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FIXED POINTS THEOREMS FOR GENERALIZED $(\alpha-\phi)$-MEIR-KEELER GREGUS QUADRATIC TYPE HYBRID CONTRACTION MAPPINGS VIA SIMULATION FUNCTION IN B-METRIC SPACES

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Abstract: In this paper, we introduce the notion of generalized (α,ϕ)-Meir-Keeler Gregus quadratic Type hybrid contractive mappings of type I and II via simulation function and establish fixed point theorems for such mappings in the setting of complete b-metric spaces. Finally, we provide an example in support of our main finding.

Keywords: fixed points; b-metric spaces; generalized (α,ϕ)-Meir-Keeler Gregus quadratic type hybrid contractive mapping.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory is one of the extremely significant topics in expansion of nonlinear and mathematical analysis in global. Also, fixed point theory has been completely used in many other category of science such as biology, physics, chemistry, economics, computer science, all engineering territory, and so on. In 1922, Banach [2] introduced a well-known fixed point result, now called Banach contraction principle, which is one of the crucial results in nonlinear analysis. Due to its importance and beneficial applications, various authors have procure many interesting extensions and generalizations of the Banach contraction principle in various direction (see, e.g., [5],

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[14]). These generalizations are attained either by using contractive conditions or by levid some additional conditions on the context spaces. For example, one of the significant and precise generalizations is due to Meir and Keeler [9]. The class of Meir-Keeler contractions consists of the class of Banach contractions and many other classes of nonlinear contractions (see, for example, [5], [14]). Meir and Keeler's theorem was the inventor of further investigation in metric fixed point theory. Later on, Meir-Keeler contraction mapping has been generalized by various authors in various ways. On the other hand, the idea of a b-metric space was presented by Bakhtin [1] and Czerwik [4] as a generalization of metric spaces. Since then, several papers have been published on the fixed point theory in such spaces which are interesting extensions and generalizations of the Banach contraction principle. For further works in the setting of b-metric spaces and their generalization. In 2020, Karapinar et al. [6] intensional fixed point results for the Meir-Keeler contraction via simulation function in the setting of metric spaces. Inspired and motivated by the work of Karapinar et al. [7], introduce the notion of generalized (α,ϕ)-Meir-Keeler hybrid contractive mappings of type I and II via simulation function and establish fixed point theorems for the introduced mappings in the setting of b-metric spaces.

Most recently Mustefa and Kidane [10] introduce the notion of generalized (α,ϕ) Meir- Keeler hybrid contractive mappings of type I and II via simulation function and establish fixed point theorems for such mappings in the setting of complete b-metric spaces.

2. Preliminaries

We recall basic definitions and results on the copies which we use in the sequel.

Throughout this paper, we denote $\mathbb{R}^+, \mathbb{R}$ *and* $\mathbb{N}$ respectively by

$\mathbb{R}^+ = [0,\infty)$ - the set of all non-negative real numbers; $\mathbb{R}$ - the set of all real numbers; $\mathbb{N}$- the set of all natural numbers.

Khojasteh et al. [8] introduced the notion of a simulation function as follows.

Definition 2.1. [8] *A weak simulation function is a mapping* $\xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *satisfying the following conditions:*

(ξ_1) $\xi(0,0) = 0$;

(ξ_2) $\xi(t,s) < s-t \ \forall \ t,s > 0$.

Note: Throughout this paper we denote by Z_w the family of all simulation functions $\xi:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$. Due to the axiom (ξ_2), we have $\xi(t,t)<0\,\forall\, t>0$. Recently, Suzuki [12] presented the following class of mappings and proved fixed point result to extend the coverage of Meir-Keeler theorem in the setting of metric spaces. Let (X,d) be a metric space and $T:X\to X$ be a self-mapping. Define a mapping $M:X\times X\to\mathbb{R}^+$ as follows:

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

And let $p:X\times X\to\mathbb{R}^+$ be a mapping satisfies the following conditions:

(1) $(P_p^1:M) x\neq y$ *and* $d(x,Tx)\leq d(x,y)\implies p(x,y)\leq M(x,y)$;

(2) $(P_p^2:c)$ $x_n\neq y, \lim_{n\to\infty} d(x_n,y)=0,$ *and* $\lim_{n\to\infty} d(x_n,Tx_n)=0$

$\implies \limsup_{n\to\infty} d(x_n,y)\leq c\, d(y,Ty),$ *where* $c\in[0,1)$.

Theorem 2.2. [13] *Let T be a self-mapping on a complete metric space (X,d). Let $p:X\times X\to\mathbb{R}^+$ be mapping that satisfies the conditions $(P_p^1:M)$ and $(P_p^2:c)$ defined above.Suppose also that the following are satisfied:*

(1) *For any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $x\neq y$ and $p(x,y)<\varepsilon+\delta(\varepsilon)\implies d(Tx,Ty)\leq\varepsilon$;*

(2) *$x\neq y$ and $p(x,y)>0\implies d(Tx,Ty)<p(x,y)$.*

Then T has a unique fixed point z. Moreover, the sequence $\{T^n x\}$ converges to z for all $x\in X$.

Bakhtin [1] and Czerwik [4] defined a b-metric space as follows.

Definition 2.3. *([1],[4]) Let X be a nonempty set and $s\geq 1$ be a given real number. A function $d:X\times X\to\mathbb{R}^+$ is said to be a b-metric if and only if $\forall x,y,z\in X$, the following conditions are satisfied:*

(1) $d(x,y)=0$ *if and only if* $x=y$;

(2) $d(x,y)=d(y,x)$;

(3) $d(x,z)\leq s[d(x,y)+d(y,z)]$.

The pair (X,d) is called a b-metric space.

Definition 2.4. [7] *Let X be a b-metric space and $\{x_n\}$ a sequence in X. We say that*

(1) *$\{x_n\}$ is b-convergent to $x\in X$ if $d(x_n,x)\to 0$ as $n\to\infty$.*

(2) $\{x_n\}$ *is a b-Cauchy sequence if* $d(x_n,x_m)\to 0$ *as* $n,m\to\infty$.

(3) (X,d) *is b-complete if every b-Cauchy sequence in* X *is b-convergent.*

Definition 2.5. [3] *Let* (X,d) *be a b-metric space with the coefficient* $s\geq 1$ *and let* $T:X\to X$ *be a given mapping. We say that* T *is b-continuous at* $x_0\in X$ *if and only if for every sequence* $x_n\in X$ *such that* $x_n\to x_0$ *as* $n\to\infty$, *we have* $Tx_n\to Tx_0$ *as* $n\to\infty$. *If* T *is b-continuous at each point* $x\in X$,*then we say that* T *is b-continuous on* X.

Definition 2.6. [11] *Let* X *be a nonempty set and* $\alpha:X\times X\to\mathbb{R}^+$ *a function. A mapping* $T:X\to X$ *is said to be* $\alpha-$*orbital admissible if,* $\forall x\in X,\alpha(x,T_x)\geq 1\implies\alpha(Tx,T^2x)\geq 1$.

Definition 2.7. [11] *Let* X *be a nonempty set,* $T:X\to X$, *and* $\alpha:X\times X\to\mathbb{R}^+$.*We say that* T *is triangular* $\alpha-$*orbital admissible if:*

(1) T *is* α*-orbital admissible;*

(2) $\forall x,y\in X,\ \alpha(x,y)\geq 1$ *and* $\alpha(y,Ty)\geq 1\implies\alpha(x,Ty)\geq 1$

In 2020, Karapinar et al. [7] introduced the following class of hybrid contraction mappings of type I and II and studied fixed point results for such mappings.

Definition 2.8. [7] *Let* T *be a self-mapping on a metric space* (X,d) *and* $\xi\in Z_{w.}$ *Suppose that* $p:X\times X\to\mathbb{R}^+$ *is a function that satisfies only* $(P^1p:M)$. *Then* T *is called a hybrid contraction of type I if the following conditions are fulfilled:*

(1)*For any* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that* $x\neq y$ *and* $p(x,y)<\varepsilon+\delta(\varepsilon)\implies d(Tx,Ty)\leq\varepsilon$*;*

(2) $x\neq y$ *and* $p(x,y)>0\implies\xi(\alpha(x,y)d(Tx,Ty),p(x,y))\geq 0$.

Let a mapping $N:X\times X\to\mathbb{R}^+$*be defined as follows:*

$$N(x,y)=\max\left\{d(y,Ty),\frac{1+d(x,Tx)}{1+d(x,y)}\right\},$$

where T *is a self-mapping defined on a metric space* (X,d).*We notice that, for any* $x,y\in X$ *with* $x=y$, *we have* $0=d(Tx,Ty)\leq N(x,y)$. *Moreover, if* $x\neq y$, *then* $N(x,y)>0$.

Definition 2.9. [10] *Let* T *be a self-mapping on a metric space* (X,d) *and* $\xi\in Z_{w.}$ *Suppose that* $p:X\times X\to\mathbb{R}^+$ *is a function that satisfies* $(P^1p:N)$ *and* $(P^2p:c)$ $\forall c\in[0,1)$. *Then* T *is called a hybrid contraction of type II if the following conditions are satisfied:*

(1) *For any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $x \neq y$ *and* $p(x,y) < \varepsilon + \delta(\varepsilon) \implies d(Tx,Ty) \leq \varepsilon$;

(2) $x \neq y$ *and* $p(x,y) > 0 \implies \xi(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0$.

3. MAIN RESULT

In this section, first we coin generalized $(\alpha,\phi)-$ Meir-Keeler Gregus quadratic type (MKGq) hybrid contractive mapping of type I in the setting of b-metric spaces and prove fixed point results for such mappings.

In this section, we denote the class of mappings ψ by $\psi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ : \phi$ is continuous, monotone non-decreasing, $\phi(t) = 0 \ iff \ t = 0\}$.

Let (X,d) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a self-mapping.We define a mapping $M_s : X \times X \to \mathbb{R}^+$ by

$$M_s(x,y) = a\, d^2(x,y) + (1-a)\max\left\{d^2(x,y), d(x,Tx).d(y,Ty), \frac{d^2(x,Ty)+d^2(y,Tx)}{2s}\right\}.$$

And let $p : X \times X \to \mathbb{R}^+$ be a mapping satisfies the following conditions:

(1) $(P_p^1 : M_s)$ $x \neq y$ *and* $d(x,Tx) \leq d(x,y) \implies p(x,y) \leq M_s(x,y)$;

(2) $(P_p^2 : sc)$ $x_n \neq y, \lim_{n\to\infty} d(x_n,y) = 0,$ *and* $\lim_{n\to\infty} d(x_n,Tx_n) = 0$
$\implies \limsup_{n\to\infty} d(x_n,y) \leq c\, d(y,Ty)$, *where* $c \in [0,1)$.

Definition 3.1. *Let* (X,d) *be a b-metric space with* $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ $p : X \times X \to \mathbb{R}^+$ *satisfy* $(P^1 p : M_s)$ *and* $\phi \in \psi$. *Then the mapping* T *is said to be a generalized* (α,ϕ)*- Meir-Keeler Gregus quadratic type hybrid contractive mapping of type I if it satisfies, for all* $x,y \in X$, *the following conditions:*

(1) *For any* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $x \neq y$ *and* $p(x,y) < \varepsilon + \delta(\varepsilon) \implies d(Tx,Ty) \leq \frac{\varepsilon}{s}$;

(2) $x \neq y$ *and* $p(x,y) > 0 \implies \xi(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0$.

Remark 3.2. *If* T *is a generalized* (α,ϕ) *MKGq type hybrid contractive mapping of type I, then*

$$\alpha(x,y)\phi(d(Tx,Ty)) < \phi(p(x,y)) \leq \phi(M_s(x,y)). \tag{3.1}$$

Indeed, we have $d(x,y) > 0$ *since* $x \neq y$. *If* $p(x,y) = 0$, *from (ii), we have* $\phi(d(Tx,Ty)) < \varepsilon$ *for any* $\varepsilon > 0$. *But* $\varepsilon > 0$ *is arbitrary, thus we obtain* $Tx = Ty$. *In this case,* $\alpha(x,y)\phi(d(Tx,Ty)) = 0 \leq \phi(p(x,y))$. *Otherwise,* $p(x,y) > 0$, *and if* $Tx \neq Ty$, *then* $d(Tx,Ty) > 0$. *If* $\alpha(x,y) = 0$, *then* (3.1) *is* *satisfied. On the other hand, from* (ξ_2) *and we get*

$0 \leq \xi(\alpha(x,y)\phi(d(Tx,Ty)), \phi\ (p(x,y))) < \phi\ (p(x,y)) - \alpha(x,y)\phi(d(Tx,Ty))$, *so* (3.1) *holds.*

Now, we present our first main result as follows:

Theorem 3.3. *Let* (X,d) *be a complete b-metric space with* $s \geq 1$, $T : X \to X$, $\alpha : X \times X \to \mathbb{R}^+$ *be mappings, and* $\phi \in \psi$. *Suppose the following conditions hold:*

(1) T *is generalized* (α,ϕ) *MKGq type hybrid contractive mapping of type I;*

(2) T *is a triangular* α*-orbital admissible mapping;*

(3) *There exists* $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$;

(4) T *is b-continuous. Then* T *has a fixed point* z. *Moreover,* $\{T^n x\}$ *converges to* z *for all* $x \in X$.

Proof. By 3 above, there survive $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$..We invent an iterative sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Assume that $x_{n_0} = x_{n_{0+1}}$for some $n_0 \in \mathbb{N}$. Since $Tx_{n_0} = xn_{0+1} = x_{n_0}$, the point x_{n_0} is a fixed point of T and this completes the proof. So from now on, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is triangular α- orbital admissible, $\alpha(x_0,Tx_0) = \alpha(x_0,x_1) \geq 1 \implies \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1 \implies \alpha(Tx_1,Tx_2) = \alpha(x_2,x_{3)} \geq 1$. Continuing in this manner, we get

$$\alpha(x_n,x_{n+1}) \geq 1\ \forall n \geq 0. \tag{3.2}$$

Again, by using the supposition that T is triangular $\alpha-$orbital admissible, for all $n \in N \cup \{0\}$, (3.2) yields that $\alpha(x_n,x_{n+1}) \geq 1$ and $\alpha(x_{n+1},x_{n+2}) \geq 1 \implies \alpha(x_n,x_{n+1}) \geq 1$. Recursively, we conclude that $\alpha(x_n,x_{n+j}) \geq 1$ for all $n, j \in \mathbb{N}$. Now we prove that the sequence $d(x_n,x_{n+1})$ is monotone decreasing. Taking $x = x_n$ and $y = x_{n+1}$ in $(P^1 p : M_s)$, we get $0 < d(x_n,x_{n+1}) = d(x_n,Tx_n) \leq d(x_n,x_{n+1}) \implies p(x_n,x_{n+1}) \leq M_s(x_n,x_{n+1})$, where

$$\begin{aligned} M_s(x_n,x_{n+1}) = ad^2(x_n,x_{n+1}) + (1-a)\max &\left\{ \begin{matrix} d^2(x_n,x_{n+1}), d(x_n,Tx_n).d(x_{n+1},Tx_{n+1}), \\ \frac{d^2(x_n,Tx_{n+1})+d^2(x_{n+1},Tx_n)}{2s} \end{matrix} \right\} \\ &= \max\left\{ d^2(x_n,x_{n+1}), \frac{d^2(x_n,x_{n+2}) + d^2(x_{n+1},x_{n+2})}{2s} \right\} \\ &= \max\left\{ d^2(x_n,x_{n+1}), \frac{d^2(x_n,x_{n+2})}{2s} \right\}, \end{aligned}$$

and, taking the b-triangle inequality we conclude that

$$\begin{aligned}\frac{d^2(x_n,x_{n+2})}{2s} &\leq \frac{sd^2(x_n,x_{n+1})+sd^2(x_{n+1},x_{n+2})}{2s}\\ &= \frac{d^2(x_n,x_{n+1})+d^2(x_{n+1},x_{n+2})}{2}\\ &\leq \max\left\{d^2(x_n,x_{n+1}),d^2(x_{n+1},x_{n+2})\right\},\end{aligned}$$

which gives

$$M_s(x_n,x_{n+1}) = \max\left\{d^2(x_n,x_{n+1}),d^2(x_{n+1},x_{n+2})\right\}.$$

By definition 2.9 (2) we conclude that

$$\begin{aligned}0 &\leq \xi(\alpha(x_n,x_{n+1})\phi(d^2(Tx_n,Tx_{n+1})),\phi\ (p(x_n,x_{n+1})))\\ &< \phi\ (p(x_n,x_{n+1})-\alpha(x_n,x_{n+1})\phi(d^2(Tx_n,Tx_{n+1})),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\phi(d^2(x_n,x_{n+1})) &= \phi(d^2(Tx_n,Tx_{n+1}))\\ &\leq \alpha(x_n,x_{n+1})\phi(d^2(Tx_n,Tx_{n+1}))\\ &< \phi(p(x_n,x_{n+1})\\ &\leq \phi(M_s(x_n,x_{n+1})).\end{aligned} \tag{3.3}$$

If $M_s(x_n,x_{n+1}) = d^2(x_{n+1},x_{n+2})$ then (3.3) surrends a contradiction. Thus, we have

$$M_s(x_n,x_{n+1}) = d^2(x_{n+1},x_{n+2}), \tag{3.4}$$

Moreover, from (3.3), we get $\phi(d^2(x_{n+1},x_{n+2})) < \phi(d^2(x_n,x_{n+1}))$, which implies, applying the monotonicity of ϕ,

$$d^2(x_{n+1},x_{n+2}) < d^2(x_n,x_{n+1})\ \forall\ n \in \mathbb{N}\cup\{0\},$$

that is, $\{d^2(x_n,x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Thus, there is some $l \geq 0$ such that $\lim_{n\to\infty} d^2(x_n,x_{n+1}) = l$. We need to show $l = 0$. Suppose, on the contrary, that $l > 0$ and set $0 < \varepsilon = l$. We also record that

$$\varepsilon = l < d^2(x_n,x_{n+1}) \forall n \in \mathbb{N}\cup\{0\}. \tag{3.5}$$

Also, from (3.3) and (3.4), we have $p(x_n,x_{n+1}) \leq d^2(x_n,x_{n+1}) < \varepsilon + \delta(\varepsilon)$ for n sufficiently large. So, applying definition 2.9(1) we have

$$d^2(Tx_n,Tx_{n+1}) \leq \frac{\varepsilon}{s} \tag{3.6}$$

Merging (3.5) together with (3.6), we obtain

$$\varepsilon < d^2(x_{n+1},x_{n+2}) = d^2(Tx_n,Tx_{n+1}) \leq \frac{\varepsilon}{s},$$

which is a contradiction.We conclude that $\varepsilon = 0$, that is,

$$\lim_{n\to\infty} d^2(x_n,x_{n+1}) = 0. \tag{3.7}$$

Now, we appear that $\{x_n\}$is a b-Cauchy sequence. Let $\varepsilon_1 > 0$ be fixed. From (3.7), we can select $k \in \mathbb{N}$ large enough such that

$$d^2(x_k,x_{k+1}) < \frac{\delta_1}{2s}, \tag{3.8}$$

for some $\delta_1 > 0$. Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon_1) < (\varepsilon_1)$. By induction, we prove that

$$d^2(x_k,x_{k+m}) < \varepsilon_1 + \frac{\delta_1}{2} \ \forall\, k,m \in \mathbb{N} \cup \{0\}. \tag{3.9}$$

Earlier we get (3.9) from (3.8), for $m = 1$. Assume that (3.9) is satisfied for some $m = j$. Now, we show that (3.9) holds for $m = j+1$. On account of (3.8) and (3.9), we first conclude that

$$\begin{aligned}
\frac{d^2(x_k,x_{k+j+1}) + d^2(x_{k+j},x_{k+1})}{2s} &\leq \frac{sd^2(x_k,x_{k+j}) + sd^2(x_{k+j},x_{k+j+1}) + sd^2(x_{k+j},x_k) + sd^2(x_k,x_{k+1})}{2s} \\
&= \frac{d^2(x_k,x_{k+j}) + d^2(x_{k+j},x_{k+j+1}) + d^2(x_{k+j},x_k) + d^2(x_k,x_{k+1})}{2} \\
&< \frac{1}{2}[2\varepsilon_1 + \delta_1 + \frac{\delta_1}{\varepsilon_1}] \\
&\leq \frac{1}{2}[2\varepsilon_1 + 2\delta_1] \\
&= \varepsilon_1 + \delta_1.
\end{aligned}$$

Thus, we have

$$\begin{aligned} M_s(x_k,x_{k+j}) &= ad^2(x_k,x_{k+j} + (1-a)\max\left\{\begin{array}{c} d^2(x_k,x_{k+j}), d(x_k,Tx_k).d(x_{k+j},Tx_{k+j}), \\ \frac{d^2(x_k,Tx_{k+j})+d^2(x_{k+j},Tx_k)}{2s} \end{array}\right\} \\ &= ad^2(x_k,x_{k+j}) + (1-a)\max\left\{\begin{array}{c} d^2(x_k,x_{k+j}), d(x_k,x_{k+1}).d(x_{k+j},x_{k+j+1}), \\ \frac{d^2(x_k,x_{k+j+1})+d^2(x_{k+j},x_{k+1})}{2s} \end{array}\right\} \\ &< \max\left\{\varepsilon_1+\frac{\delta_1}{2},\frac{\delta_1}{2s},\varepsilon_1+\delta_1\right\} \\ &= \varepsilon_1+\delta_1. \end{aligned}$$

From the above inequality, we have $p(x_k,x_{k+j}) \leq M_s(x_k,x_{k+j}) = d^2(x_k,x_{k+j}) < \varepsilon_1+\delta_1$, by Definition 2.9(1) we conclude that

$$d^2(x_k,x_{k+j+1}) = d^2(Tx_k,Tx_{k+j}) \leq \frac{\varepsilon_1}{s} \tag{3.10}$$

Now, applying the b-triangle inequality, as well as (3.8) and (3.10), we have

$$\begin{aligned} d^2(x_k,x_{k+j+1}) &\leq sd^2(x_k,x_{k+1}) + d^2(x_{k+1},x_{k+j+1}) \\ &= sd^2(x_k,x_{k+1}) + sd^2(Tx_k,Tx_{k+j}) \\ &< \frac{\delta_1}{2}+\varepsilon_1. \end{aligned}$$

So, (3.9) holds for $m = j+1$. Therefore, $d^2(x_k,x_{k+m}) < \varepsilon_1 \ \forall\, k,m \in \mathbb{N}\cup\{0\}$.

Additionally, for $m > n$, we have $\lim_{n,m\to\infty} d(x_n,x_m) = 0$ and hence the sequence $\{x_n\}$ is a b-Cauchy sequence. Since, (X,d) is a complete b-metric space, there continue $u \in X$ such that $x_n \to u$ as $n \to \infty$. By b-continuity of T, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu,$$

that is, u is a fixed point of T. □

Now, changing continuity of T by continuity of T^2 in Theorem 3.3, we prove the following fixed point result.

Theorem 3.4. *Let (X,d) be a complete b-metric space with $s \geq 1$ and let $T : X \to X$ be a generalized $(\alpha,\phi)-$ MKGq type hybrid contractive mapping of type I satisfying the following conditions:*

(1) *T is a triangular α-orbital admissible mapping;*

(2) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(3) *T^2 is b-continuous.*

Then $\{T^n x\}$is converges to z for all $x \in X$. Moreover, for $\alpha(z, Tz) \geq 1, z$ is a fixed point of T, and T is discontinuous at z if and only if $\lim_{x \to z} M_s(x, z) \neq 0$.

Proof. Coming the proof of Theorem 3.3, we discern that the sequence $\{x_n\} \in X$ defined by $x_n = Tx_{n-1} \forall n \in \mathbb{N}$ is convergent to $z \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Regarding the fact that any sub-sequence of $\{x_n\}$ converges to z, we get $\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} Tx_n = z$ and $\lim_{n \to \infty} x_{n+2} = \lim_{n \to \infty} T^2 x_n = z$. Also, due to the continuity of T^2, $T^2 z = \lim_{n \to \infty} T^2 x_n = z$. We claim that $Tz = z$. Suppose, on the contrary, that $Tz \neq z$ and $p(z, Tz) > 0$. Then we have

$$\begin{aligned} p(z,Tz) &\leq M_s(z,Tz) \\ &= a\, d^2(z,Tz) + (1-a) \max\left\{ d^2(z,Tz), d(z,Tz).d(Tz,T^2 z), \frac{d^2(z,T^2 z) + d^2(Tz,Tz)}{2s} \right\} \\ &= d^2(z,Tz). \end{aligned}$$

Thus, using (3.1) together with the hypothesis $\alpha(z, Tz) \geq 1$, we obtain

$$0 \leq \xi(\alpha(z,Tz)\phi(d^2(Tz,T^2 z)), \phi\ (p(z,Tz))),$$

and also

$$\begin{aligned} 0 < \phi(d^2(Tz,z)) &= \phi(d^2(Tz,T^2 z)) \\ &\leq \alpha(z,Tz)\phi(d^2(Tz,T^2 z)) \\ &< \phi(P(z,Tz)) \\ &\leq \phi(M_s(z,Tz)) \\ &= \phi(d^2(z,Tz)), \end{aligned}$$

which is a contradiction. So, $z = Tz$, that is, z is a fixed point of T. □

Definition 3.5. *A b-metric space (X, d) is called regular if for any sequence $\{x_n\}$ in X with*

$$\lim_{n \to \infty} d^2(x_n, z) = 0$$

and $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}\cup\{0\}$*, one has* $\alpha(x_n,z) \geq 1$ *for all* $n \in \mathbb{N}\cup\{0\}$*. In the following, we prove fixed point theorem, without continuity assumption of* T *and* T^2*.*

Theorem 3.6. *Assume* (X,d) *be a complete b-metric space with* $s \geq 1$ *and* $T: X \to X$ *be a generalized* (α,ϕ)*- MKGq type hybrid contractive mapping of type I. Suppose that* $(P^2p : sc)$ *and the following conditions hold:*

(1) T *is a triangular* α*-orbital admissible mapping;*

(2) *There exists* $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$;

(3) (X,d) *is regular.*

(4) *Then* $\{T^n x\}$ *is converges to* z *for all* $x \in X$*. Moreover,* z *is a fixed point of* T*.*

Proof. Following the proof of Theorem 3.3, we see that the sequence $\{x_n\} \in X$ defined by $x_n = Tx_{n-1} \forall\, n \in \mathbb{N}$ is convergent to $z \in X$ and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. We notice also that all adjacent terms in $\{x_n\}$ are distinct. Moreover, we note $T^n x \neq z$ for all $n \in \mathbb{N}\cup\{0\}$. Respecting the limits $\lim_{n\to\infty} d^2(x_n,z) = 0$ and $\lim_{n\to\infty} d^2(x_n,x_{n+1}) = 0$, we operate from $(P^2_p : sc)$ that

$$s \lim_{n\to\infty} \sup p(x_n,z) \leq cd^2(z,Tz) \; for\ any\ c \in [0,1). \tag{3.11}$$

So, by supposition (3), we get $\alpha(x_n,z) \geq 1$. Now, we prove that z is a fixed point of T. Assume, on the contrary, that $Tz \neq z$. Taking $x = x_n$ and $y = z$ in Definition 2.9, (2) we get

$$\begin{aligned} 0 &\leq \xi(\alpha(x_n,z)\phi(d^2(Tx_n,Tz),\phi\ (p(x_n,z))) \\ &< \phi\ (p(x_n,z))) - \alpha(x_n,z))\phi(d^2(Tx_n,Tz)), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \phi(d^2(x_{n+1},Tz)) &= \phi(d^2(Tx_n,Tz)) \\ &\leq \alpha(x_n,z)\phi(d^2(Tx_n,Tz)) \\ &< \phi(p(x_n,z)). \end{aligned} \tag{3.12}$$

Since ϕ is monotone, (3.12) gives

$$d^2(x_{n+1},Tz) < p(x_n,z). \tag{3.13}$$

Using the b-triangle inequality and using (3.13), we have

$$\begin{aligned} d^2(z,Tz) &\leq sd^2(z,x_{n+1}) + sd^2(x_{n+1},Tz) \\ &< sd^2(z,x_{n+1}) + sp(x_n,z). \end{aligned} \tag{3.14}$$

Taking the limit as $n \to \infty$ in (3.14) and using $(P_p^2 : sc)$, we conclude that

$$\begin{aligned} d^2(z,Tz) &< s \lim_{n\to\infty} \sup p(x_n,z) \\ &\leq cd^2(z,Tz) \ for \ any \ c \in [0,1), \end{aligned}$$

which is a contradiction.Therefore, z is a fixed point of T.

□

Condition(U) For all $x,y \in T$, we have $\alpha(x,y) \geq 1$, where Fix (T) denotes the set of all fixed points of T.

Theorem 3.7. *Adding Condition* (U) *to the hypotheses of Theorem* 3.3 *(resp. Theorems* 3.4 *and* 3.6*), we prove the uniqueness of fixed point of* T.

Proof. We discuss by contradiction, that is, suppose there exist $z,w \in X$ such that $z = Tz$ and $w = Tw$ with $z \neq w$. By Condition (U), we have $\alpha(z,w) \geq 1$. We observe first that the case $p(z,w) = 0$ is impossible since we have $Tz = Tw$ and $0 < d^2(z,w) = d^2(Tz,Tw) = 0$, which is a contradiction. Thus, we conclude that $p(z,w) > 0$. Since $0 = d^2(z,Tz) \leq d^2(z,w)$, by $(P_p^1 : M_s)$, we have $p(z,w) \leq M_s(z,w)$, where

$$\begin{aligned} M_s(z,w) &= a\, d^2(z,w) + (1-a)\max\left\{d^2(z,w), d(z,w).d(w,Tw), \frac{d^2(z,Tw)+d^2(Tz,w)}{2s}\right\} \\ &= d^2(z,w). \end{aligned}$$

Using Definition 2.9(2) we conclude that

$$\begin{aligned} 0 &\leq \xi(\alpha(z,w)\phi(d^2(Tz,Tw)), \phi\ (p(z,w))) \\ &< \phi\ (p(z,w)) - \alpha(z,w))\phi(d^2(Tz,Tw)), \end{aligned}$$

which imply

$$\begin{aligned} 0 < \phi(d^2(z,w)) &= \phi(d^2(Tz,Tw)) \\ &\leq \alpha(z,w)\phi(d^2(Tz,Tw)) \\ &< \phi(P(z,w)) \\ &= \phi(d^2(z,w)), \end{aligned}$$

which is a contradiction. Hence, $d^2(z,w) = 0$, that is, the fixed point of T is unique. □

Further we introduce generalized (α,ϕ)- MKGq type hybrid contractive mapping of type II and study fixed point results for such mappings.

Definition 3.8. *Suppose (X,d) be a b-metric space with $s \geq 1$, $T: X \to X$, $\alpha: X \times X \to \mathbb{R}^+$, $\xi \in Z_w, \phi \in \psi$, and suppose $p: X \times X \to \mathbb{R}^+$ is a function that satisfies $(P_p^1 : N_s)$ and $(P_p^2 : sc)$. The mapping T is said to be a generalized (α,ϕ)-MKGq type hybrid contractive mapping of type II if it satisfies for all $x,y \in X$ the following conditions:*
(1) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and $p(x,y) < \varepsilon + \delta(\varepsilon) \implies d(Tx,Ty) \leq \frac{\varepsilon}{s}$;
(2) $x \neq y$ and $p(x,y) > 0$ imply

$$\xi(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0. \tag{3.15}$$

We define a mapping $N_s : X \times X \to \mathbb{R}^+$ by

$$N_s(x,y) = ad^2(x,y) + (1-a)\max\left\{\begin{array}{c} d^2(x,y), d(x,Tx).d(y,Ty), \\ \frac{d^2(y,Ty)[1+d^2(x,Tx)]}{1+d^2(x,y)}, \\ \frac{d^2(x,Tx)[1+d^2(y,Ty)]}{1+d^2(Tx,Ty)} \end{array}\right\}.$$

We note that, for any $x,y \in X$ with $x = y$, we have $0 = d^2(Tx,Ty) \leq N_s(x,y)$. Moreover, if $x \neq y$, then $N_s(x,y) > 0$.

Now, we demonstrate the following fixed point theorem.

Theorem 3.9. *Let (X,d) be a complete b-metric space with $s \geq 1$ and let $T: X \to X$ be a generalized $(\alpha,\phi)-$ MKGq type hybrid contractive mapping of type II satisfying the following conditions:*
(1) T is a triangular α-orbital admissible mapping;

(2)*There exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(3) *either* T *is continuous;*

(4) *or* T^2*is continuous and* $\alpha(z, Tz) \geq 1$;

(5) *or* (X, d) *is a regular.*

Then T *has a fixed point of* z*, Moreover,* $\{T^n x\}$ *is convergent to* z *for all* $x \in X$.

Proof. Similar to the proof of Theorem 3.3, we construct a recursive sequence $\{x_n\}$ as follows: $x_n = Tx_{n-1}\ \forall\ n \in \mathbb{N}$. One can conclude that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, due to conditions (1) and (2). Throughout the proof, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Indeed, as it was discussed in the proof of Theorem 3.3, the other case is superficial and is omited. Now, by assuming $x = x_n$ and $y = x_{n+1}$ in $(P^1 p : N_s)$, we have

$$d^2(x_n, Tx_n) \leq d^2(x_n, x_{n+1}),$$

which implies

$$p(x_n, x_{n+1}) \leq N_s(x_n, x_{n+1}),$$

where

$$\begin{aligned} N_s(x_n, x_{n+1}) &= ad^2(x_n, x_{n+1}) + (1-a)\max\left\{\begin{array}{c} d^2(x_n, x_{n+1}), d(x_n, Tx_n).d(x_{n+1}, Tx_{n+1}), \\ \frac{d^2(x_{n+1}, Tx_{n+1})[1+d^2(x_n, Tx_n)]}{1+d^2(x_n, x_{n+1})}, \\ \frac{d^2(x_n, Tx_n)[1+d^2(x_{n+1}, Tx_{n+1})]}{[1+d^2(Tx_n, Tx_{n+1})]} \end{array}\right\} \\ &= ad^2(x_n, x_{n+1}) + (1-a)\max\left\{\begin{array}{c} d^2(x_n, x_{n+1}), d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2}), \\ \frac{d^2(x_{n+1}, x_{n+2})[1+d^2(x_n, x_{n+1})]}{1+d^2(x_n, x_{n+1})}, \\ \frac{d^2(x_n, x_{n+1})[1+d^2(x_{n+1}, x_{n+2})]}{[1+d^2(x_{n+1}, x_{n+2})]} \end{array}\right\} \\ &= ad^2(x_n, x_{n+1}) + (1-a)\max\left\{d^2(x_n, x_{n+1}), d^2(x_{n+1}, x_{n+2})\right\}. \end{aligned}$$

By definition 2.9(2), we have

$$0 \leq \xi(\alpha(x_n, x_{n+1})\phi(d^2(Tx_n, Tx_{n+1})), \phi\ (p(x_n, x_{n+1}))).$$

Consequently, the above inequality gives

$$\phi(d^2(x_{n+1}, x_{n+2})) = \phi(d^2(Tx_n, Tx_{n+1}))$$

$$
\begin{aligned}
&\leq \alpha(x_n,x_{n+1})\phi(d^2(Tx_n,Tx_{n+1})) \\
&< \phi(P(x_n,x_{n+1})) \\
&= \phi(N_s(x_n,x_{n+1})),
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
N_s(x_n,x_{n+1}) &= a\, d^2(x_n,x_{n+1}) + (1-a)\max\left\{d^2(x_n,x_{n+1}), d^2(x_{n+1},x_{n+2}\right\} \\
&= d^2(x_n,x_{n+1}).
\end{aligned}
\tag{3.17}
$$

Thus, from (3.16), (3.17) and the mono-tonocity of ϕ, $\forall\, n\in\mathbb{N}\cup\{0\}$, we have

$$d^2(x_{n+1},x_{n+2}) < d^2(x_n,x_{n+1}),$$

that is, $\{d^2(x_n,x_{n+1})\}$ is non- increasing sequence of non-negative real numbers. Consequently, there exists a real number $r\geq 0$ such that

$$d^2(x_n,x_{n+1}) \to r \ as\ n\to\infty.$$

Assume that $r=\varepsilon>0$. First, we note that $r=\varepsilon<d^2(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$. Also from (3.16), there exists $\delta>0$ such that

$$
\begin{aligned}
p(x_n,x_{n+1}) &\leq N_s(x_n,x_{n+1}) \\
&= d^2(x_n,x_{n+1}) \\
&< \varepsilon+\delta(\varepsilon),
\end{aligned}
$$

for n sufficiently large. Keeping the observations above, definition 2.9(1) gives that

$$d^2(Tx_n,Tx_{n+1}) \leq \frac{\varepsilon}{s}.$$

Thus, we have

$$\varepsilon < d^2(x_{n+1},x_{n+2}) = d^2(Tx_n,Tx_{n+1}) \leq \frac{\varepsilon}{s},$$

which is a contradiction. So, we derive that $\varepsilon=0$, that is, $\lim_{n\to\infty} d^2(x_n,x_{n+1})=0$. Now, we show that the sequence $\{x_n\}$ is b-Cauchy. For this direct, let $m\in\mathbb{N}$ be large enough to satisfy

$$d^2(x_m,x_{m+1}) < \frac{\delta_1}{s}.$$

Now, we display by induction that

$$d^2(x_m, x_{m+1}) < \varepsilon_1 + \delta_1 \ \forall\, k \in \mathbb{N}. \tag{3.18}$$

Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon) < \varepsilon$. We have already established the claim for $k = 1$. Now, we consider the following two cases:

Case(I). If $d^2(x_{m+k}, x_{m+k+1}) \leq d^2(x_m, x_{m+k})$, then we get

$$\frac{d^2(x_{m+k}, x_{m+k+1})}{1 + d^2(x_m, x_{m+k})} \leq d^2(x_{m+k}, x_{m+k+1})$$

and

$$\frac{d^2(x_{m+k}, x_{m+k+1}).d^2(x_m, x_{m+1})}{1 + d^2(x_m, x_{m+k})} < d^2(x_m, x_{m+1})$$

Hence, we have

$$\begin{aligned}
p(x_m, x_{m+1}) &\leq N_s(x_m, x_{m+k}) \\
&= a\, d^2(x_m, x_{m+k}) + (1-a) \max \left\{ \begin{array}{c} d^2(x_m, x_{m+k}), d(x_m, Tx_m).d(x_{m+k}, Tx_{m+k}), \\ \frac{d^2(x_{m+k}, Tx_{m+k})[1+d^2(x_m, Tx_m)]}{1+d^2(x_m, x_{m+k})}, \\ \frac{d^2(x_{m+k}, Tx_{m+k})[1+d^2(x_m, Tx_m)]}{1+d^2(Tx_m, Tx_{m+k})} \end{array} \right\} \\
&= a\, d^2(x_m, x_{m+k}) + (1-a) \max \left\{ \begin{array}{c} d^2(x_m, x_{m+k}), d(x_m, x_{m+1}).d(x_{m+k}, x_{m+k+1}), \\ \frac{d^2(x_{m+k}, x_{m+k+1})[1+d^2(x_m, x_{m+1})]}{1+d^2(x_m, x_{m+k})}, \\ \frac{d^2(x_{m+k}, x_{m+k+1})[1+d^2(x_m, x_{m+1})]}{1+d^2(x_{m+1}, x_{m+k+1})} \end{array} \right\} \\
&< \max\{\varepsilon_1 + \delta_1, 2\delta_1, \delta_1\} \\
&= \varepsilon_1 + \delta_1,
\end{aligned}$$

and so it follows from Definition 2.9(1) that

$$d^2(Tx_m, Tx_{m+k}) \leq \frac{\varepsilon_1}{s}.$$

Thus, by the b-triangle inequality, we have

$$\begin{aligned}
d^2(x_m, x_{m+k+1}) &\leq sd^2(x_m, x_{m+1}) + sd^2(x_{m+1}, x_{m+k+1}) \\
&= sd^2(x_m, x_{m+1}) + sd^2(Tx_m, Tx_{m+k}) \\
&< \varepsilon_1 + \delta_1.
\end{aligned}$$

Case(II.) If $d^2(x_{m+k},x_{m+k+1}) > d^2(x_m,x_{m+k})$ then we get

$$\begin{aligned} d^2(x_{m+k},x_{m+k+1}) &\leq sd^2(x_m,x_{m+k}) + sd^2(x_{m+k},x_{m+k+1}) \\ &< 2sd^2(x_{m+k},x_{m+k+1}) \\ &< 2s\frac{\delta_1}{s} \\ &= 2\delta_1 \\ &< \varepsilon_1 + \delta_1. \end{aligned}$$

Thus, by induction, (3.18) holds for every $k \in \mathbb{N}$. Since $\varepsilon_1 > 0$ is arbitrary, we get

$$\lim_{k\to\infty} \sup d^2(x_m,x_{m+k}) = 0,$$

which implies that $\{x_n\}$ is a b-Cauchy sequence in a complete b-metric space (X,d). Hence, $\{x_n\}$ b-converges to some $z \in X$. Next, we show that z is a fixed point of T. If T is continuous, then we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz \text{ and } \lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2x_n = z,$$

particularly z is a fixed point of T. If T^2 is continuous, since $x_n \to z$, we get that any subsequence of $\{x_n\}$ converges to the same limit point z, so

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z \text{ and } \lim_{n\to\infty} x_{n+2} = \lim_{n\to\infty} T^2x_n = z.$$

On the other hand, due to the continuity of T^2,

$$T^2z = \lim_{n\to\infty} T^2x_n = z.$$

We claim that $Tz = z$. Nonetheless, if $Tz \neq z$, then we have $p(z,Tz) > 0$ and

$$\begin{aligned} p(z,Tz) &\leq N_s(z,Tz) \\ &= a\, d^2(z,Tz) + (1-a)\max\left\{\begin{array}{c} d^2(z,Tz), d(z,Tz).d(Tz,T^2z), \\ \frac{d^2(Tz,T^z)[1+d^2(z,Tz)]}{1+d^2(z,Tz)}, \\ \frac{d^2(z,Tz)[1+d^2(Tz,T^2z)]}{1+d^2(Tz,T^2z)} \end{array}\right\} \end{aligned}$$

$$= a\, d^2(z,Tz) + (1-a)\max\left\{\begin{array}{c} d^2(z,Tz), d(z,Tz).d(Tz,T^2z), \\ \frac{d^2(Tz,T^2z)[1+d^2(z,Tz)]}{1+d^2(z,Tz)}, \\ \frac{d^2(z,Tz)[1+d^2(Tz,T^2z)]}{1+d^2(Tz,T^2z)} \end{array}\right\}$$

$$= d^2(z,Tz).$$

Therefore, together with the auxiliary hypothesis $\alpha(z,Tz) \geq 1$, we have

$$\begin{aligned} 0 &\leq \xi(\alpha(z,Tz)\phi(d^2(Tz,T^2z)), \phi\ (p(z,Tz))) \\ &< \phi\ (p(z,Tz)) - \alpha(z,Tz))\phi(d^2(Tz,T^2z)). \end{aligned}$$

From the above inequality, we obtain

$$\begin{aligned} \phi(d^2(Tz,z)) &= \phi(d^2(Tz,T^2z)) \\ &\leq \alpha(z,Tz)\phi(d^2(Tz,T^2z)) \\ &< \phi(P(z,Tz)) \\ &\leq \phi(N_s(z,Tz)) \\ &= \phi(d^2(z,Tz)), \end{aligned}$$

which is a contradiction. Hence, z is a fixed point of T. If $X$is regular, we deduce that $d^2(z,Tz) = 0$, using the same arguments as in the proof of Theorem 3.6 That is, z is a fixed point of T. The uniqueness of fixed point of T can be deduced as in Theorem 3.7.

□

Example 3.10. *Let $X = [0,6]$ and $d : X \times X \to \mathbb{R}^+$ be defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete b-metric space with $s = 4$ which is not a metric space. Let $T : X \to X$ be defined by*

$$T(x) = \begin{cases} 1 & \text{if } x \in [0,3) \\ \frac{x}{3} & \text{if } x \in [3,6]. \end{cases}$$

Also, we define $\alpha : X \times X \to \mathbb{R}^+$, $q : X \times X \to \mathbb{R}^+$, and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\alpha(x,y)=\begin{cases}3 \;\; if\, x,y\in[0,3)\\ 2 \;\; if\, x,y\in[3,6],\\ 0\; otherwise,\end{cases}$$

$$q(x,y)=\max\left\{d^2(x,y),\frac{d(x,Tx).d(y,Ty)}{1+d^2(x,y)},\frac{d(x,Tx).d(y,Ty)}{1+d^2(Tx,Ty)}\right\}$$

and $\phi(t)=\frac{t}{3}$. *First we note that* q *satisfies the condition* $(p_q^1:N_s)$ *and* $q(x,y)>0$ *for all* $x\neq y$. *Since, for* $x=0$ *we have* $T0=2$ *and* $\alpha(0,T0)=\alpha(0,1)=3>1$, *assumption* (2) *of Theorem* 3.9 *is satisfied. Also, it is easy to see that* T *is a triangular* $\alpha-$*orbital admissible. Suppose* $\xi\in Z_w$ *be given by* $\xi(t,s)=\frac{9}{16}s-t$. *Now, we consider the following cases:*

Case (I). *For* $x,y\in[0,3), x\neq y$, *we have* $d^2(Tx,Ty)=0$, *so*

$$\begin{aligned}\xi(\alpha(x,y)\phi(d^2(Tx,Ty)),\phi(q(x,y)))&=\frac{9\phi(q(x,y))}{16}\\&=\frac{(q(x,y))^2}{16}\\&>0.\end{aligned}$$

Case (II). *For* $x,y\in[3,6], x\neq y$, *we have* $d(Tx,Ty)=\frac{|x-y|}{3}$,

$$q(x,y)=\max\left\{|x-y|^2,\frac{x/3.y/3}{1+|x-y|^2},\frac{x/3.y/3}{1+\frac{|x-y|^2}{9}}\right\},$$

so

$$\begin{aligned}\xi(\alpha(x,y)\phi(d^2(Tx,Ty)),\phi(q(x,y)))&=\frac{9\phi(q(x,y))}{16}-\phi\left(\frac{|x-y|}{3}\right)\\&=\frac{(q(x,y))^2}{16}-\left(\frac{|x-y|^2}{9}\right)\\&\geq 0.\end{aligned}$$

Case (III) *For* $x,y\in[0,3)$, *and* $y\in[3,6]$, *we have* $\alpha(x,y)=0$ *and*

$$\begin{aligned}\xi(\alpha(x,y)\phi(d^2(Tx,Ty)),\phi(q(x,y)))&=\frac{9\phi(q(x,y))}{16}\\&=\frac{(q(x,y))^2}{16}\\&>0.\end{aligned}$$

Thus, T satisfies all the conditions of Theorem 3.9 *and has unique fixed point* $x = 1$.

Now, we give some corollaries to our main findings.

Corollary 3.11. *Suppose (X,d) be a complete b-metric space with $s \geq 1$ and suppose $T : X \to X$ be a (α,ϕ)-MKGq hybrid contractive mapping of type I with $p(x,y) = d(x,y)$. Assume that the following conditions are satisfied:*

(1) *T is triangular α-orbital admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;*

(3) *either T is continuous or T^2 is continuous and $\alpha(u,Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

Remark 3.12. *Under the conditions of Corollary* 3.11, *since $x \neq y$ implies $d^2(x,y) > 0$, it is obvious that from Definition* 2.9 *is equivalent to the following:*

$$d^2(x,y) > 0 \implies \xi(\alpha(x,y)\phi(d^2(Tx,Ty)),\phi(d^2(x,y))) \geq 0.$$

Proof. It is clear that d satisfies the conditions $(P^1 d : M_s)$, respectively $(P^2 d : 0)$, and so all the Theorems $3.3, 3.4, 3.7, 3.9$ are also satisfied. Thus, T has a fixed point. □

Corollary 3.13. *Suppose (X,d) be a complete b-metric space with $s \geq 1$, and suppose $T : X \to X$ be a (α,ϕ)-MKGq hybrid contractive mapping of type I. Let $\rho : X \times X \to \mathbb{R}^+$ be defined by $\rho(x,y) = a_1 d^2(x,y) + a_2 d^2(x,Tx) + a_3 d^2(y,Ty)$, where $a_1,a_2,a_3 \in [0,\frac{1}{s}), a_1 + a_2 \leq \frac{1}{2s}$ and $a_3 \leq \frac{1}{2s}$. Assume also that:*

(1) *T is triangular α-orbital admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;*

(3) *either T is continuous or T^2 is continuous and $\alpha(u,Tu) \geq 1$ or (X,d) is regular.*

Then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for all $x \in X$.

Proof. Suppose $x,y \in X$ be such that $x \neq y$ and $d(x,Tx) \leq d(x,y)$. Then,

$$\begin{aligned}\rho(x,y) &= a_1 d^2(x,y) + a_2 d^2(x,Tx) + a_3 d^2(y,Ty)\\ &\leq (a_1 + a_2) d^2(x,y) + a_3 d^2(y,Ty)\end{aligned}$$

$$\leq \frac{d^2(x,y)+d^2(y,Ty)}{2s}$$
$$\leq M_s(x,y),$$

which shows that $(P^1\rho : M_s)$ holds. On the other hand, if $x_n \neq y$, then

$$\lim_{n\to\infty} d^2(x_n,y)=0 \; and \; \lim_{n\to\infty} d^2(x_n,x_{n+1})=0$$

hold, and then we have

$$\lim_{n\to\infty}\sup \rho(x_n,y) = \lim_{n\to\infty}\sup a_1 d^2(x_n,y)+a_2 d^2(x_n,x_{n+1})+a_3 d^2(y,Ty)$$
$$= a_3 d^2(y,Ty).$$

Thus, $(P^2\rho : a_3)$ holds. Hence, T has a fixed point. □

4. CONCLUSION:

In this work, we introduced generalized (α,ϕ) MKGq type hybrid contractive mappings of type I and II in the setting of b-metric spaces and proved the existence and uniqueness of fixed points for such mappings. Finally, we supported the main result of this work by an illustrative example.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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