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AN INERTIAL HYBRID EXTRAGRADIENT-VISCOSITY METHOD FOR SOLVING QUASIMONOTONE VARIATIONAL INEQUALITIES

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Abstract. This research aims to solve variational inequalities involving quasimonotone operators in infinite-dimensional real Hilbert spaces numerically. The simplicity of defining step size rules using an operator explanation is the fundamental advantage of iterative strategies, rather than the Lipschitz constant or another line search method. The proposed iterative techniques replace a monotone and non-monotone step size strategy based on mapping knowledge for the Lipschitz constant or an alternative line search algorithm. The strong convergences have been proven to be consistent with the offered methodologies and to resolve specific control specification limitations. Finally, we present numerical experiments that assess the efficacy and impact of iterative approaches.

Keywords: variational inequalities; quasimonotone; extragradient-viscosity method.
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1. Introduction

The fundamental goal of this research is to investigate the iterative approaches used to solve the variational inequality problem [1] in any real Hilbert space, including quasimonotone operators. Take into the thought that $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is a mapping. Assume that Φ is a nonempty, closed, and convex subset of $\mathscr{H}$ and that $\mathscr{H}$ is an arbitrary Hilbert space. The variational

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inequality problem for mapping $\mathscr{A}$ on Φ is described by the following definition:

$$\text{Find } h^* \in \Phi \text{ such that } \langle \mathscr{A}(h^*), v - h^* \rangle \geq 0,\ \forall v \in \Phi. \tag{1.1}$$

The following conditions must be met in order to investigate the high convergence of suggested algorithms:

- $(\mathscr{C}_1)$ The solution set of a problem (1.1) is denoted by $VI(\Phi, \mathscr{A})$ and nonempty it is non-empty.
- $(\mathscr{C}_2)$ $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is said to be quasimonotone if
$$\langle \mathscr{A}(h), v - h \rangle > 0 \Rightarrow \langle \mathscr{A}(v), v - h \rangle \geq 0,\ \forall h, v \in \Phi.$$
- $(\mathscr{C}_3)$ $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is said to be Lipschitz continuous with $\mathscr{L} > 0$ such that
$$\|\mathscr{A}(h) - \mathscr{A}(v)\| \leq \mathscr{L}\|h - v\|,\ \forall h, v \in \Phi.$$
- $(\mathscr{C}_4)$ $f : \mathscr{H} \to \mathscr{H}$ is said to be κ-contraction with constant $\kappa \in [0, 1)$ such that
$$\|f(h) - f(v)\| \leq \kappa\|h - v\|,\ \forall h, v \in \Phi.$$
- $(\mathscr{C}_5)$ $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ is sequentially weakly continuous if $\{\mathscr{A}(h_m)\}$ weakly converges to $\mathscr{A}(h)$ for each $\{h_m\}$ weakly converges to h.

It is well known that the variational inequality problem (1.1) can be transformed into a fixed point problem as follows:

$$h^* = \mathscr{P}_\Phi(\mathscr{I} - \lambda\mathscr{A})h^*, \tag{1.2}$$

where $\mathscr{P}_\Phi : \mathscr{H} \to \Phi$ is the metric projection and $\lambda > 0$. The simplest projection method to find the solution of (1.1) is the projection gradient method defined as follows:

$$h_{m+1} = \mathscr{P}_\Phi(h_m - \lambda\mathscr{A}(h_m)). \tag{1.3}$$

Korpelevich [2] introduced the extragradient method as follows:

$$\begin{aligned} k_m &= \mathscr{P}_\Phi(h_m - \lambda\mathscr{A}(h_m)), \\ t_{m+1} &= \mathscr{P}_\Phi(h_m - \lambda\mathscr{A}(k_m)), \end{aligned} \tag{1.4}$$

where $\mathscr{A}$ is a monotone operator that is $\mathscr{L}$-Lipschitz continuous and $\lambda \in (0, \frac{1}{\mathscr{L}})$. In accordance with the previous technique, we applied two projections on the underlying set $\mathscr{A}$ for each iteration. If the structure of the viable set $\mathscr{A}$ is complicated, the method's computational capacity might be constrained. To overcome the drawback in (1.4), Censor et al. [3] introduced the subgradient extragradient method as follows:

$$
\begin{aligned}
k_m &= \mathscr{P}_\Phi(h_m - \lambda \mathscr{A}(h_m)),\\
\Psi_m &= \{v \in \mathscr{H} \,|\, \langle h_m - \lambda \mathscr{A}(h_m) - k_m, v - k_m\rangle \leq 0\},\\
t_{m+1} &= \mathscr{P}_\Psi(h_m - \lambda \mathscr{A}(k_m)),
\end{aligned}
\tag{1.5}
$$

where $\lambda \in (0, \frac{1}{\mathscr{L}})$.

In this study, we focus on Tseng's extragradient algorithm, which uses only one projection per iteration. The definition of this algorithm is as follows:

$$
\begin{aligned}
k_m &= \mathscr{P}_\Phi(h_m - \lambda \mathscr{A}(h_m)),\\
t_{m+1} &= k_m - \lambda[\mathscr{A}(k_m) - \mathscr{A}(h_m)],
\end{aligned}
\tag{1.6}
$$

where $\lambda \in (0, \frac{1}{\mathscr{L}})$.

2. PRELIMINARIES

Next, we provide some definitions which are used in the sequel. Let $\mathscr{H}$ be a real Hilbert space, for a nonempty closed and convex subset Φ of $\mathscr{H}$. The symbols $\to$ and $\rightharpoonup$ mean the strong convergence and the weak convergence, respectively. The metric projection, $\mathscr{P}_\Phi : \mathscr{H} \to \mathscr{H}$ is defined, for each $h \in \mathscr{H}$, as the unique element $\mathscr{P}_\Phi h \in \Phi$ such that

$$\|h - \mathscr{P}_\Phi h\| = \inf\{\|h - v\| : v \in \Phi\}.$$

It is a known fact that $\mathscr{P}_\Phi$ is nonexpansive, i.e. $\|\mathscr{P}_\Phi h - \mathscr{P}_\Phi v\| \leq \|h - v\|$, $\forall h, v \in \Phi$. Also, the mapping $\mathscr{P}_\Phi$ is firmly nonexpansive, i.e.

$$\|\mathscr{P}_\Phi h - \mathscr{P}_\Phi v\| \leq \langle \mathscr{P}_\Phi h - \mathscr{P}_\Phi v, h - v\rangle,$$

for all $h, v \in \mathscr{H}$. Some results on the metric projection map are given below.

Lemma 2.1. [4] *Let Φ be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. For any $h \in \mathscr{H}$ and $v \in \Phi$, Then,*

$$v = \mathscr{P}_\Phi h \Leftrightarrow \langle h - v, v - g \rangle \geq 0, \ \forall g \in \Phi.$$

Lemma 2.2. [5] *Let Φ be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$, $h \in \mathscr{H}$. Then*

(i) $\|\mathscr{P}_\Phi h - \mathscr{P}_\Phi v\|^2 \leq \langle h - v, \mathscr{P}_\Phi h - \mathscr{P}_\Phi v \rangle$, $\forall v \in \Phi$.

(ii) $\|h - \mathscr{P}_\Phi h\|^2 + \|v - \mathscr{P}_\Phi h\|^2 \leq \|h - v\|^2$, $\forall v \in \Phi$.

Lemma 2.3. [6] *Let $\mathscr{H}$ be a real Hilbert space. Then the following results hold:*

(1) $\|h + v\|^2 = \|h\|^2 + 2\langle v, h + v \rangle$, $\forall h, v \in \mathscr{H}$.

(2) $\|h + v\|^2 = \|h\|^2 + 2\langle h, v \rangle + \|v\|^2$, $\forall h, v \in \mathscr{H}$.

(3) $\|\alpha h + (1 - \alpha)v\|^2 = \alpha\|h\|^2 + (1 - \alpha)\|v\|^2 - \alpha(1 - \alpha)\|h - v\|^2$, $\forall h, v \in \mathscr{H}$ *and* $\alpha \in \mathbb{R}$.

Lemma 2.4. [7] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{m+1} \leq (1 - \alpha_m)a_m + \alpha_m \sigma_m,$$

$$a_{m+1} \leq a_n - \delta_m + \gamma_m,$$

for each $m \geq 0$, where $\{\alpha_m\}_{m=0}^{\infty}$ is a sequence in $(0, 1)$, $\{\delta_m\}_{m=0}^{\infty}$ is a sequence of nonnegative real numbers and $\{\sigma_m\}_{m=0}^{\infty}$ and $\{\gamma_m\}_{m=0}^{\infty}$ are two real sequences such that

(a) $\sum_{m=1}^{\infty} \alpha_m = \infty$.

(b) $\lim_{m\to\infty} \gamma_m = 0$.

(c) $\lim_{i\to\infty} \delta_{m_i} = 0$ *implies* $\limsup_{i\to\infty} \sigma_{m_i} \leq 0$ *for any subsequence* $\{m_i\}_{i=0}^{\infty} \subset \{m\}_{m=0}^{\infty}$.

Then $\lim_{m\to\infty} a_m = 0$.

Lemma 2.5. [8] *Consider the problem with Φ being a nonempty, closed, convex subset of a real Hilbert space $\mathscr{H}$ and $\mathscr{A} : \Phi \to \mathscr{H}$ being monotone and continuous. Then h^* is a solution of* (1.1) *if and only if*

$$\langle \mathscr{A}(v), v - h^* \rangle \geq 0, \ \forall v \in \Phi.$$

3. The Main Results

In this section, we offer an iterative method based on Tseng's extragradient method [9] and the viscosity scheme [10] for solving quasimonotone variational inequality problems. The main algorithm has been presented as follows:

Algorithm 3.1. *Let* $0 < \lambda < \frac{1}{\mathscr{L}}, \phi > 0, \eta \in (0,1)$ *and let* $h_0, h_1 \in \Phi$ *be arbitrary*
Step 1. *Compute*

$$k_m = (1-\rho_m)(h_m + \phi_m(h_m - h_{m-1}))$$

with ϕ_m *such that* $0 \leq \phi_m \leq \bar{\phi}_m$, *and*

$$\bar{\phi}_m = \begin{cases} \min\left\{\phi, \frac{\xi_m}{\|h_m - h_{m-1}\|}\right\}, & \text{if } h_m \neq h_{m-1}, \\ \phi, & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\xi_m = o(\rho_m)$ *is satisfying the condition* $\lim_{m\to\infty} \frac{\xi_m}{\rho_m} = 0$.
Step 2. *Compute*

$$t_m = \mathscr{P}_\Phi(k_m - \lambda \mathscr{A}(k_m)).$$

If $k_m = t_m$ *or* $\mathscr{A}(k_m) = 0$, *then stop and* k_m *is a solution. Otherwise, go to Step 3.*
Step 3. *Compute*

$$s_m = \eta b_m[t_m - \lambda(\mathscr{A}(t_m) - \mathscr{A}(k_m))] + (1-b_m)[t_m - \lambda(\mathscr{A}(t_m) - \mathscr{A}(k_m))].$$

Step 4. *Compute*

$$h_{m+1} = \rho_m f(s_m) + (1-\rho_m)s_m.$$

Set $m := m+1$ *and return to Step 1.*

Lemma 3.1. *Assume that* $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ *satisfies the conditions* $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ *and sequence* $\{h_m\}$ *generated due to Algorithm 3.1. The following inequality holds*

$$\|r_m - h^*\|^2 \leq \|k_m - h^*\|^2 - (1-\lambda^2)\mathscr{L}^2)\|t_m - k_m\|^2,$$

where $r_m = t_m - \lambda(\mathscr{A}(t_m) - \mathscr{A}(k_m))$.

Proof. It follows that $h^* \in VI(\Phi, \mathscr{A})$, such that

$$\begin{aligned}
\|r_m - h^*\|^2 &= \|t_m - \lambda(\mathscr{A}(t_m) - \mathscr{A}(k_m)) - h^*\|^2 \\
&= \|(t_m - h^*) - \lambda(\mathscr{A}(t_m) - \mathscr{A}(k_m))\|^2 \\
&= \|t_m - h^*\|^2 - 2\lambda\langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2 \\
&= \|t_m - k_m + k_m - h^*\|^2 - 2\lambda\langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2 \\
&= \|t_m - k_m\|^2 + 2\langle t_m - k_m, k_m - h^*\rangle + \|k_m - h^*\|^2 \\
&\quad - 2\lambda\langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2.
\end{aligned} \tag{3.2}$$

Since $t_m = \mathscr{P}_\Phi(k_m - \lambda\mathscr{A}(k_m))$, we have

$$\langle t_m - k_m + \lambda\mathscr{A}(k_m), t_m - h^*\rangle \leq 0$$

or, equivalently,

$$\langle t_m - k_m, t_m - h^*\rangle \leq -\lambda\langle\mathscr{A}(k_m), t_m - h^*\rangle. \tag{3.3}$$

Using (3.17) and (3.19), it follows that

$$\begin{aligned}
\|r_m - h^*\|^2 &\leq \|k_m - h^*\|^2 - \|t_m - k_m\|^2 - 2\lambda\langle\mathscr{A}(k_m), t_m - h^*\rangle \\
&\quad - 2\lambda\langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2 \\
&= \|k_m - h^*\|^2 - \|t_m - k_m\|^2 - 2\lambda\langle t_m - h^*, \mathscr{A}(t_m)\rangle + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2.
\end{aligned} \tag{3.4}$$

Given that h^* is the solution to the problem (1.1), then follows that

$$\langle\mathscr{A}(h^*), t - h^*\rangle > 0,\ \forall t \in \Phi.$$

It implies that

$$\langle\mathscr{A}(t), t - h^*\rangle \geq 0,\ \forall t \in \Phi.$$

By substituting $t = t_m \in \Phi$, we have

$$\langle\mathscr{A}(t_m), t_m - h^*\rangle \geq 0. \tag{3.5}$$

Using (3.20) and (3.21), it follows that

$$\|r_m - h^*\|^2 \leq \|k_m - h^*\|^2 - \|t_m - k_m\|^2 + \lambda^2\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2. \tag{3.6}$$

Combining (3.22) and (3.10), we obtain

$$\|r_m - h^*\|^2 \leq \|k_m - h^*\|^2 - (1-\lambda^2\mathscr{L}^2)\|t_m - k_m\|^2. \tag{3.7}$$

□

Algorithm 3.2. *Let $\lambda_1 > 0, \phi > 0, \varphi \in (0,1), \eta \in (0,1)$ and let $h_0, h_1 \in \mathscr{H}$ be arbitrary*

Step 1. *Compute*

$$k_m = (1-\rho_m)(h_m + \phi_m(h_m - h_{m-1}))$$

with ϕ_m such that $0 \leq \phi_m \leq \bar{\phi}_m$, and

$$\bar{\phi}_m = \begin{cases} \min\left\{\phi, \frac{\xi_m}{\|h_m - h_{m-1}\|}\right\}, & \text{if } h_m \neq h_{m-1}, \\ \phi, & \text{otherwise,} \end{cases} \tag{3.8}$$

where $\xi_m = o(\rho_m)$ is satisfying the condition $\lim_{m\to\infty} \frac{\xi_m}{\rho_m} = 0$.

Step 2. *Compute*

$$t_m = \mathscr{P}_\Phi(k_m - \lambda_m \mathscr{A}(k_m)).$$

If $k_m = t_m$ or $\mathscr{A}(k_m) = 0$, then stop and k_m is a solution. Otherwise, go to Step 3.

Step 3. *Compute*

$$s_m = \eta b_m[t_m - \lambda_m(\mathscr{A}(t_m) - \mathscr{A}(k_m))] + (1-b_m)[t_m - \lambda_m(\mathscr{A}(t_m) - \mathscr{A}(k_m))].$$

Step 4. *Compute*

$$h_{m+1} = \rho_m f(s_m) + (1-\rho_m)s_m.$$

Step 5. *Compute*

$$\lambda_{m+1} = \begin{cases} \min\left\{\lambda_m, \dfrac{\varphi\|t_m - k_m\|}{\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|}\right\}, & \text{if } \mathscr{A}(t_m) \neq \mathscr{A}(k_m), \\ \lambda_m, & \text{otherwise.} \end{cases} \tag{3.9}$$

Set $m := m+1$ and return to Step 1.

Lemma 3.2. *The sequence $\{\lambda_m\}$ formed by* (3.9) *is monotonically decreasing and converges to $\lambda > 0$. Moreover, we also obtain*

$$\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\| \leq \frac{\varphi}{\lambda_{m+1}}\|t_m - k_m\|, \quad \forall m. \tag{3.10}$$

Proof. According to (3.9), $\lambda_{m+1} \leq \lambda_m$ for any $m \in \mathbb{N}$. Hence, $\{\lambda_m\}$ is monotonically decreasing. Furthermore, since $\mathscr{A}$ is Lipschitz-continuous with constant $\mathscr{L} > 0$, we have

$$\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\| \leq \mathscr{L}\|t_m - k_m\|.$$

Let $\mathscr{A}(t_m) \neq \mathscr{A}(k_m)$ such that

$$\frac{\varphi\|t_m - k_m\|}{\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|} \geq \frac{\varphi\|t_m - k_m\|}{\mathscr{L}\|t_m - k_m\|} = \frac{\varphi}{\mathscr{L}}.$$

Combining this with (3.9), we get

$$\lambda_m \geq \min\left\{\lambda_1, \frac{\varphi}{\mathscr{L}}\right\}.$$

As $\{\lambda_m\}$ is monotonically decreasing and bounded below, we can argue that

$$\lim_{m\to\infty} \lambda_m = \lambda \geq \min\left\{\lambda_1, \frac{\varphi}{\mathscr{L}}\right\}.$$

By (3.9), we have

$$\|\mathscr{A}(t_m) - \mathscr{A}(k_m)\| \leq \frac{\varphi}{\lambda_{m+1}}\|t_m - k_m\|, \quad \forall m.$$

□

Lemma 3.3. *Let $\{k_m\}$ and $\{t_m\}$ be two sequences generated by Algorithm 3.1, and suppose that conditions $(\mathscr{C}_1)$-$(\mathscr{C}_3)$ hold. If there exists a subsequence $\{k_{m_i}\}$ of $\{k_m\}$ convergent weakly to $h^* \in \mathscr{H}$ and $\lim_{m\to\infty} \|k_{m_i} - t_{m_i}\| = 0$, then $h^* \in VI(\Phi, \mathscr{A})$.*

Proof. From $t_{m_i} = \mathscr{P}_\Phi(k_{m_i} - \lambda_{m_i}\mathscr{A}(k_{m_i}))$, we obtain

$$\langle k_{m_i} - \lambda_{m_i}\mathscr{A}(k_{m_i}) - t_{m_i}, v - t_{m_i}\rangle \leq 0, \quad \forall v \in \Phi,$$

which implies that

$$\frac{1}{\lambda_{m_i}}\langle k_{m_i} - t_{m_i}, v - t_{m_i}\rangle \leq \langle \mathscr{A}(k_{m_i}) - t_{m_i}, v - t_{m_i}\rangle, \quad \forall v \in \Phi.$$

From this we obtain

$$\frac{1}{\lambda_{m_i}}\langle k_{m_i} - t_{m_i}, v - t_{m_i}\rangle + \langle \mathscr{A}(k_{m_i}), t_{m_i} - k_{m_i}\rangle \leq \langle \mathscr{A}(k_{m_i}) - t_{m_i}, v - t_{m_i}\rangle, \quad \forall v \in \Phi. \tag{3.11}$$

Because $\{k_{m_i}\}$ converges weakly to $h^* \in \mathscr{H}$, we have that $\{k_{m_i}\}$ is bounded. From the Lipschitz continuity of $\mathscr{A}$ and $\|k_{m_i} - t_{m_i}\| \to 0$, we obtain that $\{\mathscr{A}(k_{m_i})\}$ and $\{t_{m_i}\}$ are also bounded. Since $\lambda_{m_i} \geq \min\left\{\lambda_1, \frac{\varphi}{\mathscr{L}}\right\}$, from (3.11) it follows $\mathscr{L}$ that

$$\liminf_{i\to\infty} \langle \mathscr{A}(k_{m_i}), v - k_{m_i}\rangle \geq 0, \quad \forall v \in \Phi. \tag{3.12}$$

Moreover, we obtain

$$\langle \mathscr{A}(t_{m_i}), v - t_{m_i}\rangle = \langle \mathscr{A}(t_{m_i}) - \mathscr{A}(k_{m_i}), v - k_{m_i}\rangle + \langle \mathscr{A}(k_{m_i}), v - k_{m_i}\rangle + \langle \mathscr{A}(t_{m_i}), k_{m_i} - t_{m_i}\rangle. \tag{3.13}$$

Because $\lim_{i\to\infty} \|k_{m_i} - t_{m_i}\| = 0$, then by the Lipschitz continuity of $\mathscr{A}$ we have

$$\lim_{i\to\infty} \|\mathscr{A}(k_{m_i}) - \mathscr{A}(t_{m_i})\| = 0.$$

This together with (3.12) and (3.13) gives

$$\liminf_{i\to\infty} \langle \mathscr{A}(t_{m_i}), v - t_{m_i}\rangle \geq 0, \quad \forall v \in \Phi.$$

Now, choose a decreasing sequence xi_i of positive numbers such that $\xi_i \to 0$ as $i \to \infty$. For any i, we represent the smallest positive integer with N_i such that

$$\langle \mathscr{A}(t_{m_j}), v - t_{m_i}\rangle + \xi_i \geq 0, \quad \forall j \geq N_i. \tag{3.14}$$

It is clear that the sequence $\{N_i\}$ is increasing since ξ_i is decreasing. Furthermore, for any i, from $\{t_{N_i}\} \subset \Phi$, we can assume $\mathscr{A}(t_{N_i}) \neq 0$ (otherwise, t_{N_i} is a solution) and letting

$$x_{N_i} = \frac{\mathscr{A}(t_{N_i})}{\|\mathscr{A}(t_{N_i})\|^2},$$

we have $\langle \mathscr{A}(t_{N_i}), x_{N_i}\rangle = 1$, for each i. From (3.14), one can easily deduce that

$$\langle \mathscr{A}(t_{m_j}), v + \xi_i x_{N_i} - t_{N_i}\rangle \geq 0.$$

By the quasimonotone of $\mathscr{A}$, we have

$$\langle \mathscr{A}(v + \xi_i x_{N_i}), v + \xi_i x_{N_i} - t_{N_i}\rangle > 0,$$

which implies that

$$\langle \mathscr{A}(v), v - t_{N_i}\rangle \geq \langle \mathscr{A}(v) - \mathscr{A}(v + \xi_i x_{N_i}), v + \xi_i x_{N_i} - t_{N_i}\rangle - \xi_i \langle \mathscr{A}(v), x_{N_i}\rangle. \tag{3.15}$$

Next, we show that $\lim_{i\to\infty}\xi_i x_{N_i}=0$. Because $k_{m_i}\rightharpoonup h^*$ and $\lim_{i\to\infty}\|k_{m_i}-t_{m_i}\|=0$, we obtain $t_{N_i}\rightharpoonup h^*$ as $i\to\infty$. From $\{t_m\}\subset\Phi$, we obtain $h^*\in\Phi$. By the sequentially weakly continuity of $\mathscr{A}$ on Φ, we get $\mathscr{A}(t_{m_i})\rightharpoonup\mathscr{A}(h^*)$. We can assume that $\mathscr{A}(h^*)\neq 0$ (otherwise, h^* is a solution). Because the norm mapping is sequentially weakly lower semicontinuous, we have

$$0<\|\mathscr{A}(h^*)\|\leq\lim_{i\to\infty}\|\mathscr{A}(t_{m_i})\|.$$

In fact, $\{t_{N_i}\}\subset\{t_{m_i}\}$ and $\xi_i\to 0$ as $i\to\infty$, we get

$$0\leq\limsup_{i\to\infty}\|\xi_i x_{N_i}\|\leq\limsup_{i\to\infty}\frac{\xi_i}{\|\mathscr{A}(t_{N_i})\|}\leq\frac{\limsup_{i\to\infty}\xi_i}{\liminf_{i\to\infty}\|\mathscr{A}(t_{m_i})\|}=0,$$

this implies that $\limsup_{i\to\infty}\xi_i x_{N_i}=0$. In facts, $\mathscr{A}$ is Lipschitz continuous, sequences $\{t_{N_i}\},\{x_{N_i}\}$ are bounded and $\lim_{i\to\infty}\xi_i x_{N_i}=0$, we conclude from (3.15) that

$$\liminf_{i\to\infty}\langle\mathscr{A}(v),v-t_{N_i}\rangle\geq 0. \tag{3.16}$$

Thus,

$$\langle\mathscr{A}(v),v-h^*\rangle=\lim_{i\to\infty}\langle\mathscr{A}(v),v-t_{N_i}\rangle=\liminf_{i\to\infty}\langle\mathscr{A}(v),v-t_{N_i}\rangle\geq 0,\quad\forall v\in\Phi.$$

Hence, by Lemma 2.5, $h^*\in VI(\Phi,\mathscr{A})$ as required. □

Lemma 3.4. *Assume that $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. The following inequality holds*

$$\|r_m-h^*\|^2\leq\|k_m-h^*\|^2-(1-\varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m-k_m\|^2,$$

where $r_m=t_m-\lambda_m(\mathscr{A}(t_m)-\mathscr{A}(k_m))$.

Proof. It follows that $h^*\in VI(\Phi,\mathscr{A})$, such that

(3.17)

$$\begin{aligned}\|r_m-h^*\|^2&=\|t_m-\lambda_m(\mathscr{A}(t_m)-\mathscr{A}(k_m))-h^*\|^2\\&=\|(t_m-h^*)-\lambda_m(\mathscr{A}(t_m)-\mathscr{A}(k_m))\|^2\\&=\|t_m-h^*\|^2-2\lambda_m\langle t_m-h^*,\mathscr{A}(t_m)-\mathscr{A}(k_m)\rangle+\lambda_m^2\|\mathscr{A}(t_m)-\mathscr{A}(k_m)\|^2\\&=\|t_m-k_m+k_m-h^*\|^2-2\lambda_m\langle t_m-h^*,\mathscr{A}(t_m)-\mathscr{A}(k_m)\rangle+\lambda_m^2\|\mathscr{A}(t_m)-\mathscr{A}(k_m)\|^2\end{aligned}$$

$$\begin{aligned} &= \|t_m - k_m\|^2 + 2\langle t_m - k_m, k_m - h^*\rangle + \|k_m - h^*\|^2 \\ &\quad - 2\lambda_m \langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda_m^2 \|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2. \end{aligned} \tag{3.18}$$

Since $t_m = \mathscr{P}_\Phi(k_m - \lambda_m \mathscr{A}(k_m))$, we have

$$\langle t_m - k_m + \lambda_m \mathscr{A}(k_m), t_m - h^*\rangle \leq 0$$

or, equivalently,

$$\langle t_m - k_m, t_m - h^*\rangle \leq -\lambda_m \langle \mathscr{A}(k_m), t_m - h^*\rangle. \tag{3.19}$$

Using (3.17) and (3.19), it follows that

$$\begin{aligned} \|r_m - h^*\|^2 &\leq \|k_m - h^*\|^2 - \|t_m - k_m\|^2 - 2\lambda_m \langle \mathscr{A}(k_m), t_m - h^*\rangle \\ &\quad - 2\lambda_m \langle t_m - h^*, \mathscr{A}(t_m) - \mathscr{A}(k_m)\rangle + \lambda_m^2 \|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2 \\ &= \|k_m - h^*\|^2 - \|t_m - k_m\|^2 - 2\lambda_m \langle t_m - h^*, \mathscr{A}(t_m)\rangle + \lambda_m^2 \|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2. \end{aligned} \tag{3.20}$$

Given that h^* is the solution to the problem (1.1), then follows that

$$\langle \mathscr{A}(h^*), t - h^*\rangle > 0, \ \forall t \in \Phi.$$

It implies that

$$\langle \mathscr{A}(t), t - h^*\rangle \geq 0, \ \forall t \in \Phi.$$

By substituting $t = t_m \in \Phi$, we have

$$\langle \mathscr{A}(t_m), t_m - h^*\rangle \geq 0. \tag{3.21}$$

Using (3.20) and (3.21), it follows that

$$\|r_m - h^*\|^2 \leq \|k_m - h^*\|^2 - \|t_m - k_m\|^2 + \lambda_m^2 \|\mathscr{A}(t_m) - \mathscr{A}(k_m)\|^2. \tag{3.22}$$

Combining (3.22) and (3.10), we obtain

$$\|r_m - h^*\|^2 \leq \|k_m - h^*\|^2 - (1 - \varphi^2 \frac{\lambda_m^2}{\lambda_{m+1}^2}) \|t_m - k_m\|^2. \tag{3.23}$$

□

Lemma 3.5. *Let $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. Then the sequence $\{h_m\}$ is bounded.*

Proof. Consider that

$$\begin{aligned} \|k_m - h^*\| &= \|(1-\rho_m)(h_m + \phi_m(h_m - h_{m-1})) - h^*\| \\ &= \|(1-\rho_m)(h_m - h^*) + (1-\rho_m)\phi_m(h_m - h_{m-1}) - \rho_m h^*\| \\ &\leq (1-\rho_m)\|h_m - h^*\| + (1-\rho_m)\phi_m\|h_m - h_{m-1}\| + \rho_m\|h^*\| \\ &= (1-\rho_m)\|h_m - h^*\| + \rho_m\left[(1-\rho_m)\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| + \|h^*\|\right]. \end{aligned} \tag{3.24}$$

From (3.1) we have

$$\lim_{m\to\infty}\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| \leq \lim_{m\to\infty}\frac{\xi_m}{\rho_m} = 0. \tag{3.25}$$

Then

$$\lim_{m\to\infty}\left[(1-\rho_m)\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| + \|h^*\|\right] = \|h^*\|. \tag{3.26}$$

Since $\mathscr{G}_1 > 0$, we have

$$(1-\rho_m)\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| + \|h^*\| \leq \mathscr{G}_1. \tag{3.27}$$

From (3.24) and (3.27), we obtain

$$\begin{aligned} \|k_m - h^*\| &\leq (1-\rho_m)\|h_m - h^*\| + \rho_m\mathscr{G}_1 \\ &\leq \|h_m - h^*\| + \rho_m\mathscr{G}_1. \end{aligned} \tag{3.28}$$

From Lemma 3.3 and $\lim_{m\to\infty}\lambda_m$ for $\varphi \in (0,1)$, we have

$$\lim_{m\to\infty}(1 - \varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2}) = 1 - \varphi^2 > 0. \tag{3.29}$$

This implies that there exists $m_0 \in \mathbb{N}$ such that $1 - \varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2} > 0$ for all $m \geq m_0$. Hence, from (3.23) we obtain

$$\|r_m - h^*\| \leq \|k_m - h^*\|, \quad \forall m \geq m_0. \tag{3.30}$$

From (3.28) and (3.30) will be rewritten as:

$$\|r_m - h^*\| \leq \|k_m - h^*\| \leq \|h_m - h^*\| + \rho_m\mathscr{G}_1, \quad \forall m \geq m_0. \tag{3.31}$$

Furthermore, we be rewritten s_m are $s_m = \eta b_m r_m + (1-b_m)r_m$, and

$$\begin{aligned}
\|s_m - h^*\| &= \|\eta b_m r_m + (1-b_m)r_m - h^*\| \\
&= \|\eta b_m (r_m - h^*) + (1-b_m)(r_m - h^*)\| \\
&\leq \eta b_m \|r_m - h^*\| + (1-b_m)\|r_m - h^*\| \\
&\leq b_m \|r_m - h^*\| + (1-b_m)\|r_m - h^*\| \\
&= \|r_m - h^*\|.
\end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we have

$$\|s_n - h^*\| \leq \|r_m - h^*\| \leq \|k_m - h^*\| \leq \|h_m - h^*\| + \rho_m \mathscr{G}_1, \quad \forall m \geq m_0. \tag{3.33}$$

and

$$\begin{aligned}
\|h_{m+1} - h^*\| &= \|\rho_m f(s_m) + (1-\rho_m)s_m - h^*\| \\
&= \|\rho_m (f(s_m) - h^*) + (1-\rho_m)(s_m - h^*)\| \\
&\leq \rho_m \|f(s_m) - h^*\| + (1-\rho_m)\|s_m - h^*\| \\
&\leq \rho_m \|f(s_m) - f(h^*)\| + \rho_m \|f(h^*) - h^*\| + (1-\rho_m)\|s_m - h^*\| \\
&\leq \rho_m \kappa \|s_m - h^*\| + \rho_m \|f(h^*) - h^*\| + (1-\rho_m)\|s_m - h^*\| \\
&= (1-(1-\kappa)\rho_m)\|s_m - h^*\| + \rho_m \|f(h^*) - h^*\|, \quad \forall m \geq m_0.
\end{aligned} \tag{3.34}$$

Substituting (3.33) in (3.34) for all $m \geq m_0$ we have

$$\begin{aligned}
\|h_{m+1} - h^*\| &\leq (1-(1-\kappa)\rho_m)(\|h_m - h^*\| + \rho_m \mathscr{G}_1) + \rho_m \|f(h^*) - h^*\| \\
&\leq (1-(1-\kappa)\rho_m)\|h_m - h^*\| + (1-\kappa)\rho_m \frac{\mathscr{G}_1 + \|f(h^*) - h^*\|}{1-\kappa} \\
&\leq \max\left\{\|h_m - h^*\|, \frac{\mathscr{G}_1 + \|f(h^*) - h^*\|}{1-\kappa}\right\} \\
&\vdots \\
&\leq \max\left\{\|h_{m_0} - h^*\|, \frac{\mathscr{G}_1 + \|f(h^*) - h^*\|}{1-\kappa}\right\}.
\end{aligned}$$

Thus the sequence $\{h_m\}$ is bounded. □

Lemma 3.6. *Assume that $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. The following inequality holds*

$$\|h_{m+1}-h^*\|^2 \leq \|h_m-h^*\|^2+\rho_m\mathscr{G}_4-(1-\varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m-k_m\|^2,$$

for some $\mathscr{G}_4>0$.

Proof. It following (3.23) and (3.32), we have

$$\|s_m-h^*\|^2 \leq \|k_m-h^*\|^2-(1-\varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m-k_m\|^2. \tag{3.35}$$

Consider

$$\begin{aligned}
\|h_{m+1}-h^*\|^2 &= \|\rho_m f(s_m)+(1-\rho_m)s_m-h^*\|^2\\
&= \|\rho_m(f(s_m)-f(h^*)+f(h^*)-h^*)+(1-\rho_m)(s_m-h^*)\|^2\\
&\leq \rho_m\|f(s_m)-f(h^*)+f(h^*)-h^*\|^2+(1-\rho_m)\|s_m-h^*\|^2\\
&\leq \rho_m\left(\|f(s_m)-f(h^*)\|+\|f(h^*)-h^*\|\right)^2+(1-\rho_m)\|s_m-h^*\|^2\\
&\leq \rho_m\left(\kappa\|s_m-h^*\|+\|f(h^*)-h^*\|\right)^2+(1-\rho_m)\|s_m-h^*\|^2\\
&\leq \rho_m\left(\|s_m-h^*\|+\|f(h^*)-h^*\|\right)^2+(1-\rho_m)\|s_m-h^*\|^2\\
&= \rho_m\|s_m-h^*\|^2+\rho_m\left(\|f(h^*)-h^*\|^2+2\|s_m-h^*\|\|f(h^*)-h^*\|\right)\\
&\quad +(1-\rho_m)\|s_m-h^*\|^2\\
&\leq \rho_m\|s_m-h^*\|^2+\rho_m\mathscr{G}_3+(1-\rho_m)\|s_m-h^*\|^2\\
&= \|s_m-h^*\|^2+\rho_m\mathscr{G}_2,
\end{aligned} \tag{3.36}$$

for some $\mathscr{G}_2>0$. Substituting (3.35) in (3.36), we get

$$\|h_{m+1}-h^*\|^2 \leq \|k_m-h^*\|^2+\rho_m\mathscr{G}_2-(1-\varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m-k_m\|^2. \tag{3.37}$$

From (3.33), we have

$$\begin{aligned}
\|k_n-h^*\|^2 &\leq \left(\|h_m-h^*\|+\rho_m\mathscr{G}_1\|\right)^2\\
&= \|h_m-h^*\|^2+2\rho_m\mathscr{G}_1\|h_m-h^*\|+\rho_m^2\mathscr{G}_1^2
\end{aligned} \tag{3.38}$$

$$= \|h_m - h^*\|^2 + \rho_m \left(2\mathscr{G}_1 \|h_m - h^*\| + \rho_m \mathscr{G}_1^2\right)$$
$$\leq \|h_m - h^*\|^2 + \rho_m \mathscr{G}_3,$$

for some $\mathscr{G}_3 > 0$. Substituting (3.38) in (3.37), we get

$$\|h_{m+1} - h^*\|^2 \leq \|h_m - h^*\|^2 + \rho_m \mathscr{G}_2 + \rho_m \mathscr{G}_2 - (1 - \varphi^2 \frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m - k_m\|^2. \tag{3.39}$$

This implies that

$$\|h_{m+1} - h^*\|^2 \leq \|h_m - h^*\|^2 + \rho_m \mathscr{G}_4 - (1 - \varphi^2 \frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m - k_m\|^2, \tag{3.40}$$

where $\mathscr{G}_4 := \mathscr{G}_2 + \mathscr{G}_3$. □

Lemma 3.7. *Assume that $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. The following inequality holds*

$$\|r_m - h^*\|^2 \leq \|h_m - h^*\|^2 + \rho_m \Big[\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|\phi_m\|h_m - h_{m-1}\|$$
$$+ 2(1-\rho_m)\|h_m - h^*\|\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| + 2\|h^*\|\|k_m - h_{m+1}\|$$
$$+ 2\|h^*\|\|k_m - s_m\| + 2\|h^*\|\|s_m - h^*\|\Big].$$

Proof. The following inequality holds for all $h^* \in VI(\Phi, \mathscr{A})$, $m \in \mathbb{N}$ and Lemma 2.3, we obtain

(3.41)

$$\|k_m - h^*\|^2 = \|(1-\rho_m)(h_m + \phi_m(h_m - h_{m-1})) - h^*\|^2$$
$$= \|(1-\rho_m)(h_m - h^*) + (1-\rho_m)\phi_m(h_m - h_{m-1}) - \rho_m h^*\|^2$$
$$\leq \|(1-\rho_m)(h_m - h^*) + (1-\rho_m)\phi_m(h_m - h_{m-1})\|^2 + 2\rho_m\langle -h^*, k_m - h^*\rangle$$
$$= (1-\rho_m)^2\|h_m - h^*\|^2 + (1-\rho_m)^2\phi_m^2\|h_m - h_{m-1}\|^2$$
$$+ 2\phi_m(1-\rho_m)^2\langle h_m - h^*, h_m - h_{m-1}\rangle + 2\rho_m\langle -h^*, k_m - s_m\rangle + 2\rho_m\langle -h^*, s_m - h^*\rangle$$
$$\leq \|h_m - h^*\|^2 + \phi_m^2\|h_m - h_{m-1}\|^2 + 2\phi_m(1-\rho_m)\|h_m - h^*\|\|h_m - h_{m-1}\|$$
$$+ 2\rho_m\|h^*\|\|k_m - s_m\| + 2\rho_m\|h^*\|s_m - h^*\|$$
$$= \|h_m - h^*\|^2 + \rho_m\Big[\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|\phi_m\|h_m - h_{m-1}\|$$
$$+ 2(1-\rho_m)\|h_m - h^*\|\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\| + 2\|h^*\|\|k_m - s_m\| + 2\|h^*\|\|s_m - h^*\|\Big].$$

From (3.23) and (3.41), we obtain

(3.42)
$$\|r_m-h^*\|^2 \le \|h_m-h^*\|^2+\rho_m\Big[\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|\phi_m\|h_m-h_{m-1}\| + 2(1-\rho_m)\|h_m-h^*\|\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|+2\|h^*\|\|k_m-s_m\|+2\|h^*\|\|s_m-h^*\|\Big].$$

□

Lemma 3.8. *Assume that $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. The following inequality holds*

$$\begin{aligned}\|h_{m+1}-h^*\|^2 \le{}& (1-(1-\kappa)\rho_m)\|h_m-h^*\|^2+(1-\kappa)\rho_m\Big[\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|\frac{\phi_m}{1-\kappa}\|h_m-h_{m-1}\|\\ &+\frac{2}{1-\kappa}(1-\rho_m)\|h_m-h^*\|\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|\\ &+\frac{2}{1-\kappa}\|h^*\|\|k_m-s_m\|+\frac{2}{1-\kappa}\|h^*\|s_m-h^*\|\\ &+\frac{2}{1-\kappa}\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\Big],\quad \forall m\ge m_0.\end{aligned}$$

Proof. The following inequality holds for all $h^*\in VI(\Phi,\mathscr{A})$, $m\in\mathbb{N}$, Lemma 2.3 and Lemma 3.7, we obtain

(3.43)
$$\begin{aligned}\|h_{m+1}-h^*\|^2 &= \|\rho_m f(s_m)+(1-\rho_m)s_m-h^*\|^2\\ &= \|\rho_m(f(s_m)-f(h^*))+(1-\rho_m)(s_m-h^*)+\rho_m(f(h^*)-h^*)\|^2\\ &\le \|\rho_m(f(s_m)-f(h^*))+(1-\rho_m)(s_m-h^*)\|^2+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\\ &\le \rho_m\|f(s_m)-f(h^*)\|^2+(1-\rho_m)\|s_m-h^*\|^2+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\\ &\le \rho_m\kappa^2\|s_m-h^*\|^2+(1-\rho_m)\|s_m-h^*\|^2+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\\ &\le (1-(1-\kappa)\rho_m)\|s_m-h^*\|^2+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\\ &\le (1-(1-\kappa)\rho_m)\|r_m-h^*\|^2+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\\ &\le (1-(1-\kappa)\rho_m)+\rho_m\Big[\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|\phi_m\|h_m-h_{m-1}\|\\ &\quad+2(1-\rho_m)\|h_m-h^*\|\frac{\phi_m}{\rho_m}\|h_m-h_{m-1}\|+2\|h^*\|\|k_m-s_m\|\\ &\quad+2\|h^*\|\|s_m-h^*\|\Big]+2\rho_m\langle f(h^*)-h^*,h_{m+1}-h^*\rangle\end{aligned}$$

$$= (1-(1-\kappa)\rho_m)\|h_m - h^*\|^2 + (1-\kappa)\rho_m \Big[\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|\frac{\phi_m}{1-\kappa}\|h_m - h_{m-1}\|$$
$$+ \frac{2}{1-\kappa}(1-\rho_m)\|h_m - h^*\|\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|$$
$$+ \frac{2}{1-\kappa}\|h^*\|\|k_m - s_m\| + \frac{2}{1-\kappa}\|h^*\|\|s_m - h^*\|$$
$$+ \frac{2}{1-\kappa}\langle f(h^*) - h^*, h_{m+1} - h^*\rangle\Big], \quad \forall m \geq m_0.$$

□

Theorem 3.1. *Let $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ satisfies the conditions $(\mathscr{C}_1)$-$(\mathscr{C}_5)$ and sequence $\{h_m\}$ generated due to Algorithm 3.1. Then the sequence $\{h_m\}$ weakly converges to h^*, where $h^* = \mathscr{P}_{VI(\Phi,\mathscr{A})} \circ f(h^*)$.*

Proof. From Lemma 3.8 and Lemma 3.6, let

$$a_m = \|h_m - h^*\|^2, \quad \alpha_m = (1-\kappa)\rho_m,$$
$$\sigma_m = \frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|\frac{\phi_m}{1-\kappa}\|h_m - h_{m-1}\| + \frac{2}{1-\kappa}(1-\rho_m)\|h_m - h^*\|\frac{\phi_m}{\rho_m}\|h_m - h_{m-1}\|$$
$$+ \frac{2}{1-\kappa}\|h^*\|\|k_m - s_m\| + \frac{2}{1-\kappa}\|h^*\|\|s_m - h^*\| + \frac{2}{1-\kappa}\langle f(h^*) - h^*, h_{m+1} - h^*\rangle,$$
$$\delta_m = (1 - \varphi^2\frac{\lambda_m^2}{\lambda_{m+1}^2})\|t_m - k_m\|^2, \quad \gamma_m = \rho_m \mathscr{G}_4.$$

We can be rewritten as follows:

$$a_{m+1} \leq (1-\alpha_m)a_m + \alpha_m\sigma_m, \quad m \geq 0,$$
$$a_{m+1} \leq a_n - \delta_m + \gamma_m, \quad m \geq 0.$$

Obviously, conditions (a) and (b) of Lemma 2.4 are satisfied. To complete the proof, we must prove that condition (c) of Lemma 2.4 is also satisfied. This implies that there is some subsequence $\{m_i\}_{i=0}^{\infty} \subset \{m\}_{m=0}^{\infty}$ such that $\lim_{i\to\infty}\delta_{m_i} = 0$, then $\limsup_{i\to\infty}\sigma_{m_i} \leq 0$. In fact, $\lim_{i\to\infty}\delta_{m_i} = 0$, it suffices to show that

$$\lim_{i\to\infty}\|t_{m_i} - k_{m_i}\| = 0. \tag{3.44}$$

From (3.44), it follows that

$$\begin{aligned}\|s_{m_i}-k_{m_i}\| &= \|\eta b_{m_i} r_{m_i} + (1-b_{m_i})r_{m_i} - k_{m_i}\| \\ &\le b_{m_i}\|r_{m_i}-k_{m_i}\| + (1-b_{m_i})\|r_{m_i}-k_{m_i})\| \\ &= \|r_{m_i}-k_{m_i}\| \\ &\le \|t_{m_i} - \lambda_{m_i}(\mathscr{A}(t_{m_i}) - \mathscr{A}(k_{m_i})) - k_{m_i}\| \\ &\le \|t_{m_i}-k_{m_i}\| + \lambda_{m_i}\|\mathscr{A}(t_{m_i}) - \mathscr{A}(k_{m_i})\| \\ &\le \left(1+\varphi\frac{\lambda_{m_i}}{\lambda_{m_i+1}}\right)\|t_{m_i}-k_{m_i}\| \to 0.\end{aligned} \tag{3.45}$$

Moreover, we have

$$\|h_{m_i+1}-s_{m_i}\| = \rho_{m_i}\|f(s_{m_i})-s_{m_i}\| \to 0. \tag{3.46}$$

Using (3.45) and (3.46), we have

$$\|h_{m_i+1}-k_{m_i}\| \le \|h_{m_i+1}-s_{m_i}\| + \|s_{m_i}-k_{m_i}\| \to 0. \tag{3.47}$$

Next, we estimate

$$\begin{aligned}\|k_{m_i}-h_{m_i}\| &= \|(1-\rho_{m_i})(h_{m_i}+\phi_{m_i}(h_{m_i}-h_{m_i-1})) - h_{m_i}\| \\ &= \|\phi_{m_i}(h_{m_i}-h_{m_i-1}) - \rho_{m_i}(h_{m_i}+\phi_{m_i}(h_{m_i}-h_{m_i-1}))\| \\ &\le \phi_{m_i}\|h_{m_i}-h_{m_i-1}\| + \rho_{m_i}\|h_{m_i}\| + \rho_{m_i}\phi_{m_i}\|h_{m_i}-h_{m_i-1}\| \\ &= \rho_{m_i}\frac{\phi_{m_i}}{\rho_{m_i}}\|h_{m_i}-h_{m_i-1}\| + \rho_{m_i}\|h_{m_i}\| + \rho_{m_i}^2\frac{\phi_{m_i}}{\rho_{m_i}}\|h_{m_i}-h_{m_i-1}\| \to 0.\end{aligned} \tag{3.48}$$

Combining this together with the fact that $\rho_{m_i} \to 0$ as $i \to \infty$, we obtain

$$\|h_{m_i+1}-h_{m_i}\| \le \|h_{m_i+1}-s_{m_i}\| + \|s_{m_i}-k_{m_i}\| + \|k_{m_i}-h_{m_i}\| \to 0. \tag{3.49}$$

Since the sequence $\{h_{m_i}\}$ is bounded, it follows that there exists a subsequence $\{h_{m_{i_e}}\}$ of $\{h_{m_i}\}$, which converges weakly to some $h \in \mathscr{H}$, such that

$$\limsup_{i\to\infty}\langle f(h^*)-h^*, h_{m_i}-h^*\rangle = \lim_{e\to\infty}\langle f(h^*)-h^*, h_{m_{i_e}}-h^*\rangle = \langle f(h^*)-h^*, h-h^*\rangle. \tag{3.50}$$

Using (3.48), we obtain

$$h_{m_i} \rightharpoonup h \text{ as } i \to \infty.$$

By Lemma 3.3, $h \in VI(\Phi, \mathscr{A})$. From (3.50) and the definition of $h^* = \mathscr{P}_{VI(\Phi,\mathscr{A})} \circ f(h^*)$, we have

$$\limsup_{i\to\infty} \langle f(h^*) - h^*, h_{m_i} - h^* \rangle = \langle f(h^*) - h^*, h - h^* \rangle \leq 0. \tag{3.51}$$

Using (3.49) and (3.51), we have

$$\begin{aligned}\limsup_{i\to\infty} \langle f(h^*) - h^*, h_{m_i+1} - h^* \rangle &\leq \limsup_{i\to\infty} \langle f(h^*) - h^*, h_{m_i} - h^* \rangle \\ &= \langle f(h^*) - h^*, h - h^* \rangle \\ &\leq 0.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\limsup_{i\to\infty} \sigma_{m_i} = \limsup_{i\to\infty} \Bigg[&\frac{\phi_{m_i}}{\rho_{m_i}} \|h_{m_i} - h_{m_i-1}\| \frac{\phi_{m_i}}{1-\kappa} \|h_{m_i} - h_{m_i-1}\| \\ &+ \frac{2}{1-\kappa}(1-\rho_{m_i})\|h_{m_i} - h^*\| \frac{\phi_{m_i}}{\rho_{m_i}} \|h_{m_i} - h_{m_i-1}\| \\ &+ \frac{2}{1-\kappa}\|h^*\|\|k_{m_i} - s_{m_i}\| + \frac{2}{1-\kappa}\|h^*\|\|s_{m_i} - h^*\| \\ &+ \frac{2}{1-\kappa}\langle f(h^*) - h^*, h_{m_i+1} - h^* \rangle \Bigg] \leq 0.\end{aligned}$$

By Lemma 2.4, we have $\lim_{m\to\infty} \|h_m - h^*\| = 0$. □

4. Numerical Example

The numerical results for the proposed method are described in this section. All codes were written in Matlab 2016b and run on Dell i-5 Core laptop.

Example 1. *Let $\mathscr{H} = l_2$ be a real Hilbert space containing sequences of real numbers that satisfy the following condition*

$$\|h_1\|^2 + \|h_2\|^2 + \cdots + \|h_m\|^2 + \cdots < \infty. \tag{4.1}$$

Assume that $\mathscr{A} : \Phi \to \Phi$ is defined by

$$\mathscr{A}(h) = (8 - \|h\|)h, \quad \forall h \in \mathscr{H},$$

where $\Phi = \{h \in \mathscr{H} : \|h\| \leq 6\}$*. It is easy to see that* $\mathscr{A}$ *is weakly sequentially continuous on* $\mathscr{H}$ *and* $VI(\Phi, \mathscr{A}) = \{0\}$*. For any* $h, v \in \mathscr{H}$*, we have*

$$
\begin{aligned}
\|\mathscr{A}(h) - \mathscr{A}(v)\| &= \|(8 - \|h\|)h - (8 - \|v\|)v\| \\
&= \|8(h - v) - \|h\|(h - v) - (\|h\| - \|v\|)v\| \\
&\leq 8\|h - v\| + \|h\|\|h - v\| + \|h\|\,|\|h\| - \|v\||\,\|v\| \\
&\leq 8\|h - v\| + 6\|h - v\| + 6\|h - v\| \\
&= 20\|h - v\|.
\end{aligned} \tag{4.2}
$$

Thus, $\mathscr{A}$ *is* $\mathscr{L}$*-Lipschitz continuous with* $\mathscr{L} = 20$*. For any* $h, v \in \mathscr{H}$*, let* $\langle \mathscr{A}(h), v - h \rangle > 0$ *such that*

$$(8 - \|h\|)\langle h, v - h \rangle > 0.$$

Since $\|h\| \leq 6$ *implies that*

$$\langle h, v - h \rangle > 0.$$

Hence, we have

$$
\begin{aligned}
\langle \mathscr{A}(v), v - h \rangle &= (8 - \|v\|)\langle v, v - h \rangle \\
&\geq (8 - \|v\|)\langle v, v - h \rangle - (8 - \|v\|)\langle h, v - h \rangle \\
&\geq 2\|h - v\|^2 \\
&\geq 0.
\end{aligned} \tag{4.3}
$$

Consequently, we shown that $\mathscr{A}$ *is quasimonotone on* Φ*. A projection on the set* Φ *is computed explicitly as follows:*

$$
\mathscr{P}_{\Phi}(h) = \begin{cases} h, & \text{if } \|h\| \leq 6, \\ \dfrac{6h}{\|h\|}, & \text{otherwise}, \end{cases}
$$

and let $f : \Phi \to \Phi$ *is a* κ*-contraction with constant* $\kappa \in [0, 1)$ *which*

$$f(h) = \frac{h}{2}.$$

The control conditions have been taken as follows:

$$\lambda = \frac{0.5}{\mathscr{L}}, \lambda_1 = 0.02,$$

$$\varphi = 0.001, b_n = 0.5,$$

$$\eta = 0.3, \phi = \phi_1 = 0.25,$$

$$\text{and } \xi_m = \frac{1}{(m+1)^2}.$$

We terminate the iterations if $\|k_m - t_m\| \leq \varepsilon$, *where* $\varepsilon = 10^{-5}$. *The results are reported in Table 1-2 and Figures 1-6.*

TABLE 1. Comparison of Algorithm 3.1 and Algorithm 3.2 with difference constants

		Number of Iteration		*Execution Time in Seconds*	
ρ_m	$h_0 = h_1$	*Algorithm 3.1*	*Algorithm 3.2*	*Algorithm 3.1*	*Algorithm 3.2*
0.55	$(1,1,\ldots,1_{500},0,0,\ldots)$	*7*	*6*	*0.061593*	*0.023909*
0.66	$(1,1,\ldots,1_{5000},0,0,\ldots)$	*7*	*5*	*0.274731*	*0.104215*
0.76	$(1,1,\ldots,1_{50000},0,0,\ldots)$	*8*	*5*	*0.144871*	*0.048751*
0.94	$(1,1,\ldots,1_{50000},0,0,\ldots)$	*5*	*3*	*0.228054*	*0.137483*

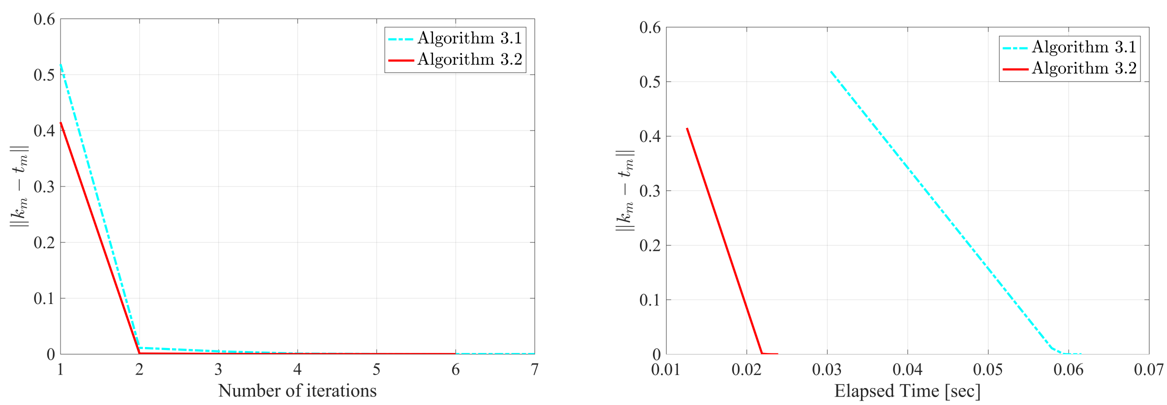

FIGURE 1. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = 0.55$

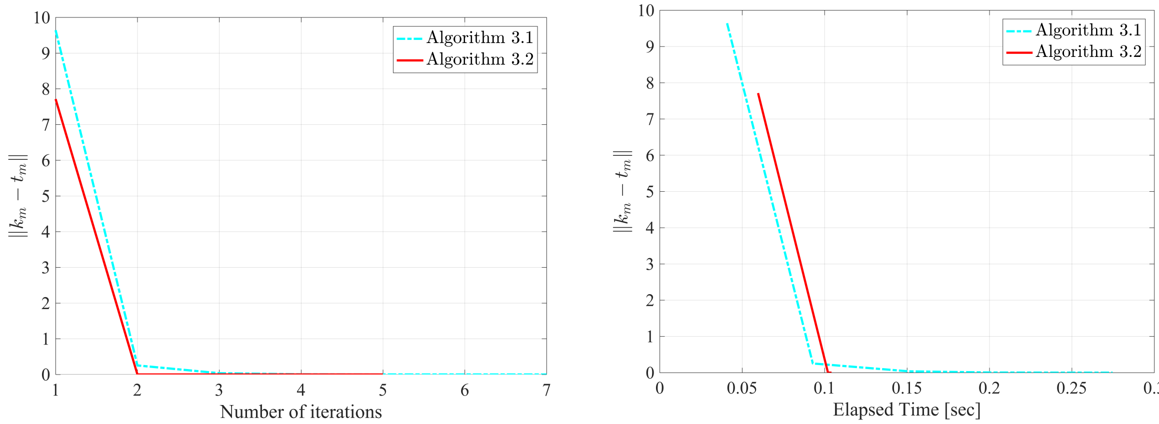

FIGURE 2. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = 0.66$

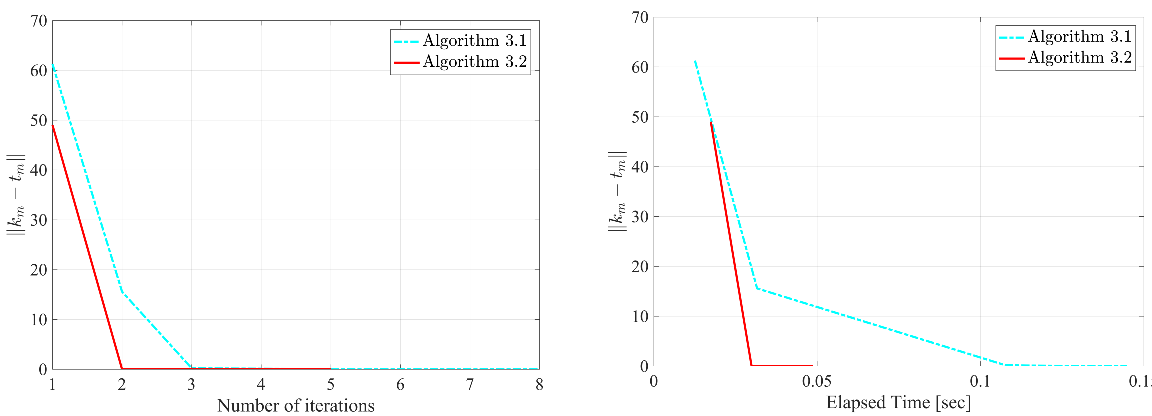

FIGURE 3. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = 0.76$

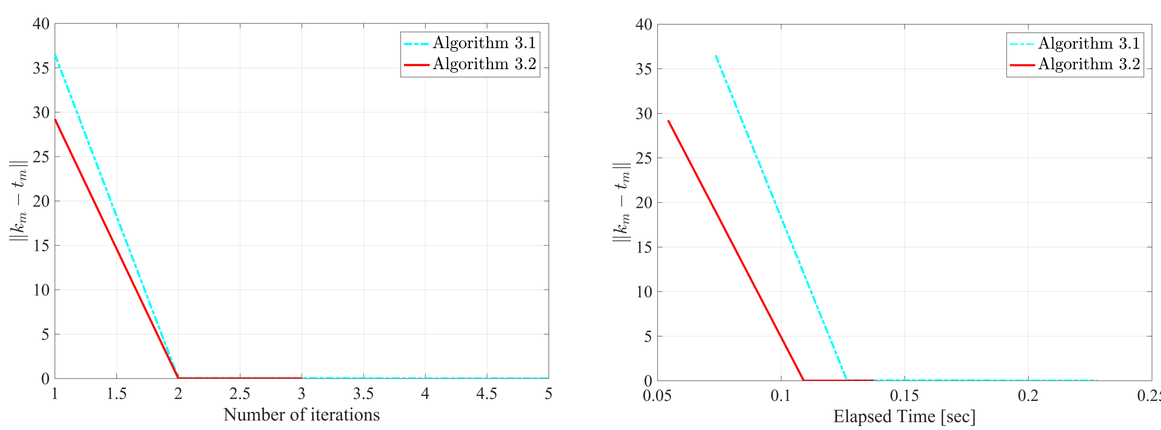

FIGURE 4. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = 0.94$

TABLE 2. Comparison of Algorithm 3.1 and Algorithm 3.2 with difference sequences

		Number of Iteration		*Execution Time in Seconds*	
ρ_m	$h_0 = h_1$	*Algorithm 3.1*	*Algorithm 3.2*	*Algorithm 3.1*	*Algorithm 3.2*
$\frac{55m}{100m+1}$	$(2,2,\ldots,2_{500},0,0,\ldots)$	*9*	*6*	*0.043179*	*0.015539*
$\frac{66m}{100m+2}$	$(2,2,\ldots,2_{5000},0,0,\ldots)$	*10*	*5*	*0.035098*	*0.016635*
$\frac{85m}{100m+3}$	$(2,2,\ldots,2_{50000},0,0,\ldots)$	*9*	*4*	*0.068068*	*0.026746*
$\frac{96m}{100m+4}$	$(2,2,\ldots,2_{500000},0,0,\ldots)$	*7*	*4*	*0.203115*	*0.143335*

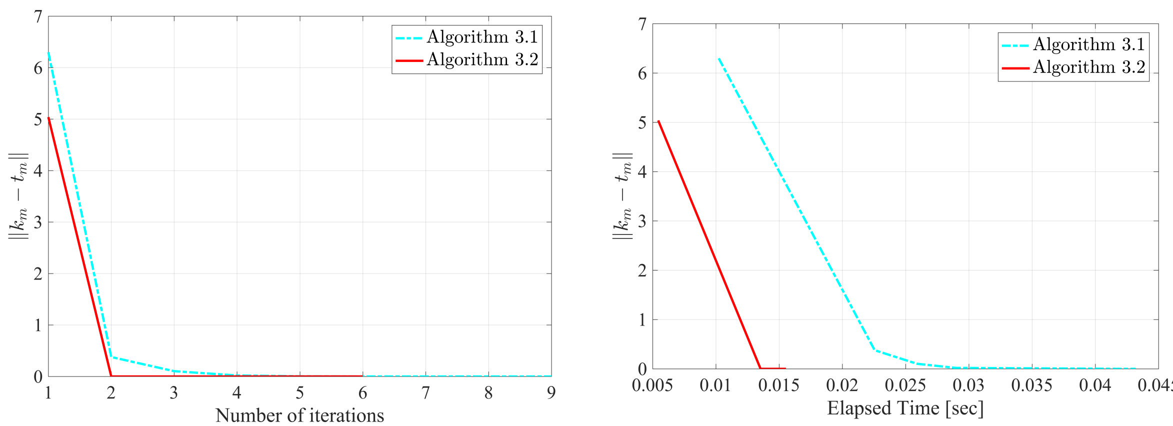

FIGURE 5. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = \frac{55m}{100m+1}$

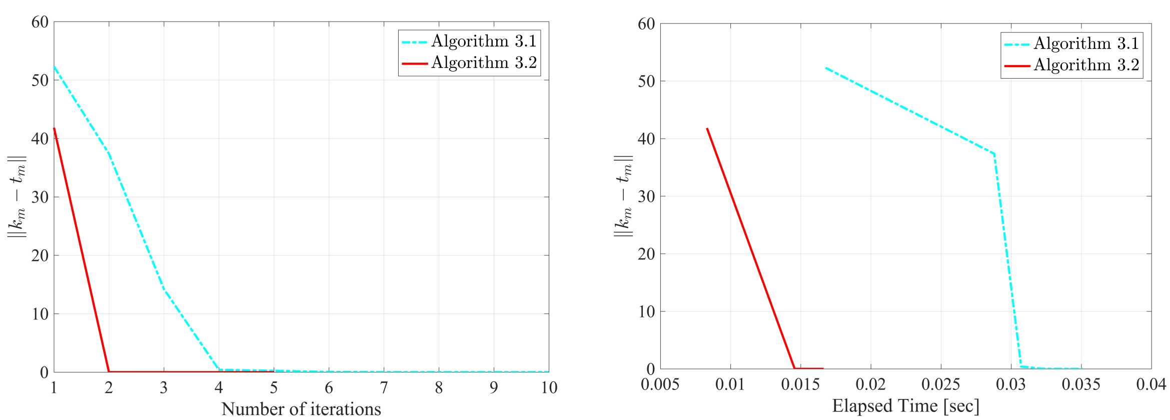

FIGURE 6. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = \frac{66m}{100m+2}$

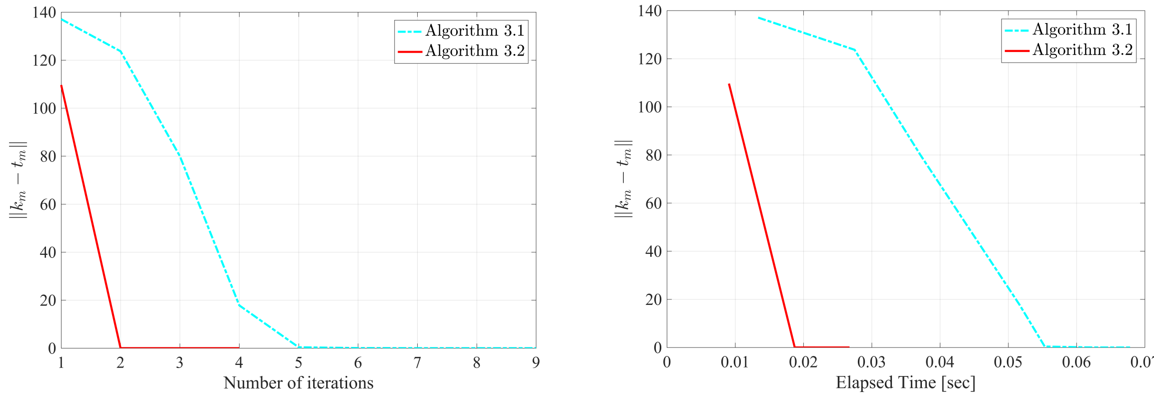

FIGURE 7. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = \frac{85m}{100m+3}$

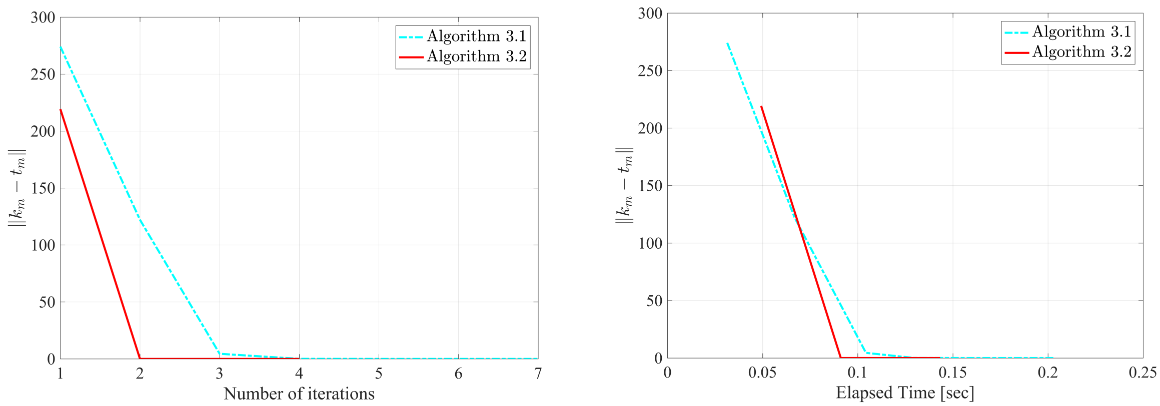

FIGURE 8. Comparison of Algorithm 3.1 and Algorithm 3.2 when $\rho_m = \frac{96m}{100m+4}$

5. Conclusion

We established two improved extragradient type algorithms to provide numerical solutions to quasimonotone variational inequality problems in the real Hilbert space. Numerical results are provided to demonstrate the numerical effectiveness of our algorithm. These computational investigations have shown that the variable step size influences the efficiency of the iterative sequence in this situation.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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