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A FIXED POINT OF A CONSTRAINED PROBLEM OF STATE-DEPENDENT FUNCTIONAL EQUATION

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Abstract. Our main objective in this paper is to prove the existence of a unique solution (fixed point) of a constrained problem of a state-dependent functional equation constrained by its conjugate. The continuous dependence of the solution will be proved. The Hyres-Ulam stability of our problem will be studied.

Keywords: unique solution; fixed point; constrained problem; state-dependent; continuous dependence; Hyres-Ulam stability.

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1. INTRODUCTION

Functional equations arise in many fields of mathematics, such as mechanics, geometry, statistics, measure theory ,algebraic geometry, group theory. Functional equations have many interesting applications in characterization problems of probability theory, which have been studied in several papers and monographs (see for example [1, 2, 3, 7, 9]).

Usually differential and integral equations with deviating arguments that appear in many literature have deviation of the argument involves only the time itself, However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of

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importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations (see [1, 2, 4, 7, 8, 9, 10, 11, 12, 13, 15, 16, 19]).

In [14], the authors studied the existence of a unique solution for some functional equations with state-dependent deviated arguments

$$x(t) = f(t, x(\phi(x(t)))),\ t \in [0,b],$$

$$x(t) = f(t, x(g(t, x(t)))),\ t \in [0,b],$$

$$x(t) = f(t, x(x(\phi(t))), x(g(t, x(t)))),\ t \in [0,b].$$

Several coupled systems of integral and differential equations have been studied in many papers,(see [6, 17, 18, 21, 22, 23]).

In this article, we are concerning with the constrained problem of a state-dependent functional equation

$$x(t) = f_1(t, y(\phi_1(y(t)))),\ t \in [0,T] \tag{1}$$

constrained by

$$y(t) = f_2(t, x(\phi_2(x(t)))),\ t \in [0,T]. \tag{2}$$

The continuous dependence of the unique solution $x \in C[0,T]$ on the functions ϕ_i and f_i will be analyzed. Also the continuous dependence of x on y and of y on x will be studied. Moreover, the Hyers-Ulam stability of our problem will be established.
Let $C[0,T]$ be the class of all continuous functions define on $[0,T]$ and $X = C[0,T] \times C[0,T]$ be the Banach space with the norm

$$\|(u,v)\|_X = \|u\|_C + \|v\|_C,$$

where

$$\|u\|_C = \sup_{t \in [0,T]} |u(t)|.$$

2. Existence Theorem

Consider the problem (1)-(2) under the following assumptions:

(i) $f_i : [0,T] \times [0,T] \to [0,T]$ are continuous and there exist positive constants K_i such that

$$|f_i(t,x) - f_i(s,y)| \leq K_i(|t-s| + |x-y|),\ i = 1,2.$$

(ii) $\phi_i : [0,T] \to [0,T]$ satisfies $\phi_i(0) = 0$ and

$$|\phi_i(t) - \phi_i(s)| \leq |t-s|,\ \forall\, t,s \in [0,T].$$

(iii) There exists a real positive root $L \in (0,1)$ of the algebraic equation $KL^2 - L + K = 0$, where $K = \max\{K_1, K_2\}$.

And define the subset S_L

$$S_L = \{(x,y) \in X : |x(t) - x(s)| \leq L|t-s|,\ |y(t) - y(s)| \leq L|t-s|\ \forall\, t,s \in [0,T]\}$$

and the operator F by

$$F(x,y) = (F_1 y, F_2 x).$$

Where

$$F_1 y(t) = f_1(t, y(\phi_1(y(t)))),$$

$$F_2 x(t) = f_2(t, x(\phi_2(x(t)))).$$

Theorem 1. *Let the assumptions (i)-(iii) be satisfied, if $KL + K < 1$, then the problem (1)-(2) has a unique solution $x \in C[0,T]$.*

Let $(x,y) \in X$, $t_1, t_2 \in [0,T]$, we have

$$\begin{aligned}
|F_1 y(t_2) - F_1 y(t_1)| &= |f_1(t_2, y(\phi_1(y(t_2)))) - f_1(t_1, y(\phi_1(y(t_1))))| \\
&\leq K_1|t_2 - t_1| + K_1|y(\phi_1(y(t_2))) - y(\phi_1(y(t_1)))| \\
&\leq K_1|t_2 - t_1| + K_1 L|\phi_1(y(t_2)) - \phi_1(y(t_1))| \\
&\leq K_1|t_2 - t_1| + K_1 L|y(t_2) - y(t_1)| \\
&\leq K|t_2 - t_1| + KL^2|t_2 - t_1| \\
&\leq (K + KL^2)|t_2 - t_1| \\
&\leq L|t_2 - t_1|.
\end{aligned}$$

Then

$$F_1 : S_L \to S_L.$$

Similarly,

$$\begin{aligned} |F_2x(t_2) - F_2x(t_1)| &\le |f_2(t_2, x(\phi_2(x(t_1)))) - f_2(t_1, x(\phi_2(x(t_2))))| \\ &\le L|t_2 - t_1|. \end{aligned}$$

This prove that $F_2 : S_L \to S_L$ and we deduce that

$$F(x,y) = (F_1y, F_2x) : S_L \to S_L.$$

Now, let $u = (x,y) \in X,\ v = (\bar{x}, \bar{y}) \in X$, then

$$F(x,y) = (F_1y, F_2x),\ F(\bar{x}, \bar{y}) = (F_1\bar{y}, F_2\bar{x})$$

and

$$\begin{aligned} |F_1y(t) - F_1\bar{y}(t)| &= |f_1(t, y(\phi_1(y(t)))) - f_1(t, \bar{y}(\phi_1(\bar{y}(t))))| \\ &= |f_1(t, y(\phi_1(y(t)))) - f_1(t, y(\phi_1(\bar{y}(t)))) + f_1(t, y(\phi_1(\bar{y}(t)))) - f_1(t, \bar{y}(\phi_1(\bar{y}(t))))| \\ &\le |f_1(t, y(\phi_1(y(t)))) - f_1(t, y(\phi_1(\bar{y}(t))))| + |f_1(t, y(\phi_1(\bar{y}(t)))) - f_1(t, \bar{y}(\phi_1(\bar{y}(t))))| \\ &\le K_1L|\phi_1(y(t)) - \phi_1(\bar{y}(t))| + K_1|y(\phi_1(\bar{y}(t))) - \bar{y}(\phi_1(\bar{y}(t)))| \\ &\le KL|y(t) - \bar{y}(t)| + K\|y - \bar{y}\| \\ &\le KL\|y - \bar{y}\| + K\|y - \bar{y}\| \end{aligned}$$

and

$$\|F_1y - F_1\bar{y}\| \le (KL + K)\|y - \bar{y}\|.$$

Similarly, we can prove that

$$\|F_2x - F_2\bar{x}\| \le (KL + K)\|x - \bar{x}\|.$$

Hence

$$\begin{aligned} \|F(x,y) - F(\bar{x}, \bar{y})\|_X &= \|(F_1y, F_2x) - (F_1\bar{y}, F_2\bar{x})\|_X \\ &= \|(F_1y - F_1\bar{y}, F_2x - F_2\bar{x})\| \end{aligned}$$

$$
\begin{aligned}
&= \|F_1y - F_1\bar{y}\| + \|F_2x - F_2\bar{x}\| \\
&\leq (KL+K)(\|y-\bar{y}\| + \|x-\bar{x}\|) \\
&\leq (KL+K)\|((x-\bar{x}),(y-\bar{y}))\| \\
&\leq (KL+K)\|(x,y)-(\bar{x},\bar{y})\|.
\end{aligned}
$$

Then F is a contraction mapping and by Banach fixed point Theorem [20], F has a unique fixed point. Consequently the problem $(1)-(2)$ has a unique solution $x \in C[0,T]$.

3. Continuous Dependence

Here we shall study the continuous dependence of solution of the problem $(1)-(2)$ on some functions.

Definition 1. *The solution $x \in C[0,T]$ of (1)-(2) depends continuously on the functions f_i and ϕ_i if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that*

$$\max\{|f_i - f_i^*|,\ |\phi_i - \phi_i^*|\} < \delta \Rightarrow \|x - x^*\| < \varepsilon,$$

where

$$x^*(t) = f_1^*(t, y^*(\phi_1^*(y^*(t)))), \tag{3}$$

$$y^*(t) = f_2^*(t, x^*(\phi_2^*(x^*(t)))). \tag{4}$$

Theorem 2. *Let the assumptions of Theorem 1 be satisfied, then $x \in C[0,T]$ depends continuously on f_i and ϕ_i .*

Proof. Let x and x^* be two solutions of the problems (1)-(2) and (5)-(6) respectively, then

$$
\begin{aligned}
&|x(t)-x^*(t)| \\
=\ & |f_1(t,y(\phi_1(y(t)))) - f_1^*(t,y^*(\phi_1^*(y^*(t))))| \\
=\ & |f_1(t,y(\phi_1(y(t)))) - f_1^*(t,y^*(\phi_1^*(y^*(t)))) + f_1(t,y^*(\phi_1^*(y^*(t)))) - f_1(t,y^*(\phi_1^*(y^*(t))))| \\
\leq\ & |f_1(t,y(\phi_1(y(t)))) - f_1(t,y^*(\phi_1^*(y^*(t))))| + |f_1(t,y^*(\phi_1^*(y^*(t)))) - f_1^*(t,y^*(\phi_1^*(y^*(t))))|
\end{aligned}
$$

$$
\begin{aligned}
&\leq K_1|y(\phi_1(y(t))) - y^*(\phi_1^*(y^*(t)))| + \delta \\
&\leq K_1|y(\phi_1(y(t))) - y(\phi_1^*(y^*(t))) + y(\phi_1^*(y^*(t))) - y^*(\phi_1^*(y^*(t)))| + \delta \\
&\leq K_1|y(\phi_1(y(t))) - y(\phi_1^*(y^*(t)))| + K_1|y(\phi_1^*(y^*(t))) - y^*(\phi_1^*(y^*(t)))| + \delta \\
&\leq K_1L|\phi_1(y(t)) - \phi_1^*(y^*(t))| + K_1\|y - y^*\| + \delta \\
&\leq K_1L|\phi_1(y(t)) - \phi_1(y^*(t)) + \phi_1(y^*(t)) - \phi_1^*(y^*(t))| + K_1\|y - y^*\| + \delta \\
&\leq K_1L|\phi_1(y(t)) - \phi_1(y^*(t))| + K_1L|\phi_1(y^*(t)) - \phi_1^*(y^*(t))| + K_1\|y - y^*\| + \delta \\
&\leq KL\|y - y^*\| + KL\delta + K\|y - y^*\| + \delta \\
&\leq (KL + K)\|y - y^*\| + KL\delta + \delta.
\end{aligned}
$$

Then

$$\|x - x^*\| \leq (KL + K)\|y - y^*\| + (KL + 1)\delta. \tag{5}$$

Similarly,

$$\|y - y^*\| \leq (KL + K)\|x - x^*\| + (KL + 1)\delta. \tag{6}$$

By addition (5) and (6), we get

$$\|x - x^*\| + \|y - y^*\| \leq (KL + K)(\|x - x^*\| + \|y - y^*\|) + 2(KL + 1)\delta.$$

Then

$$(1 - (KL + K))(\|x - x^*\| + \|y - y^*\|) \leq 2(KL + 1)\delta.$$

Hence

$$\|x - x^*\| + \|y - y^*\| \leq \frac{2(KL + 1)}{1 - (KL + K)}\delta = \varepsilon.$$

Now

$$
\begin{aligned}
\|(x,y) - (x^*,y^*)\|_X &= \|(x - x^*), (y - y^*)\|_X \\
&= \|(x - x^*)\|_C + \|(y - y^*)\|_C \leq \varepsilon.
\end{aligned}
$$

Then

$$\|(x,y) - (x^*,y^*)\|_X < \varepsilon.$$

□

Definition 2. *The solution of the functional equation (1) depends continuously on y if* $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ *such that*

$$|y - y^*| < \delta \ \Rightarrow\ |x - x^*| < \varepsilon,$$

where

$$x^*(t) = f_1(t, y^*(\phi_1(y^*(t)))),$$

$$y^*(t) = f_2(t, x^*(\phi_1(x^*(t)))).$$

Theorem 3. *Let the assumptions of Theorem 1 be satisfied, then* $x \in C[0,T]$ *depends continuously on y .*

Proof. Let x and x^* be two solutions of (1), then

$$\begin{aligned}
|x(t) - x^*(t)| &= |f_1(t, y(\phi_1(y(t)))) - f_1(t, y^*(\phi_1(y^*(t))))| \\
&\leq K_1|y(\phi_1(y(t))) - y^*(\phi_1(y^*(t)))| \\
&\leq K_1|y(\phi_1(y(t))) - y(\phi_1(y^*(t))) + y(\phi_1(y^*(t))) - y^*(\phi_1(y^*(t)))| \\
&\leq K_1|y(\phi_1(y(t))) - y(\phi_1(y^*(t)))| + K_1|y(\phi_1(y^*(t))) - y^*(\phi_1(y^*(t)))| \\
&\leq K_1 L|\phi_1(y(t)) - \phi_1(y^*(t))| + K_1\|y - y^*\| \\
&\leq KL\|y - y^*\| + K\|y - y^*\| \\
&\leq (KL + K)\delta = \varepsilon.
\end{aligned}$$

Then

$$\|x - x^*\| \leq \varepsilon.$$

□

By the same way we can prove that the solution y of (2) depends continuously on x

4. HYRES-ULAM STABILITY

Definition 3. [5] *Let the solution* $x \in C[0,T]$ *of the problem (1)-(2) be exists, then the problem (1)-(2) is Hyers-Ulam stable if* $\forall \varepsilon > 0\ \exists\ \delta(\varepsilon) > 0$ *such that for any* δ*-approximate solution* $x_s \in C[0,T$ *of (1)-(2) satisfies*

$$\max\{|x_s(t)-f_1(t,y_s(\phi_1(y_s(t))))|,\ |y_s(t)-f_2(t,x_s(\phi_2(x_s(t))))|\}<\delta.$$

Then

$$\|x-x_s\|<\varepsilon.$$

Theorem 4. *Let the assumptions of Theorem 1 be satisfied, then the problem (1)-(2) is Hyers-Ulam stable.*

Proof. Let

$$\max\{|x_s(t)-f_1(t,y_s(\phi_1(y_s(t))))|,\ |y_s(t)-f_2(t,x_s(\phi_2(x_s(t))))|\}<\delta,$$

Then

$$\begin{aligned}
|x(t)-x_s(t)| &= |f_1(t,y(\phi_1(y(t))))-x_s(t)| \\
&= |f_1(t,y(\phi_1(y(t))))-x_s(t)+f_1(t,y_s(\phi_1(y_s(t))))-f_1(t,y_s(\phi_1(y_s(t))))| \\
&\leq |f_1(t,y_s(\phi_1(y_s(t))))-x_s(t)|+|f_1(t,y(\phi_1(y(t))))-f_1(t,y_s(\phi_1(y_s(t))))| \\
&\leq \delta+K_1|y(\phi_1(y(t)))-y_s(\phi_1(y_s(t)))| \\
&\leq \delta+K_1|y(\phi_1(y(t)))-y(\phi_1(y_s(t)))+y(\phi_1(y_s(t)))-y_s(\phi_1(y_s(t)))| \\
&\leq \delta+K_1L|\phi_1(y(t))-\phi_1(y_s(t))|+K_1\|y-y_s\| \\
&\leq \delta+KL|y(t)-y_s(t)|+K\|y-y_s\| \\
&\leq \delta+KL\|y-y_s\|+K\|y-y_s\|,
\end{aligned}$$

and

$$\|x-x_s\|\leq\delta+(KL+K)\|y-y_s\|. \tag{7}$$

Similarly,

$$\|y-y_s\|\leq\delta+(KL+K)\|x-x_s\|. \tag{8}$$

By addition (7) and (8), we get

$$(\|x-x_s\|+\|y-y_s\|)\leq 2\delta+(KL+K)(\|y-y_s\|+\|x-x_s\|),$$

and

$$(1-(KL+K))(\|y-y_s\|+\|x-x_s\|)\leq 2\delta.$$

Hence

$$(\|y-y_s\|+\|x-x_s\|)\leq \frac{2}{1-(KL+K)}\delta=\varepsilon.$$

Since

$$\begin{aligned}\|(x,y)-(x_s,y_s)\|_X &= \|(x-x_s),(y-y_s)\|_X\\ &= \|(x-x_s)\|_C+\|(y-y_s)\|_C\leq\varepsilon,\end{aligned}$$

and

$$\|(x,y)-(x_s,y_s)\|_X<\varepsilon.$$

Then

$$\|x-x_s\|<\varepsilon.$$

□

Example 1. Consider the problem

(9) $$x(t)=\frac{2}{7}\ln(1+t)+\frac{1}{8}y(\gamma_1 y(t)),\ t\in[0,\frac{1}{5}]$$

constrained by

(10) $$y(t)=\frac{1}{10}t+e^{-t}\frac{x(\gamma_2 x(t))}{6},\ t\in[0,\frac{1}{5}].$$

Where $\phi_1(t)=\gamma_1 t,\phi_2(t)=\gamma_2 t$ and $\gamma_1,\ \gamma_2\in(0,1)$. Set

$$f_1(t,y)=\frac{2}{7}\ln(1+t)+\frac{1}{8}y(\gamma_1 y(t)),$$

$$f_2(t,x)=\frac{1}{10}t+e^{-t}\frac{x(\gamma_2 x(t))}{6}.$$

Thus

$$\begin{aligned}|f_1(t,u)-f_1(s,v)| &= |\frac{2}{7}(\ln(1+t)-\ln(1+s))|+\frac{1}{8}|u-v|\\ &\leq \frac{2}{7}|t-s|+\frac{1}{8}|u-v|\\ &\leq \frac{2}{7}(|t-s|+|u-v|),\end{aligned}$$

and

$$\begin{aligned}
|f_2(t,u) - f_2(s,v)| &= |\frac{t}{10} + e^{-t}\frac{u}{6} - \frac{s}{10} - e^{-s}\frac{v}{6}| \\
&\leq \frac{1}{10}|t-s| + \frac{1}{6}|e^{-t}u - e^{-t}v + e^{-t}v - e^{-s}v| \\
&\leq \frac{1}{10}|t-s| + \frac{1}{6}|e^{-t}u - e^{-t}v| + \frac{1}{6}|e^{-t}v - e^{-s}v| \\
&\leq \frac{1}{10}|t-s| + \frac{1}{6}|u-v| + \frac{e^{-s}}{30}|t-s| \\
&\leq \frac{2}{15}|t-s| + \frac{1}{6}|u-v| \\
&\leq \frac{1}{6}(|t-s| + |u-v|).
\end{aligned}$$

Where $K_1 = \frac{2}{7}$, $K_2 = \frac{1}{6}$, $K = \max\{\frac{2}{7}, \frac{1}{6}\} = \frac{2}{7}$. Thus we have $L = \frac{1 \mp \sqrt{1-4K^2}}{2K} = 0.3138 < 1$ and $K + KL = 0.37537 < 1$. It is clear that all assumptions of Theorem (1) are satisfied. Hence there exist unique solution $(x,y) \in X$ of the problem 9-10.

Conflict of Interests

The authors declare that there is no conflict of interests.

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