha_\kappa\},\{\zeta_\kappa\})$ converges and thus it biconverges to a point $\wp\in\mathscr{S}\cap\mathscr{T}$ such that

$$\lim_{\kappa\to\infty}\alpha_\kappa=\wp=\lim_{\kappa\to\infty}\zeta_\kappa. \tag{11}$$

That is

$$\lim_{\kappa\to\infty}\Gamma u_\kappa=\lim_{\kappa\to\infty}\Lambda u_{\kappa+1}=\lim_{\kappa\to\infty}\Gamma p_\kappa=\lim_{\kappa\to\infty}\Lambda p_{\kappa+1}=\wp.$$

Since Λ is continuous function, we have

$$\begin{aligned}&\lim_{\kappa\to\infty}\Lambda\Gamma u_\kappa=\Lambda\wp \quad \lim_{\kappa\to\infty}\Lambda^2 u_{\kappa+1}=\Lambda\wp\\ &\lim_{\kappa\to\infty}\Lambda\Gamma p_\kappa=\Lambda\wp \quad \lim_{\kappa\to\infty}\Lambda^2 p_{\kappa+1}=\Lambda\wp.\end{aligned} \tag{12}$$

Since the pair $\{\Gamma,\Lambda\}$ is compatible, we have
$\lim_{\kappa\to\infty}d(\Gamma\Lambda u_{\kappa+1},\Lambda\Gamma p_\kappa)=\lim_{\kappa\to\infty}d(\Lambda\Gamma u_\kappa,\Gamma\Lambda p_{\kappa+1})=0.$
Therefore,

$$\lim_{\kappa\to\infty}\Lambda\Gamma p_\kappa=\lim_{\kappa\to\infty}\Gamma\Lambda u_{\kappa+1}=\Lambda\wp \quad \lim_{\kappa\to\infty}\Lambda\Gamma u_\kappa=\lim_{\kappa\to\infty}\Gamma\Lambda p_{\kappa+1}=\Lambda\wp. \tag{13}$$

Taking $u=\Lambda u_{\kappa+1}$ and $p=p_\kappa$ in (1), we get

$$\psi_\star(d(\Gamma\Lambda u_{\kappa+1},\Gamma p_\kappa))\le\psi_\star\left(d(\Lambda^2 u_{\kappa+1},\Lambda p_\kappa)\right)^\ell.$$

Letting $\kappa\to\infty$ in this inequality, by (11), (12),(13) and the continuity of $\psi_\star$ we obtain

$$\psi_\star(d(\Lambda\wp,\wp))\le\psi_\star(d(\Lambda\wp,\wp))^\ell<\psi_\star(d(\Lambda\wp,\wp))$$

a contradiction. Consequently, $d(\Lambda\wp,\wp)=0$. That is $\Lambda\wp=\wp$.
By using the condition (1) and (B_4), we obtain

$$\begin{aligned}\psi_\star(d(\Gamma\wp,\wp)) &\le \psi_\star(d(\Gamma\wp,\zeta_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\zeta_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\wp))\\ &\le \psi_\star(d(\Gamma\wp,\Gamma p_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\zeta_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\wp))\\ &\le \psi_\star(d(\Lambda\wp,\Lambda p_{\kappa+1}))^\ell+\psi_\star(d(\alpha_{\kappa+1},\zeta_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\wp))\\ &\le \psi_\star(d(\Lambda\wp,\zeta_\kappa))^\ell+\psi_\star(d(\alpha_{\kappa+1},\zeta_{\kappa+1}))+\psi_\star(d(\alpha_{\kappa+1},\wp))\end{aligned}$$

$\to 0$ as $\kappa \to \infty$.

Thus $\Gamma\wp = \wp$. Hence $\Gamma\wp = \Lambda\wp = \wp$. Now we prove the uniqueness; we begin by taking $\aleph$ to be another fixed point of covariant maps Γ and Λ. Then $\Gamma\aleph = \Lambda\aleph = \aleph$ implies $\aleph \in \mathscr{S} \cap \mathscr{T}$ and we have

$$\psi_\star(d(\wp,\aleph)) = \psi_\star(d(\Gamma\wp,\Gamma\aleph)) \leq \psi_\star(d(\Lambda\wp,\Lambda\aleph))^\ell < \psi_\star(d(\wp,\aleph))$$

a contradiction. Consequently, we have $\wp = \aleph$. This shows that $\wp$ is the unique fixed point of Γ and Λ. The proof is completed.

Corollary 3.3: Let $(\mathscr{S},\mathscr{T},d)$ be a complete bipolar metric space. Suppose that $\Gamma : (\mathscr{S},\mathscr{T}) \rightrightarrows (\mathscr{S},\mathscr{T})$ be a covariant mapping satisfy *JS*-quasi contraction with $\psi_\star \in \Omega$. Then Γ has a unique fixed point in $\mathscr{S} \cup \mathscr{T}$.

Corollary 3.4: Let $(\mathscr{S},\mathscr{T},d)$ be a complete bipolar metric space. Suppose that $\Gamma : (\mathscr{S},\mathscr{T},d) \leftrightarrows (\mathscr{S},\mathscr{T},d)$ be a contravariant mapping satisfy

$$\psi_\star(d(\Gamma p,\Gamma u)) \leq \psi_\star(d(u,p))^\ell$$

for all $u \in \mathscr{S}$, $p \in \mathscr{T}$ and $\psi_\star \in \Omega$ with $\ell \in (0,1)$ Then there is a unique fixed point of Γ in $\mathscr{S} \cup \mathscr{T}$.

Example 3.5: Let $\mathscr{S} = \mathbb{R}^2$ and $\mathscr{T} = \mathbb{R} \times \{0\}$. Define $d : \mathscr{S} \times \mathscr{T} \to [0,\infty)$ as $d((p,q),(r,0)) = \sqrt{(p-r)^2+q^2}$ for all $p,q,r \in \mathbb{R}$. Then obviously $(\mathscr{S},\mathscr{T},d)$ is a complete bipolar-metric space. And define $\Gamma,\Lambda : (\mathscr{S},\mathscr{T},d) \rightrightarrows (\mathscr{S},\mathscr{T},d)$ as $\Gamma(x,y) = (\frac{x+5}{6},\frac{y}{6})$ and $\Lambda(x,y) = (\frac{x+1}{2},\frac{y}{2})$ and also define $\psi_\star : (0,+\infty) \to (1,+\infty)$ as $\psi_\star(t) = e^t$. Then obviously, $\Gamma(\mathscr{S} \cup \mathscr{T}) \subseteq \Lambda(\mathscr{S} \cup \mathscr{T})$ and observe that the pairs (Γ,Λ) is a compatible. Let $((\alpha_\kappa,\beta_\kappa),(\gamma_\kappa,0_\kappa))$ be a bisequence in $(\mathscr{S},\mathscr{T})$ such that, for some $(\wp,0) \in \mathscr{S} \cap \mathscr{T}$,
$\lim_{\kappa\to\infty} d(\Lambda(\alpha_\kappa,\beta_\kappa),(\wp,0)) = 0$, $\lim_{\kappa\to\infty} d((\wp,0),\Lambda(\gamma_\kappa,0_\kappa)) = 0$,
$\lim_{\kappa\to\infty} d(\Gamma(\alpha_\kappa,\beta_\kappa),(\wp,0)) = 0$, $\lim_{\kappa\to\infty} d((\wp,0),\Gamma(\gamma_\kappa,0_\kappa)) = 0$.
Since Γ and Λ are continuous, we have

$$\begin{aligned}\lim_{\kappa\to\infty} d(\Lambda\Gamma(\alpha_\kappa,\beta_\kappa),\Gamma\Lambda(\gamma_\kappa,0\kappa)) &= d(\lim_{\kappa\to\infty}\Lambda\Gamma(\alpha_\kappa,\beta_\kappa),\lim_{\kappa\to\infty}\Gamma\Lambda(\gamma_\kappa,0\kappa))\\ &= d(\Lambda(\wp,0),\Gamma(\wp,0))\end{aligned}$$

$$= d\left((\frac{\wp+1}{2},0),(\frac{\wp+5}{6},0)\right)$$
$$= \frac{\wp-1}{3}.$$

But $\frac{\wp-1}{3}=0 \Leftrightarrow \wp=1$. Similarly, we prove $\lim_{\kappa\to\infty} d(\Gamma\Lambda(\alpha_\kappa,\beta_\kappa),\Lambda\Gamma(\gamma_\kappa,0\kappa))=0$.
In fact, we have for any elements $(p,q)\in\mathscr{S}$, $(r,0)\in\mathscr{T}$

$$\begin{aligned}\psi_\star(d(\Gamma(p,q),\Gamma(r,0))) &= e^{d(\Gamma(p,q),\Gamma(r,0))}\\ &= e^{d\left((\frac{p+5}{6},\frac{q}{6}),(\frac{r+5}{6},0)\right)}\\ &= e^{\frac{1}{6}\sqrt{(p-r)^2+q^2}}\\ &\leq (e^{\frac{1}{2}\sqrt{(p-r)^2+q^2}})^{\frac{1}{2}}\\ &\leq \psi_\star(d(\Lambda(p,q),\Lambda(r,0)))^{\ell}.\end{aligned}$$

Hence all the conditions of the Theorem (3.2) are satisfied and $\ell\in(0,1)$. So, Γ and Λ must have unique common fixed point. In fact (1, 0) is the unique common fixed point of Γ and Λ.

3.1. Application to Integral Equations.

We will apply Corollary 3.3 to resolve the integral equation

$$\eta(x)=f(x)+\int_{G_1\cup G_2}\Omega(x,y)\Delta(y,\eta(y))dy,\quad x\in G_1\cup G_2. \tag{14}$$

where $G_1\cup G_2$ is a Lebesgue measurable set.
Let $\mathscr{S}=L^\infty(G_1),\mathscr{T}=L^\infty(G_2)$ be two normed linear spaces, where G_1,G_2 are Lebesgue measurable sets with $m(G_1\cup G_2)<\infty$. Define $d:\mathscr{S}\times\mathscr{T}\to R^+$ as $d(\ell,\sigma)=||\ell-\sigma||_\infty$ for all $\ell\in\mathscr{S},\sigma\in\mathscr{T}$. Obviously, $(\mathscr{S},\mathscr{T},d)$ is a complete bipolar metric space.
Define $\Gamma: L^\infty(E_1)\cup L^\infty(E_2)\to L^\infty(E_1)\cup L^\infty(E_2)$ by

$$\Gamma\eta(x)=f(x)+\int_{G_1\cup G_2}\Omega(x,y)\Delta(y,\eta(y))dy\quad x\in G_1\cup G_2.$$

Then Γ is a covariant mapping.

Theorem 3.1.1: Assume that the following conditions are fulfilled

(i) $\Omega : G_1^2 \cup G_2^2 \to R^+$, $\Delta : (G_1 \cup G_2) \times [0,\infty) \to R^+$ and $f : G_1 \cup G_2 \to R^+$ are continuous functions. Let $\psi_\star : (0,+\infty) \to (1,+\infty)$ as $\psi_\star(t) = e^t$.

(ii) There exists a continuous function $\chi : G_1 \cup G_2 \to R^+$ such that for all $\eta \in \mathscr{S}, \zeta \in \mathscr{T}$, and $y \in G_1 \cup G_2$, we get that
$|\Delta(y,\eta(y)) - \Delta(y,\zeta(y))| \leq |\chi(\zeta)||\eta(y) - \zeta(y)|^\ell$ where $\ell \in (0,1)$

(iii) $||\int_{G_1 \cup G_2} \Omega(x,y)\chi(\zeta)dy|| \leq 1$.

Then the integral equation (14) has a solution in $L^\infty(G_1) \cup L^\infty(G_2)$.

Proof The existence of a solution of Eq.(14) is equivalent to the existence of a unique solution of Γ. Using the inequalities, (i), (ii) and (iii), we have

$$\begin{aligned}
|\Gamma\eta(x) - \Gamma\zeta(x)| &= |\int_{G_1 \cup G_2} \Omega(x,y)\,[\Delta(y,\eta(y)) - \Delta(y,\zeta(y))]dy| \\
&\leq \int_{G_1 \cup G_2} \Omega(x,y)|\Delta(y,\eta(y)) - \Delta(y,\zeta(y))|dy \\
&\leq \int_{G_1 \cup G_2} \Omega(x,y)|\chi(\zeta(y))||\eta(y) - \zeta(y)|^\ell dy \\
&\leq \int_{G_1 \cup G_2} \Omega(x,y)|\chi(\zeta(y))|||\eta - \zeta||_\infty^\ell dy \\
&\leq ||\eta - \zeta||_\infty^\ell \left(\int_{G_1 \cup G_2} \Omega(x,y)|\chi(\zeta(y))|dy \right)
\end{aligned}$$

Then

$$\begin{aligned}
||\Gamma\eta - \Gamma\zeta||_\infty &\leq ||\eta - \zeta||_\infty^\ell || \int_{G_1 \cup G_2} \Omega(x,y)|\chi(\zeta(y))|dy||_\infty \\
&\leq ||\eta - \zeta||_\infty^\ell
\end{aligned}$$

Choose $\psi_\star(t) = e^t$. Then Consequently, for all $\eta \in \mathscr{S}, \zeta \in \mathscr{T}$, we deduce that

$$\psi_\star(d(\Gamma\eta,\Gamma\zeta) \leq \psi_\star(d(\eta,\zeta))^\ell$$

Hence, all the conditions of Corollary (3.3) hold, we conclude that Γ has a unique solution in $\mathscr{S} \cup \mathscr{T}$ to the integral equation (14)..

3.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: Let $(\mathscr{S},\mathscr{T},d)$ be complete bipolar metric space, $(\mathscr{P},\mathscr{Q})$ and $(\overline{\mathscr{P}},\overline{\mathscr{Q}})$ be an open and closed subset of $(\mathscr{S},\mathscr{T})$ such that $(\mathscr{P},\mathscr{Q})\subseteq(\overline{\mathscr{P}},\overline{\mathscr{Q}})$. Suppose
$\mathscr{H}:\left(\overline{\mathscr{P}}\cup\overline{\mathscr{Q}}\right)\times[0,1]\to\mathscr{S}\cup\mathscr{T}$ be an operator with following conditions are satisfying,
$i)$ $\wp\neq\mathscr{H}(\wp,s)$ for each $\wp\in\partial\mathscr{P}\cup\partial\mathscr{Q}$ and $s\in[0,1]$ (here $\partial\mathscr{P}\cup\partial\mathscr{Q}$ is boundary of $\mathscr{P}\cup\mathscr{Q}$ in $\mathscr{S}\cup\mathscr{T}$);
$ii)$ for all $\wp\in\overline{\mathscr{P}},\imath\in\overline{\mathscr{Q}}$, $s\in[0,1]$ and $\psi_\star\in\Omega$ and $\ell\in(0,1)$ such that

$$\psi_\star\left(d\left(\mathscr{H}(\wp,s),\mathscr{H}(\imath,s)\right)\right)\leq\psi_\star\left(d(\wp,\imath)\right)^{\ell}$$

$iii)$ $\exists\, M\geq0\ni d(\mathscr{H}(\wp,s),\mathscr{H}(\imath,t))\leq M|s-t|$ for every $\wp\in\overline{\mathscr{P}}$, $\imath\in\overline{\mathscr{Q}}$ and $s,t\in[0,1]$.
Then $\mathscr{H}(.,0)$ has a fixed point $\iff$ $\mathscr{H}(.,1)$ has a fixed point.

Proof Let the sets

$$\Theta=\left\{\ s\in[0,1]:\mathscr{H}(\wp,s)=\wp\text{ for some }\wp\in\mathscr{P}\ \right\}.$$

$$\Upsilon=\left\{\ t\in[0,1]:\mathscr{H}(\varkappa,t)=\varkappa\text{ for some }\varkappa\in\mathscr{Q}\ \right\}.$$

Suppose that $\mathscr{H}(.,0)$ has a fixed point in $\mathscr{P}\cup\mathscr{Q}$, we have that $0\in\Theta\cap\Upsilon$. So that $\Theta\cap\Upsilon\neq\phi$. Now we show that $\Theta\cap\Upsilon$ is both closed and open in $[0,1]$ and hence by the connectedness $\Theta=\Upsilon=[0,1]$. As a result, $\mathscr{H}(.,0)$ has a fixed point in $\Theta\cap\Upsilon$. First we show that $\Theta\cap\Upsilon$ closed in $[0,1]$. To see this, Let $(\{a_p\}_{p=1}^{\infty},\{x_p\}_{p=1}^{\infty})\subseteq(\Theta,\Upsilon)$ with $(a_p,x_p)\to(\alpha,\alpha)\in[0,1]$ as $p\to\infty$. We must show that $\alpha\in\Theta\cap\Upsilon$. Since $(a_p,x_p)\in(\Theta,\Upsilon)$ for $p=0,1,2,3,\cdots$, there exists sequences $\left(\{\wp_p\},\{\varkappa_p\}\right)$ with $\wp_{p+1}=\mathscr{H}(\wp_p,a_p)$, $\varkappa_{p+1}=\mathscr{H}(\varkappa_p,x_p)$
Consider

$$\begin{aligned}
\psi_\star\left(d(\wp_p,\varkappa_{p+1})\right) &= \psi_\star\left(d\left(\mathscr{H}(\wp_{p-1},a_{p-1}),\mathscr{H}(\varkappa_p,b_p)\right)\right)\\
&\leq \psi_\star\left(d\left(\mathscr{H}(\wp_{p-1},a_{p-1}),\mathscr{H}(\varkappa_p,a_{p-1})\right)\right)\\
&\quad+\psi_\star\left(d\left(\mathscr{H}(\wp_p,b_{p-1}),\mathscr{H}(\varkappa_p,a_{p-1})\right)\right)\\
&\quad+\psi_\star\left(d\left(\mathscr{H}(\wp_p,b_{p-1}),\mathscr{H}(\varkappa_p,b_p)\right)\right)\\
&\leq \psi_\star\left(d\left(\wp_{p-1},\varkappa_p\right)\right)^{\ell}+M|b_{p-1}-a_{p-1}|+M|b_{p-1}-b_p|
\end{aligned}$$

$$< \quad \psi_{\star}\left(d\left(\wp_{p-1}, \varkappa_{p}\right)\right) \tag{15}$$

By using property of $\psi_{\star}$, we have

$$d(\wp_{p}, \varkappa_{p+1}) \quad < \quad d\left(\wp_{p-1}, \varkappa_{p}\right) \tag{16}$$

Similar lines we can prove that

$$d(\wp_{p+1}, \varkappa_{p}) \quad < \quad d\left(\wp_{p}, \varkappa_{p-1}\right) \tag{17}$$

and

$$d(\wp_{p}, \varkappa_{p}) \quad < \quad d\left(\wp_{p-1}, \varkappa_{p-1}\right) \tag{18}$$

The inequalities (16), (17) and (18) yield that the bisequence
$\{dn := d(\wp_p, \varkappa_p)\}$ is non-increasing, so it converges to $\delta \geq 0$. Assume that $\delta > 0$. Taking $p \to \infty$ in equations (15), we get a contradiction. Therefore,

$$\lim_{p\to\infty} d(\wp_p, \varkappa_p) = 0 \tag{19}$$

We will prove $(\{\wp_p\}, \{\varkappa_p\})$ is a Cauchy bisequence. Assume there are $\varepsilon > 0$ and $\{q_k\}, \{p_k\}$ so that for $p_k > q_k > k$,

$$d(\wp_{p_k}, \varkappa_{q_k}) \geq \varepsilon, \quad d(\wp_{p_k-1}, \varkappa_{q_k}) < \varepsilon \tag{20}$$

and

$$d(\wp_{q_k}, \varkappa_{p_k}) \geq \varepsilon, \quad d(\wp_{q_k}, \varkappa_{p_k-1}) < \varepsilon \tag{21}$$

By view of (20) and triangle inequality, we get

$$\begin{aligned}
\varepsilon &\leq d(\wp_{p_k}, \varkappa_{q_k}) \\
&\leq d(\wp_{p_k}, \varkappa_{p_k-1}) + d(\wp_{p_k-1}, \varkappa_{p_k-1}) + d(\wp_{p_k-1}, \varkappa_{q_k}) \\
&< d(\wp_{p_k}, \varkappa_{p_k-1}) + d\left(\mathscr{H}(\wp_{p-2}, a_{p-2}), \mathscr{H}(\varkappa_{p-2}, x_{p-2})\right) + \varepsilon \\
&< d(\wp_{p_k}, \varkappa_{p_k-1}) + M|a_{p-2} - x_{p-2}| + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$, and using (19), we obtain

$$\lim_{p\to\infty} d(\wp_{p_k}, \varkappa_{q_k}) = \varepsilon \tag{22}$$

Using (21), one can prove

$$\lim_{p\to\infty} d(\wp_{q_k},\varkappa_{p_k})=\varepsilon \tag{23}$$

For all $k\in\mathbb{N}$, by (ii) we have

$$\psi_\star\left(d(\wp_{p_k+1},\varkappa_{q_k+1})\right)<\psi_\star\left(d(\wp_{p_k},\varkappa_{q_k})\right)$$

and

$$\psi_\star\left(d(\wp_{q_k+1},\varkappa_{p_k+1})\right)<\psi_\star\left(d(\wp_{q_k},\varkappa_{p_k})\right).$$

Applying (22) and (23), we get at the limit, $\psi_\star(\varepsilon)<\psi_\star(\varepsilon)$. That is $\varepsilon=0$ which is a contradiction. Hence $(\{\wp_p\},\{\varkappa_p\})$ is a Cauchy bi-sequences in $(\mathscr{P},\mathscr{Q})$. By completeness, there exist $\tau\in\mathscr{P}\cap\mathscr{Q}$ with

$$\lim_{p\to\infty}\wp_{p+1}=\tau=\lim_{p\to\infty}\varkappa_{p+1} \tag{24}$$

we have

$$\begin{aligned}\psi_\star\left(d\left(\mathscr{H}(\tau,\alpha),\varkappa_{p+1}\right)\right) &= \psi_\star\left(d\left(\mathscr{H}(\tau,\alpha),\mathscr{H}(\varkappa_p,x_p)\right)\right)\\ &\le \psi_\star\left(d\left(\tau,\varkappa_p\right)\right)^{\ell}\\ &< \psi_\star\left(d\left(\tau,\varkappa_p\right)\right).\end{aligned}$$

By taking the limsup on both sides and $\psi_\star$ is continuous and non-decreasing, we have $d\left(\mathscr{H}(\tau,\alpha),\tau\right)=0$ implies that $\mathscr{H}(\tau,\alpha)=\tau$. Therefore, $\alpha\in\Theta\cap\Upsilon$. Clearly, $\Theta\cap\Upsilon$ closed in $[0,1]$. Let $(a_0,x_0)\in\Theta\times\Upsilon$, there exists bisequences $(\wp_0,\varkappa_0)$ with $\wp_0=\mathscr{H}(\wp_0,a_0)$, $\varkappa_0=\mathscr{H}(\varkappa_0,x_0)$. Since $\mathscr{P}\cup\mathscr{Q}$ is open, then there exist $\delta>0$ such that $B_d(\wp_0,\delta)\subseteq\mathscr{P}\cup\mathscr{Q}$ and $B_d(\delta,\varkappa_0)\subseteq\mathscr{P}\cup\mathscr{Q}$.

Choose $a\in(a_0-\varepsilon,a_0+\varepsilon)$, $x\in(x_0-\varepsilon,x_0+\varepsilon)$ such that $|a-x_0|\le\frac{1}{M^p}<\frac{\varepsilon}{2}$, $|x-a_0|\le\frac{1}{M^p}<\frac{\varepsilon}{2}$ and $|a_0-x_0|\le\frac{1}{M^p}<\frac{\varepsilon}{2}$.

Then for, $\varkappa\in\overline{B}_{\mathscr{P}\cup\mathscr{Q}}(\wp_0,\delta)=\{\varkappa,\varkappa_0\in\mathscr{Q}/d(\wp_0,\varkappa)\le d(\wp_0,\varkappa_0)+\delta\}$,
$\wp\in\overline{B}_{\mathscr{P}\cup\mathscr{Q}}(\delta,\varkappa_0)=\{\wp,\wp_0\in\mathscr{P}/d(\wp,\varkappa_0)\le d(\wp_0,\varkappa_0)+\delta\}$

$$\begin{aligned}d\left(\mathscr{H}(\wp,a),\varkappa_0\right) &= d\left(\mathscr{H}(\wp,a),\mathscr{H}(\varkappa_0,x_0)\right)\\ &\le d\left(\mathscr{H}(\wp,a),\mathscr{H}(\varkappa,x_0)\right)+d\left(\mathscr{H}(\wp_0,a),\mathscr{H}(\varkappa,x_0)\right)\end{aligned}$$

$$
\begin{aligned}
&+d\left(\mathscr{H}(\wp_0,a),\mathscr{H}(\varkappa_0,x_0)\right)\\
\leq\ & 2M|a-x_0|+d\left(\mathscr{H}(\wp_0,a),\mathscr{H}(\varkappa,x_0)\right)\\
\leq\ & \frac{2}{M^{p-1}}+d\left(\mathscr{H}(\wp_0,a),\mathscr{H}(\varkappa,x_0)\right).
\end{aligned}
$$

Letting $p\to\infty$ and using $\psi_\star$ property, then we have

$$
\begin{aligned}
\psi_\star\left(d\left(\mathscr{H}(\wp,a),\varkappa_0\right)\right) &\leq \psi_\star\left(d\left(\mathscr{H}(\wp_0,a),\mathscr{H}(\varkappa,x_0)\right)\right)\\
&\leq \psi_\star\left(d\left(\wp_0,\varkappa\right)\right)^{\ell}\\
&< \psi_\star\left(d\left(\wp_0,\varkappa\right)\right)
\end{aligned}
$$

Using the property of $\psi_\star$, we get

$$
\begin{aligned}
d\left(\mathscr{H}(\wp,a),\varkappa_0\right) &< d\left(\wp_0,\varkappa\right)\\
&\leq d(\wp_0,\varkappa_0)+\delta
\end{aligned}
$$

Similarly we can prove

$$
\begin{aligned}
d\left(\wp,\mathscr{H}(\varkappa,x)\right) &< d\left(\wp,\varkappa_0\right)\\
&\leq d(\wp_0,\varkappa_0)+\delta
\end{aligned}
$$

On the other hand,

$$
d(\wp_0,\varkappa_0)=d\left(\mathscr{H}(\wp_0,a_0),\mathscr{H}(\varkappa_0,x_0)\right)\leq M|a_0-x_0|<\frac{1}{M^{p-1}}\to 0 \text{ as } p\to\infty.
$$

So $\wp_0=\varkappa_0$ and hence $a=x$. Thus for each fixed $a\in(a_0-\varepsilon,a_0+\varepsilon)$,
$\mathscr{H}(.,a):\overline{B}_{\Theta\cup\Upsilon}(\wp_0,\delta)\to\overline{B}_{\Theta\cup\Upsilon}(\wp_0,\delta)$. Thus, we conclude that $\mathscr{H}(.,a)$ has a fixed point in $\overline{\mathscr{P}}\cap\overline{\mathscr{Q}}$. But this must be in $\mathscr{P}\cup\mathscr{Q}$. Therefore, $a\in\Theta\cap\Upsilon$ for
$a\in(a_0-\varepsilon,a_0+\varepsilon)$.Hence, $(a_0-\varepsilon,a_0+\varepsilon)\subseteq\Theta\cap\Upsilon$. Clearly, $\Theta\cap\Upsilon$ is open in $[0,1]$. For the reverse implication, we use the same strategy.

Theorem 3.2.2: Let $(\mathscr{S},\mathscr{T},d)$ be complete bipolar metric space, $(\mathscr{P},\mathscr{Q})$ and $(\overline{\mathscr{P}},\overline{\mathscr{Q}})$ be an open and closed subset of $(\mathscr{S},\mathscr{T})$ such that $(\mathscr{P},\mathscr{Q})\subseteq(\overline{\mathscr{P}},\overline{\mathscr{Q}})$. Suppose
$\mathscr{H}:\left(\overline{\mathscr{P}},\overline{\mathscr{Q}},d\right)\times[0,1]\rightleftharpoons(\mathscr{S},\mathscr{T},d)$ be an contravariant operator with following conditions are satisfying,

i) $\wp\neq\mathscr{H}(\wp,s)$ for each $\wp\in\partial\mathscr{P}\cup\partial\mathscr{Q}$ and $s\in[0,1]$ (here $\partial\mathscr{P}\cup\partial\mathscr{Q}$ is boundary of $\mathscr{P}\cup\mathscr{Q}$

in $\mathscr{S} \cup \mathscr{T}$);

ii) for all $\wp \in \overline{\mathscr{P}}, \iota \in \overline{\mathscr{Q}}$, $s \in [0,1]$ and $\psi_\star \in \Omega$ and $\ell \in (0,1)$ such that

$$\psi_\star \left(d\left(\mathscr{H}(\iota,s),\mathscr{H}(\wp,s)\right)\right) \leq \psi_\star \left(d(\wp,\iota)\right)^\ell$$

iii) $\exists\, M \geq 0 \ni d(\mathscr{H}(\iota,s),\mathscr{H}(\wp,t)) \preceq M|s-t|$ for every $\wp \in \overline{\mathscr{P}}$, $\iota \in \overline{\mathscr{Q}}$ and $s,t \in [0,1]$.

Then $\mathscr{H}(.,0)$ has a fixed point $\iff$ $\mathscr{H}(.,1)$ has a fixed point.

4. Conclusions

By utilising *JS*-quasi contractive conditions that are defined on complete bipolar metric spaces and appropriate examples that corroborate the main findings, this study presents some fixed point conclusions. Applications to integral equations and Homotopy theory are also given.

Author contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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