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SOLVING INTEGRO-DIFFERENTIAL EQUATION IN ORTHOGONAL PARTIAL B-METRIC SPACES VIA SIMULATION FUNCTION

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Abstract. In our research, we delve into exploring the presence and singular nature of a stable point amidst nearly orthogonal almost $\mathscr{Z}$-contractions, facilitated by simulation functions within fully developed orthogonal partial $\mathfrak{b}$-metric spaces, employing orthogonal $(\hat{\propto}, \hat{\beta})$-admissibility. Additionally, we provide demonstrative instances to substantiate the outcomes. Moreover, we offer an application to integro-differential equations, thereby contributing to the expansion and enhancement of various prior studies in the field.

Keywords: orthogonal almost $\mathscr{Z}$-contractions; fixed point; orthogonal-Cauchy sequence; orthogonal complete partial $\mathfrak{b}$-metric space.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The exploration of fixed points (f.p.) stands as a cornerstone in mathematical theory, offering solutions to a broad spectrum of problems. Originating from the renowned Banach contraction principle [1], the establishment of fixed points in complete metric spaces is a well-explored territory (also referenced in [2], [3], [4], [5]). Since its inception, numerous scholars have expanded upon this concepts by introducing various types of contractions within traditional metric

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spaces, such as $\mathfrak{b}$-metric spaces, partial metric spaces, metric-like spaces, and partial $\mathfrak{b}$-metric spaces (abbreviated as $\mathscr{P}_b MS$). In 1989, Bakhtin [6] introduced the concept of $\mathfrak{b}$-metric spaces. In 1994, Matthews (see, [7], [8]) initiated the concept of partial metric spaces, which is the conventional metric is replaced by a partial metric with the intriguing property that the self-distance of any point in the space may not be zero. Shukla [9] generalized both the concepts of $\mathfrak{b}$-metric and partial metric spaces by introducing $\mathscr{P}_b MS$. He proved the Banach contraction principle as well as the Kannan-type fixed-point theorem in $\mathscr{P}_b MS$. Additionally, Shukla provided illustrative examples to demonstrate the results obtained in this novel space. Subsequently, Mustafa and colleagues [10] established several shared fixed-point theorems within the framework of $\mathscr{P}_b MS$. In recent times, various researchers have achieved fixed-point outcomes in $\mathscr{P}_b MS$, as evidenced by publications such as [11], [12], [13], and similar works. In 2012, the concept of $\hat{\propto}$-admissible maps was introduced by Samet et al. [14], which has since found application in various studies, as demonstrated in ([15], [16]). In 2013, Karapinar et al. [17] extended this concept to triangular $\hat{\propto}$-admissible maps. More recently, Chandok [18] proposed the idea of $(\hat{\propto}, \hat{\beta})$-admissible Geraghty-type contractive maps, establishing sufficient conditions for the existence of f.p. within this class of generalized nonlinear contractive maps in metric spaces and providing several f.p. results (also see [19], [20], [21]). Berinde [22, 23] further broadened the scope by proposed the notion of almost contractions as an extension of contractive maps. Additionally, Khojasteh et al. [24] introduced the concept of $\mathscr{Z}$-contraction, which involves a new class of maps known as simulation functions, to establish f.p. results. Isik et al. [25] demonstrated fixed point theorems for nearly $\mathscr{Z}$-contraction with an application in 2018. Recently, Saluja [27] showcased fixed point results for nearly $\mathscr{Z}$-contractions in partial $\mathfrak{b}$-metric spaces with simulation functions. Gordji and Habibi [28], [29] delve into the concept of orthogonality within complete metric spaces. Additionally, Arul Joseph et al. [30], [31] introduce the notion of orthogonally triangular $\hat{\propto}$-admissible maps and present some fixed-point results for self-maps in orthogonal complete metric spaces. Recently, in 2023, Senthil Kumar et al. [32] enhance the concept of orthogonally modified F-contractions of type-I and type-II, along with certain fixed-point theorems for self-maps in orthogonal metrics. Subsequently, Mani et al. [33] concentrate on advancing fixed-point theorems related to orthogonal F-contractive type maps, orthogonal Kannan F-contractive type maps, and orthogonal F-expanding type maps. Furthermore, numerous researchers are interested in developing concepts related to orthogonality, as evidenced by citations [34] and [35].

In this investigation, we explore the existence and uniqueness of f.p. arising from orthogonal almost $\mathscr{Z}$-contractions, employing simulation functions in the context of extensive orthogonal $\mathscr{P}_bMS$, utilizing orthogonal $(\hat{\propto},\hat{\beta})$-admissibility. Furthermore, we provide several illustrative examples to corroborate our findings. Additionally, we apply our results to integro-differential equations. Our study extends, expands, and enriches various conclusions drawn from the existing literature.

2. Preliminaries

This section necessitates the inclusion of the following notion to support our primary discoveries.

Definition 2.1. *[9] A partial* $\mathfrak{b}$*-metric on a non-void set* $\Finv$ *is a function* $\mathsf{d}_{\mathfrak{bm}} : \Finv \times \Finv \to \mathbb{R}^+$ *such that for all* $\check{\intercal}, \tilde{\mathsf{O}}, \mathfrak{z} \in \Finv$*:*

(1) $\check{\intercal} = \tilde{\mathsf{O}}$ *if and only if* $\mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\check{\intercal}) = \mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\tilde{\mathsf{O}}) = \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathsf{O}},\tilde{\mathsf{O}})$*;*

(2) $\mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\check{\intercal}) \leq \mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\tilde{\mathsf{O}})$*;*

(3) $\mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\tilde{\mathsf{O}}) = \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathsf{O}},\check{\intercal})$*;*

(4) There exists $\aleph \geq 1$ *such that* $\mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\tilde{\mathsf{O}}) \leq \aleph[\mathsf{d}_{\mathfrak{bm}}(\check{\intercal},\mathfrak{z}) + \mathsf{d}_{\mathfrak{bm}}(\mathfrak{z},\tilde{\mathsf{O}})] - \mathsf{d}_{\mathfrak{bm}}(\mathfrak{z},\mathfrak{z})$.

A $\mathscr{P}_bMS$ *is a pair* $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ *such that* $\Finv$ *is a non-void set and* $\mathsf{d}_{\mathfrak{bm}}$ *is a partial* $\mathfrak{b}$*-metric on* $\Finv$*. The real number* $\aleph$ *is said to be a constant of* $(\Finv, \mathsf{d}_{\mathfrak{bm}})$.

Remark 2.1. *Every* $\mathscr{P}_bMS$ *is a generalization of the partial metric space and the* $\mathfrak{b}$*-metric space. However, converse is not true in general.*

Definition 2.2. *[18] Let* $\Finv$ *be a non-void set. Let* $\mathtt{A} : \Finv \to \Finv$ *and* $\hat{\propto}, \hat{\beta} : \Finv \times \Finv \to [0,1)$ *be given maps. We say that* $\mathtt{A}$ *is an* $(\hat{\propto},\hat{\beta})$*-admissible if* $\hat{\propto}(\check{\intercal},\tilde{\mathsf{O}}) \geq 1$ *and* $\hat{\beta}(\check{\intercal},\tilde{\mathsf{O}}) \geq 1$ *implies* $\hat{\propto}(\mathtt{A}\check{\intercal},\mathtt{A}\tilde{\mathsf{O}}) \geq 1$ *and* $\hat{\beta}(\mathtt{A}\check{\intercal},\mathtt{A}\check{\intercal}) \geq 1$ *for all* $\check{\intercal},\tilde{\mathsf{O}} \in \Finv$.

Definition 2.3. *[23] Let $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ be a metric space. A self-map $\mathtt{A} : \Finv \to \Finv$ is said to be an almost contraction if there exist $\lambda \in (0,1)$ and $\hat{\propto} \geq 0$ such that*

$$\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}\check{\mathtt{ţ}}, \mathtt{A}\tilde{\mathtt{O}}) \leq \lambda \mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}}) + \hat{\propto} \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\check{\mathtt{ţ}}),$$

for all $\check{\mathtt{ţ}}, \tilde{\mathtt{O}} \in \Finv$.

Khojasteh et al. [24] initiate the simulation function as follows:

Definition 2.4. *[24] Let $\zeta : [0,1) \times [0,1) \to \mathbb{R}$ be a function, if ζ satisfies the given conditions:*

- *($\zeta 1$) $\zeta(0,0) = 0$.*
- *($\zeta 2$) $\zeta(è, \aleph) < \aleph - è$ for all $è, \aleph > 0$.*
- *($\zeta 3$) If $(è_{\breve{ß}})$ and $(\aleph_{\breve{ß}})$ are sequences in $(0,1)$ such that $\lim_{\breve{ß} \to \infty} è_{\breve{ß}} = \lim_{\breve{ß} \to \infty} \aleph_{\breve{ß}} > 0$, then $\limsup_{\breve{ß} \to \infty} \zeta(è_{\breve{ß}}, \aleph_{\breve{ß}}) < 0$.*

Then ζ is called a simulation function.

Definition 2.5. *[25] Let $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ be a metric space and $\zeta \in \mathbb{Z}$. We say that $\mathtt{A} : \Finv \to \Finv$ is an almost $\mathscr{Z}$-contraction if there is a constant $\hat{\propto} \geq 0$ such that*

$$\zeta(\hat{\propto}(\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}\check{\mathtt{ţ}}, \mathtt{A}\tilde{\mathtt{O}})), \mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}}) + \hat{\propto}\mathscr{N}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}})) \leq 0,$$

for all $\check{\mathtt{ţ}}, \tilde{\mathtt{O}} \in \Finv$, where $\mathscr{N}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}}) = \min\{\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \mathtt{A}\check{\mathtt{ţ}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\check{\mathtt{ţ}})\}$.

Definition 2.6. *[27] Let $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ be a complete $\mathscr{P}_b$MS with constant $\aleph \geq 1$, let $\mathtt{A} : \Finv \to \Finv$ be a map, and $\hat{\propto}, \hat{\beta} : \Finv \times \Finv \to [0, +\infty)$ be given functions. Then $\mathtt{A}$ is called a $(\hat{\propto}, \hat{\beta})$-admissible almost $\mathscr{Z}$-contraction if it fulfills the below constrains:*

1. *$\mathtt{A}$ is $(\hat{\propto}, \hat{\beta})$-admissible,*
2. *there exists a simulation function $\zeta \in \mathbb{Z}$ such that*

$$0 \leq \zeta(\hat{\propto}(\mathtt{A}(\check{\mathtt{ţ}}), \mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\check{\mathtt{ţ}}), \mathtt{A}(\tilde{\mathtt{O}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{ţ}})\mathtt{A}(\tilde{\mathtt{O}})), \mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}})) + \theta \mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}}),$$

for all $\check{\mathtt{ţ}}, \tilde{\mathtt{O}} \in \Finv$, where

$$\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{ţ}}, \tilde{\mathtt{O}}) = \min\{\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \mathtt{A}\check{\mathtt{ţ}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{ţ}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\check{\mathtt{ţ}})\}.$$

Lemma 2.2. *[26]*

(1) A sequence $(\breve{\mathsf{T}}_{\check{ß}})$ is a $\mathfrak{b}$-Cauchy sequence in a $\mathscr{P}_bMS$ $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ iff it is a $\mathfrak{b}$-Cauchy sequence in the $\mathfrak{b}$-metric space $(\Finv, \mathsf{d}_{\mathfrak{bm}})$.

(2) A $\mathscr{P}_bMS$ is complete iff the $\mathfrak{b}$-metric space $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ is $\mathfrak{b}$-complete. Moreover, $\lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}, \breve{\mathsf{T}}_{\check{ß}}) = 0$ if and only if

$$\lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}, \breve{\mathsf{T}}_{\check{ß}}) = \lim_{\check{ß},\check{x}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\check{ß}}, \breve{\mathsf{T}}_{\check{x}}) = \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}, \breve{\mathsf{T}}).$$

Definition 2.7. *[28] Let $\Finv$ be a non-void and $\perp \subseteq \Finv \times \Finv$ be an binary relation. If $\perp$ fulfills the following condition:*

$$\exists\ \breve{\mathsf{T}}_0 : (\forall\ \tilde{\mathsf{O}}, \tilde{\mathsf{O}} \perp \mathfrak{r}_0)\ or\ (\forall\ \tilde{\mathsf{O}}, \breve{\mathsf{T}}_0 \perp \tilde{\mathsf{O}}),$$

then $(\Finv, \perp)$ is called an orthogonal set (O_{set}).

Definition 2.8. *[28] Let $(\Finv, \perp)$ be an O_{set}. A sequence $\{\breve{\mathsf{T}}_{\check{ß}}\}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall \check{ß} \in \mathbb{N}, \breve{\mathsf{T}}_{\check{ß}} \perp \breve{\mathsf{T}}_{\check{ß}+1}) \quad or \quad (\forall \check{ß} \in \mathbb{N}, \breve{\mathsf{T}}_{\check{ß}+1} \perp \breve{\mathsf{T}}_{\check{ß}}).$$

Definition 2.9. *Let $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ be an orthogonal $\mathscr{P}_bMS$ if $(\Finv, \perp)$ is an O_{set} and $(\Finv, \mathsf{d}_{\mathfrak{bm}})$ is a $\mathscr{P}_bMS$.*

Now, let's revisit the given concept within orthogonal $\mathscr{P}_bMS$.

Definition 2.10. *Let $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ be an orthogonal $\mathscr{P}_bMS$ with constant $\aleph$. Then:*

(1) An orthogonal sequence $(\breve{\mathsf{T}}_{\check{ß}})$ in $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ is called a convergent with respect to $\mathsf{d}_{\mathfrak{bm}}$ and converges to a point $\breve{\mathsf{T}} \in \Finv$, if $\lim\limits_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\check{ß}}, \breve{\mathsf{T}}) = \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}, \breve{\mathsf{T}})$.

(2) An orthogonal sequence $(\breve{\mathsf{T}}_{\check{ß}})$ is said to be an orthogonal Cauchy sequence in $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ if $\lim\limits_{\check{ß},\check{x}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\check{ß}}, \breve{\mathsf{T}}_{\check{x}})$ exists and is finite.

(3) $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ is said to be an orthogonal complete $\mathscr{P}_bMS$ if for every orthogonal Cauchy sequence $(\breve{\mathsf{T}}_{\check{ß}})$ in $\Finv$ there exists $\breve{\mathsf{T}} \in \Finv$ such that

$$\lim_{\check{ß},\check{x}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\check{ß}}, \breve{\mathsf{T}}_{\check{x}}) = \lim_{n\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\check{ß}}, \breve{\mathsf{T}}) = \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}, \breve{\mathsf{T}}).$$

(4) A function $\mathtt{A}\colon \Finv \to \Finv$ is said to be an O-preserving if $\mathtt{A}(\breve{\mathsf{T}}) \perp \mathtt{A}(\tilde{\mathsf{O}})$ whenever $\breve{\mathsf{T}} \perp \tilde{\mathsf{O}}$.

3. Main Results

Definition 3.1. *Let $\Finv$ be a non-void set. Let $\mathtt{A} : \Finv \to \Finv$ and $\hat{\propto}, \hat{\beta} : \Finv \times \Finv \to [0,1)$ be given maps. We say that $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible if $\hat{\propto}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) \geq 1$ and $\hat{\beta}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) \geq 1$ implies $\hat{\propto}(\mathtt{A}\breve{\mathtt{T}}, \mathtt{A}\tilde{\mathtt{O}}) \geq 1$ and $\hat{\beta}(\mathtt{A}\breve{\mathtt{T}}, \mathtt{A}\breve{\mathtt{T}}) \geq 1$ for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv$ with $\breve{\mathtt{T}} \perp \tilde{\mathtt{O}}$.*

Definition 3.2. *Let $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ be an orthogonal complete $\mathscr{P}_b$MS with constant $\aleph \geq 1$, let $\mathtt{A} : \Finv \to \Finv$ be a map, and $\hat{\propto}, \hat{\beta} : \Finv \times \Finv \to [0, +\infty)$ be given functions. We say that $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible almost $\mathscr{Z}$-contraction if it fulfills the below constrains:*

(1) $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible,

(2) there exists an orthogonal simulation function $\zeta \in \mathbb{Z}$ such that

$$
\begin{aligned}
0 \leq \zeta(\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}})), \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \tilde{\mathtt{O}})) \\
+ \theta \mho_{\mathsf{d}}^{\mathfrak{bm}}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}), \ \forall \ \breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv \ \text{ with } \breve{\mathtt{T}} \perp \tilde{\mathtt{O}},
\end{aligned} \tag{1}
$$

where

$$
\mho_{\mathsf{d}}^{\mathfrak{bm}}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) = \min\{\mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \mathtt{A}\breve{\mathtt{T}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \mathtt{A}\tilde{\mathtt{O}}), \mathsf{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}}, \mathtt{A}\breve{\mathtt{T}})\}.
$$

Lemma 3.1. *(1) An orthogonal sequence $(\breve{\mathtt{T}}_{\breve{ß}})$ is an orthogonal $\mathfrak{b}$-Cauchy sequence in orthogonal $\mathscr{P}_b$MS if and only if it is an orthogonal $\mathfrak{b}$-Cauchy sequence in the orthogonal $\mathfrak{b}$-metric space $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$.*

(2) An orthogonal $\mathscr{P}_b$MS is complete iff the orthogonal $\mathfrak{b}$-metric space $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ is an orthogonal $\mathfrak{b}$-complete. Moreover, $\lim_{\breve{ß} \to \infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \breve{\mathtt{T}}_{\breve{ß}}) = 0$ if and only if

$$
\lim_{\breve{ß} \to \infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \breve{\mathtt{T}}_{\breve{ß}}) = \lim_{\breve{ß}, \check{x} \to \infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}_{\breve{ß}}, \breve{\mathtt{T}}_{\check{x}}) = \mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \breve{\mathtt{T}}).
$$

Presently, we are prepared to demonstrate our main outcome.

Theorem 3.2. *Let $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ be an orthogonal complete $\mathscr{P}_b$MS with orthogonal element $\breve{\mathtt{T}}_0$ and a coefficient $\aleph \geq 1$, and let a continuous map $\mathtt{A} : \Finv \to \Finv$ be a $(\hat{\propto}, \hat{\beta})$ satisfies following conditions*

(1) $\mathtt{A}$ is an O-preserving,

(2) $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible almost $\mathscr{Z}$-contraction,

(3) there exists $\breve{\mathtt{T}}_0 \in \Finv$ such that $\hat{\propto}(\breve{\mathtt{T}}_0, \mathtt{A}\breve{\mathtt{T}}_0) \geq 1$ and $\hat{\beta}(\breve{\mathtt{T}}_0, \mathtt{A}\breve{\mathtt{T}}_0) \geq 1$.

Then $\mathtt{A}$ *has a unique f.p. of* $\natural \in \daleth$ *such that* $\mathsf{d}_{\mathfrak{bm}}(\natural, \natural) = 0$.

Proof. By the Definition of orthogonality, let $(\daleth, \perp)$ is an orthogonal set, there exist

$$\breve{\tau}_0 \in \daleth : \forall\ \breve{\tau} \in \daleth,\ \breve{\tau} \perp \breve{\tau}_0\ (or)\ \forall\ \breve{\tau} \in \daleth,\ \breve{\tau}_0 \perp \breve{\tau}.$$

It follows that $\breve{\tau}_0 \perp \mathtt{A}\breve{\tau}_0$ or $\mathtt{A}\breve{\tau}_0 \perp \breve{\tau}_0$. Let

$$\breve{\tau}_1 = \mathtt{A}\breve{\tau}_0,\ \breve{\tau}_2 = \mathtt{A}\breve{\tau}_1 = \mathtt{A}^2\breve{\tau}_0 \cdots \breve{\tau}_{\breve{\text{ß}}+1} = \mathtt{A}\breve{\tau}_{\breve{\text{ß}}} = \mathtt{A}^{\breve{\text{ß}}+1}\breve{\tau}_0,$$

$\forall\ \breve{\text{ß}} \in \mathbb{N}$. For any $\breve{\tau}_0 \in \daleth$, set $\breve{\tau}_{\breve{\text{ß}}} = \mathtt{A}\breve{\tau}_{\breve{\text{ß}}-1}$. Now, we consider the following two cases:

- (i) If there exists $\breve{\text{ß}} \in \mathbb{N} \cup \{0\}$ such that $\breve{\tau}_{\breve{\text{ß}}} = \breve{\tau}_{\breve{\text{ß}}+1}$, then we have $\mathtt{A}\breve{\tau}_{\breve{\text{ß}}} = \breve{\tau}_{\breve{\text{ß}}}$. It is easy to see that $\breve{\tau}_{\breve{\text{ß}}}$ is a f.p. of $\mathtt{A}$. Therefore, the proof is finished.
- (ii) If $\breve{\tau}_{\breve{\text{ß}}} \neq \breve{\tau}_{\breve{\text{ß}}+1}$, for any $\breve{\text{ß}} \in \mathbb{N} \cup \{0\}$, then we have $\mathfrak{d}_{\mathfrak{b}}(\breve{\tau}_{\breve{\text{ß}}+1}, \breve{\tau}_{\breve{\text{ß}}}) > 0$, for each $\breve{\text{ß}} \in \mathbb{N}$.

Since $\mathtt{A}$ is $\perp$-preserving, we have

$$\breve{\tau}_{\breve{\text{ß}}} \perp \breve{\tau}_{\breve{\text{ß}}+1}\ \text{(or)}\ \breve{\tau}_{\breve{\text{ß}}+1} \perp \breve{\tau}_{\breve{\text{ß}}}.$$

This implies that $\{\breve{\tau}_{\breve{\text{ß}}}\}$ is an O-sequence.
Since $\hat{\propto}(\breve{\tau}_0, \mathtt{A}\breve{\tau}_0) \geq 1$ implies $\hat{\propto}(\breve{\tau}_0, \breve{\tau}_1) \geq 1$ and $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible, so $\hat{\propto}(\mathtt{A}\breve{\tau}_0, \mathtt{A}\breve{\tau}_1) \geq 1$ implies $\hat{\beta}(\breve{\tau}_1, \breve{\tau}_2) \geq 1$.
Now, continuing in the same manner, we get for all $\breve{\text{ß}} \geq 0$,

$$\hat{\propto}(\breve{\tau}_{\breve{\text{ß}}}, \breve{\tau}_{\breve{\text{ß}}+1}) \geq 1. \tag{2}$$

Likewise, for all $\breve{\text{ß}} \geq 0$, we obtain

$$\hat{\beta}(\breve{\tau}_{\breve{\text{ß}}}, \breve{\tau}_{\breve{\text{ß}}+1}) \geq 1. \tag{3}$$

By orthogonal simulation function in (1), we obtain

$$\begin{aligned}0 \leq \zeta(&\hat{\propto}(\mathtt{A}(\breve{\tau}_{\breve{\text{ß}}-1}), \mathtt{A}(\breve{\tau}_{\breve{\text{ß}}}))\hat{\beta}(\mathtt{A}(\breve{\tau}_{\breve{\text{ß}}-1}), \mathtt{A}(\breve{\tau}_{\breve{\text{ß}}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\tau}_{\breve{\text{ß}}-1}), \mathtt{A}(\breve{\tau}_{\breve{\text{ß}}})), \mathsf{d}_{\mathfrak{bm}}(\breve{\tau}_{\breve{\text{ß}}-1}, \breve{\tau}_{\breve{\text{ß}}}))\\ &+ \theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\breve{\tau}_{\breve{\text{ß}}-1}, \breve{\tau}_{\breve{\text{ß}}}),\end{aligned}$$

that is,

$$0 \leq \zeta(\hat{\propto}(\breve{\tau}_{\breve{\text{ß}}}, \breve{\tau}_{\breve{\text{ß}}+1})\hat{\beta}(\breve{\tau}_{\breve{\text{ß}}}, \breve{\tau}_{\breve{\text{ß}}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\tau}_{\breve{\text{ß}}}, \breve{\tau}_{\breve{\text{ß}}+1}), \mathsf{d}_{\mathfrak{bm}}(\breve{\tau}_{\breve{\text{ß}}-1}\breve{\tau}_{\breve{\text{ß}}})) + \theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\breve{\tau}_{\breve{\text{ß}}-1}, \breve{\tau}_{\breve{\text{ß}}}), \tag{4}$$

where

$$\begin{aligned}\mho_{\mathsf{d}}^{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}) &= \min\{\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\mathtt{A}\breve{\mathsf{T}}_{\breve{ß}}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\mathtt{A}\breve{\mathsf{T}}_{\breve{ß}}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\mathtt{A}\breve{\mathsf{T}}_{\breve{ß}}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\mathtt{A}\breve{\mathsf{T}}_{\breve{ß}-1})\}\\ &= \min\{\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}+1}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}+1}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}})\}\\ &= 0.\end{aligned}$$

Therefore, from equations (3) and Definition 2.4 condition (2), we have

$$\begin{aligned}0 \leq& \zeta(\hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}),\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}))\\ <& \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}) - \hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}),\end{aligned}$$

that is,

$$\hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) < \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}). \tag{5}$$

Now, we know that

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) \leq \hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}), \tag{6}$$

since $\hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) \geq 1$ and $\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) \geq 1$.
From equations (5) and (6) for all $\breve{ß} \geq 0$, we have

$$\begin{aligned}\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) &\leq \hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\\ &< \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}),\end{aligned}$$

that is,

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) < \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}-1},\breve{\mathsf{T}}_{\breve{ß}}). \tag{7}$$

The orthogonal sequence $(\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}))$ is non decreasing. So, there exist $\propto \geq 0$ such that $\lim_{\breve{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) = \propto$. We clear that $\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) = 0$.
At this moment, let us consider the opposite scenario where $\propto > 0$. By (6) we get

$$\lim_{\breve{ß}\to\infty}(\hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})) = \propto . \tag{8}$$

Since $\propto > 0$, letting

$$\aleph_{\breve{ß}} = \hat{\propto}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\hat{\beta}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}},\breve{\mathsf{T}}_{\breve{ß}+1}) \text{ and}$$

$$\grave{e}_{\breve{ß}} = \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}}, \breve{\mathsf{T}}_{\breve{ß}+1}),$$

such that $\lim_{\breve{ß}\to\infty} \aleph_{\breve{ß}} = \lim_{\breve{ß}\to\infty} \grave{e}_{\breve{ß}} = \propto$, then by Definition 2.4 condition (3), $\limsup_{\breve{ß}\to\infty} \zeta(\aleph_{\breve{ß}}, \grave{e}_{\breve{ß}}) < 0$. Since $\zeta(\aleph_{\breve{ß}}, \grave{e}_{\breve{ß}}) \geq 0$. So

$$0 \leq \limsup_{\breve{ß}\to\infty} \zeta(\aleph_{\breve{ß}}, \grave{e}_{\breve{ß}}) < 0,$$

which is a contradiction.

Therefore, our assertion that $\propto > 0$ is incorrect. Hence $\propto = 0$.

Now, we prove that $\{\breve{\mathsf{T}}_{\breve{ß}}\}$ is an orthogonal Cauchy sequence in $(\dashv, \mathsf{d}_{\mathfrak{bm}})$, i.e.,

$$\lim_{\breve{ß},\breve{x}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}}, \breve{\mathsf{T}}_{\breve{x}}) = 0. \tag{9}$$

Suppose the contrary, that is, $\{\breve{\mathsf{T}}_{\breve{ß}}\}$ is not an orthogonal Cauchy sequence. Then there exist $\varepsilon > 0$ for which we can find two orthogonal sub sequences $(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}})$ of $(\breve{\mathsf{T}}_{\breve{ß}})$ such that $\breve{x}_{(\pi)}$ is the smallest index for which

$$\breve{x}_{(\pi)} > \breve{ß}_{(\pi)} > \pi, \quad \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}) \geq \varepsilon. \tag{10}$$

This means that

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) < \varepsilon. \tag{11}$$

From equation (10) and using the triangular inequality, we obtain

$$\begin{aligned}
\varepsilon &\leq \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}) \\
&\leq \aleph[\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) + \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}})] - \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \\
&\leq \aleph[\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) + \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}})].
\end{aligned} \tag{12}$$

Letting the limit as $\pi \to \infty$ in (12) and using equations (9) and (11), we obtain

$$\frac{\varepsilon}{\aleph} \leq \liminf_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \varepsilon. \tag{13}$$

Also from (10) and (12), we have

$$\varepsilon \leq \limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{ß}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}) \leq \aleph\varepsilon. \tag{14}$$

Again, we have

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}) \leq \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}) + \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{x}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}),$$

and hence

$$\limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}) \leq \aleph^2\varepsilon. \tag{15}$$

Further,

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}) \leq \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}) + \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{x}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}),$$

and hence

$$\limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}+1}) \leq \aleph\varepsilon. \tag{16}$$

Again, we have

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}) + \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}),$$

and hence

$$\limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \aleph\varepsilon. \tag{17}$$

Finally, we have

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}) + \aleph\, \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}),$$

and hence

$$\limsup_{\pi\to\infty} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \leq \aleph^2\varepsilon. \tag{18}$$

Taking limit as $\pi \to 1$ in (18) and using equation (9), we obtain

$$\limsup_{\pi\to 1} \mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_(\pi)-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1}) \to 0 \text{ as } \pi \to 1, \tag{19}$$

From equations (1) and (19), we have

$$0 \leq \limsup_{\pi\to 1} \zeta(\breve{\alpha}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}})\breve{\beta}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}})\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}}),$$

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathsf{T}}_{\breve{\text{ß}}_{(\pi)}-1}, \breve{\mathsf{T}}_{\breve{x}_{(\pi)}-1})) < 0.$$

This contradicts our assumption. So, $\{\breve{\mathsf{T}}_{\breve{\text{ß}}}\}$ is an orthogonal $\mathfrak{b}$-Cauchy sequence in $(\dashv, \perp, \mathsf{d}_{\mathfrak{bm}})$.

Next, we verify that $\mathtt{A}$ has a f.p. Since $(\daleth, \perp, \mathsf{d}_{\mathfrak{bm}})$ is an orthogonal complete $\mathscr{P}_bMS$, then by using Lemma 3.1, such that the orthogonal sequence converges to some $\natural \in \daleth$, i.e.,

$$\lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\check{\intercal}_{\check{ß}}, \natural) = 0.$$

Again, from Lemma 3.1, we obtain

$$\lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\natural, \check{\intercal}_{\check{ß}}) = \lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\check{\intercal}_{\check{ß}}, \check{\intercal}_{\check{ß}}) = \mathsf{d}_{\mathfrak{bm}}(\natural, \natural). \tag{20}$$

Now, since $\lim_{\check{ß}\to\infty} \mathsf{d}_{\mathfrak{bm}}(\natural, \check{\intercal}_{\check{ß}}) = 0$ or $\check{\intercal}_{\check{ß}} \to \natural$ as $\check{ß} \to \infty$, the continuity of $\mathtt{A}$ implies that $\mathtt{A}\check{\intercal}_{2\check{ß}} \to \mathtt{A}(\natural)$.

Since $\check{\intercal}_{2\check{ß}+1} = \mathtt{A}\check{\intercal}_{2\check{ß}}$ and $\check{\intercal}_{2\check{ß}+1} \to \natural$ as $\check{ß} \to \infty$, by uniqueness of limit, we have $\mathtt{A}(\natural) = \natural$. So, $\natural$ is a f.p. of $\mathtt{A}$.

Next, we will demonstrate the uniqueness of the f.p. $\natural$.

Uniqueness : Consider that $\mathfrak{z}, \natural \in \daleth$ are two f.p.s of $\mathtt{A}$ such that $\natural \neq \mathfrak{z}$. Therefore, we have

$$\eth(\mathfrak{z}, \natural) = \eth(\mathtt{A}\mathfrak{z}, \mathtt{A}\natural) > 0.$$

By choice of $\check{\intercal}^*$, we obtain

$$(\check{\intercal}^* \perp \natural,\ \check{\intercal}^* \perp \mathfrak{z}) \text{ or } (\natural \perp \check{\intercal}^*,\ \mathfrak{z} \perp \check{\intercal}^*).$$

Since $\mathtt{A}$ is $\perp$-preserving, we have

$$(\mathtt{A}^{\check{ß}}\check{\intercal}^* \perp \mathtt{A}^{\check{ß}}\natural,\ \mathtt{A}^{\check{ß}}\check{\intercal}^* \perp \mathtt{A}^{\check{ß}}\mathfrak{z}) \text{ or } (\mathtt{A}^{\check{ß}}\natural \perp \mathtt{A}^{\check{ß}}\check{\intercal}^*,\ \mathtt{A}^{\check{ß}}\mathfrak{z} \perp \mathtt{A}^{\check{ß}}\check{\intercal}^*)$$

for all $\check{ß} \in \mathbb{N}$.

Since $\mathtt{A}$ is an orthogonal $(\hat{\propto}, \hat{\beta})$-admissible almost $\mathscr{Z}$-contraction in (1), and taking $\check{\intercal} = \natural$ and $\tilde{\mathsf{O}} = \natural_0$, we get

$$\begin{aligned} 0 &\leq \zeta(\hat{\propto}(\mathtt{A}(\natural), \mathtt{A}(\natural_0))\hat{\beta}(\mathtt{A}(\natural), \mathtt{A}(\natural_0))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\natural), \mathtt{A}(\natural_0)), \mathsf{d}_{\mathfrak{bm}}(\natural, \natural_0)) + \theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\natural, \natural_0) \\ &= \zeta(\hat{\propto}(\mathtt{A}(\natural), \mathtt{A}(\natural_0))\hat{\beta}(\mathtt{A}(\natural), \mathtt{A}(\natural_0))\mathsf{d}_{\mathfrak{bm}}(\natural, \natural_0), \mathsf{d}_{\mathfrak{bm}}(\natural, \natural_0)) + \theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\natural, \natural_0). \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mho_{\mathsf{d}}^{\mathfrak{bm}} &= \min\{\mathsf{d}_{\mathfrak{bm}}(\natural, \mathtt{A}\natural), \mathsf{d}_{\mathfrak{bm}}(\natural_0, \mathtt{A}\natural_0), \mathsf{d}_{\mathfrak{bm}}(\natural, \mathtt{A}\natural_0), \mathsf{d}_{\mathfrak{bm}}(\natural_0, \mathtt{A}\natural)\} \\ &= \min\{\mathsf{d}_{\mathfrak{bm}}(\natural, \natural), \mathsf{d}_{\mathfrak{bm}}(\natural_0, \natural_0), \mathsf{d}_{\mathfrak{bm}}(\natural, \natural_0), \mathsf{d}_{\mathfrak{bm}}(\natural_0, \natural)\} \\ &= 0. \end{aligned} \tag{22}$$

By using (21) and (22), we get

$$
\begin{aligned}
0 &\le \zeta\big(\hat{\propto}(\mathtt{A}(\natural),\mathtt{A}(\natural_0))\hat{\beta}(\mathtt{A}(\natural),\mathtt{A}(\natural_0))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\natural),\mathtt{A}(\natural_0)),\mathsf{d}_{\mathfrak{bm}}(\natural,\natural_0)\big) \\
&\le \mathsf{d}_{\mathfrak{bm}}(\natural,\natural_0)-\hat{\propto}(\mathtt{A}(\natural),\mathtt{A}(\natural_0))\hat{\beta}(\mathtt{A}(\natural),\mathtt{A}(\natural_0))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\natural),\mathtt{A}(\natural_0)) \\
&= \mathsf{d}_{\mathfrak{bm}}(\natural,\natural_0)-\hat{\propto}(\natural,\natural_0)\hat{\beta}(\natural,\natural_0)\mathsf{d}_{\mathfrak{bm}}(\natural_0,\natural) \\
&= \mathsf{d}_{\mathfrak{bm}}(\natural,\natural_0)[1-\hat{\propto}(\natural,\natural_0)\hat{\beta}(\natural,\natural_0)]<0,
\end{aligned}
$$

which is a contradiction, since $\hat{\propto}(\natural,\natural_0)\ge 1$ and $\hat{\beta}(\natural,\natural_0)\ge 1$.

Hence, we clear that $\mathsf{d}_{\mathfrak{bm}}(\natural,\natural_0)=0$, i.e., $\natural=\natural_0$.

Therefore $\mathtt{A}$ has a unique f.p. in $\daleth$. The proof is now concluded. □

As a Corollary of Theorem 3.2, we derive the subsequent result.

Corollary 1. *If the orthogonal contractive conditions provided below hold, we reach the same conclusion as stated in Theorem* (3.2)*.*

$$
\begin{aligned}
0 &\le \zeta\big(\hat{\propto}(\check{\mathtt{T}},\mathtt{A}(\check{\mathtt{T}}))\hat{\beta}(\tilde{\mathtt{O}},\mathtt{A}(\tilde{\mathtt{O}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})+\theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})\big) \\
0 &\le \zeta\big(\hat{\propto}(\check{\mathtt{T}},\tilde{\mathtt{O}})\hat{\beta}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})+\theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})\big) \\
0 &\le \zeta\big(\hat{\propto}(\check{\mathtt{T}},\tilde{\mathtt{O}})\hat{\beta}(\check{\mathtt{T}},\tilde{\mathtt{O}})\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})+\theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})\big) \\
0 &\le \zeta\big(\hat{\propto}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\check{\mathtt{T}},\tilde{\mathtt{O}})\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})+\theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})\big)
\end{aligned}
$$

with $\check{\mathtt{T}}\perp\tilde{\mathtt{O}}$ *for all* $\check{\mathtt{T}},\tilde{\mathtt{O}}\in\daleth$*, where* $\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})$ *is as in Theorem 3.2.*

The subsequent Corollary is a consequence of the Shukla type [9].

Corollary 2. *Let* $(\daleth,\perp,\mathsf{d}_{\mathfrak{bm}})$ *be an orthogonal complete* $\mathscr{P}_b$*MS with orthogonal element* $\check{\mathtt{T}}_0$ *and a constant* $\aleph\ge 1$*, and let* $\mathtt{A}:\daleth\to\daleth$ *be a map. Assume that there exists* $\lambda\in[0,1)$ *such that*

$$
\hat{\propto}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\check{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\le\lambda\mathsf{d}_{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})+\theta\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}}),
$$

with $\check{\mathtt{T}}\perp\tilde{\mathtt{O}}$ *for all* $\check{\mathtt{T}},\tilde{\mathtt{O}}\in\daleth$*, where* $\mho_{\mathsf{d}}^{\mathfrak{bm}}(\check{\mathtt{T}},\tilde{\mathtt{O}})$ *is as in Theorem 3.2. Then* $\mathtt{A}$ *possesses a unique f.p.* $\natural\in\daleth$ *such that* $\mathsf{d}_{\mathfrak{bm}}(\natural,\natural)=0$.

Proof. By taking the simulation function

$$\zeta(\grave{e}, \aleph) = \lambda \aleph - \grave{e},$$

for all $\grave{e}, \aleph \geq 0$ and $\lambda \in [0,1)$. $\square$

If we take $\zeta(\grave{e}, \aleph) = \varphi(\aleph) - \grave{e}$, $\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}})) = \hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}})) = 1$ for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv$ with $\breve{\mathtt{T}} \perp \tilde{\mathtt{O}}$, and $\theta = 0$ in Theorem 3.2, the subsequent result is derived.

Corollary 3. *Let $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ be an orthogonal complete $\mathscr{P}_b MS$ with orthogonal element $\breve{\mathtt{T}}_0$ and a constant $\aleph \geq 1$, and let $\mathtt{A} : \Finv \to \Finv$ be a given map fulfilling*

$$\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}), \mathtt{A}(\tilde{\mathtt{O}})) \leq \varphi(\mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \tilde{\mathtt{O}})),$$

for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in X$ with $\breve{\mathtt{T}} \perp \tilde{\mathtt{O}}$, where $\varphi : [0,1) \to [0,1)$ is an upper semi-continuous function with $\varphi(\grave{e}) < \grave{e}$, $\forall\, \grave{e} > 0$ and $\varphi(0) = 0$. Then, $\mathtt{A}$ has a unique f.p. $\natural \in \Finv$ such that $\mathsf{d}_{\mathfrak{bm}}(\natural, \natural) = 0$.

Example 3.3. *Consider the set $\Finv = [0, +1)$ and let the binary relation $\perp$ on $\Finv$ by $\breve{\mathtt{T}} \perp \tilde{\mathtt{O}}$ if $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \geq 0$, for every $\breve{\mathtt{T}}\tilde{\mathtt{O}} \in \Finv$. Define $\mathsf{d}_{\mathfrak{bm}} : \Finv \times \Finv \to \mathbb{R}^+$ by*

$$\mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) = [\max\{\breve{\mathtt{T}}, \tilde{\mathtt{O}}\}]^2,$$

for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv$. Then $(\Finv, \perp, \mathsf{d}_{\mathfrak{bm}})$ is an orthogonal complete $\mathscr{P}_b MS$ with constant $\aleph = 4 > 1$. Now, define the map $\mathtt{A} : \Finv \to \Finv$ as follows

$$\mathtt{A}(\breve{\mathtt{T}}) = \begin{cases} \frac{\breve{\mathtt{T}}}{4}, & \text{if } \breve{\mathtt{T}} \in [0,1], \\ \breve{\mathtt{T}} + 2, & \text{if } \breve{\mathtt{T}} > 1, \ \forall\, \breve{\mathtt{T}} \in \Finv. \end{cases}$$

Define two functions $\hat{\propto}, \hat{\beta} : \Finv \times \Finv \to [0, +1)$ by:

$$\hat{\propto}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) = \hat{\beta}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) = \begin{cases} 1, & \text{if } \breve{\mathtt{T}}, \tilde{\mathtt{O}} \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv$ with $\breve{\mathtt{T}} \perp \tilde{\mathtt{O}}$.

Let $\zeta(\grave{e}, \aleph) = \lambda \aleph - \grave{e}$, $\forall\, \grave{e}, \aleph \in [0,1)$ and $\lambda \in [0,1)$. Since for all $\breve{\mathtt{T}}, \tilde{\mathtt{O}} \in \Finv$ such that $\hat{\propto}(\breve{\mathtt{T}}, \tilde{\mathtt{O}}) \geq 1$

and $\hat{\beta}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \geq 1$. *By definition of orthogonal* $(\hat{\propto},\hat{\beta})$*-admissible this implies that* $\breve{\mathtt{T}},\tilde{\mathtt{O}} \in [0,1]$. *So,*

$$\hat{\propto}(\mathtt{A}\breve{\mathtt{T}},\mathtt{A}\tilde{\mathtt{O}}) = \hat{\propto}(\frac{\breve{\mathtt{T}}}{4},\frac{\tilde{\mathtt{O}}}{4}) = 1.$$

Likewise

$$\hat{\beta}(\mathtt{A}\breve{\mathtt{T}},\mathtt{A}\tilde{\mathtt{O}}) = 1.$$

Now, we verify that the orthogonal contraction condition (1). *Let* $\breve{\mathtt{T}},\tilde{\mathtt{O}} \in \daleth$ *and suppose* $\breve{\mathtt{T}} \geq \tilde{\mathtt{O}}$ *such that* $\hat{\propto}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \geq 1$ *and* $\hat{\beta}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \geq 1$. *So,* $\breve{\mathtt{T}},\tilde{\mathtt{O}} \in [0,1]$. *In this case, we get*

$$\begin{aligned}
&\zeta(\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathtt{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}})) + \theta\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \\
&\qquad = \zeta(\frac{\breve{\mathtt{T}}^2}{4},\breve{\mathtt{T}}^2) + \theta\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}),
\end{aligned} \tag{23}$$

Here $\theta \geq 0$ *and*

$$\begin{aligned}
\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) &= \min\{\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\mathtt{A}\breve{\mathtt{T}}),\mathtt{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}},\mathtt{A}\tilde{\mathtt{O}}),\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\mathtt{A}\tilde{\mathtt{O}}),\mathtt{d}_{\mathfrak{bm}}(\tilde{\mathtt{O}},\mathtt{A}\breve{\mathtt{T}})\} \\
&= \min\left\{\breve{\mathtt{T}}^2,\tilde{\mathtt{O}}^2,\breve{\mathtt{T}}^2,\tilde{\mathtt{O}}^2\right\} = 0.
\end{aligned} \tag{24}$$

From equations (23) *and* (24)*, we have*

$$\begin{aligned}
&\zeta(\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathtt{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}})) + \theta\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \\
&\qquad = \zeta(\frac{\breve{\mathtt{T}}^2}{4},\breve{\mathtt{T}}^2) \\
&\qquad = \lambda\breve{\mathtt{T}}^2 - \frac{\breve{\mathtt{T}}^2}{4}.
\end{aligned}$$

If we take $\lambda = \frac{1}{2}$*, we get*

$$\begin{aligned}
&\zeta(\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathtt{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}})) + \theta\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \\
&\qquad = \frac{\breve{\mathtt{T}}^2}{2} - \frac{\breve{\mathtt{T}}^2}{4} \geq 0,
\end{aligned}$$

that is

$$\zeta(\hat{\propto}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\hat{\beta}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\mathtt{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}})),\mathtt{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}})) + \theta\mho_{\mathtt{d}}^{\mathfrak{dm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}}) \geq 0.$$

It is easy to see that $\mathtt{A}$ *is an orthogonal continuous with* $\natural = 0$.

Therefore, we have confirmed all the hypothesis stated in Theorem 3.2. Hence, $\natural = 0$ *is the unique f.p. of* $\mathtt{A}$.

4. APPLICATIONS

We investigate the presence and uniqueness of solutions to the provided impulsive integro-differential equation with fractional relaxation, employing Corollary 3 as a guiding framework.

(25)
$$\begin{cases} D^{\tau\ \mathscr{L}\mathscr{C}}D^{\Omega}\breve{\intercal}(\sigma)+\aleph_1\breve{\intercal}(\sigma)=\mathsf{q}((\sigma),\breve{\intercal}(\sigma),\mathscr{I}^{\ell}\breve{\intercal}(\sigma)), & \sigma\neq\sigma_{\mathfrak{z}\imath},\sigma\in(0,\wp),\ \aleph_1\in\mathbb{R},\\ \Delta\breve{\intercal}(\sigma_{\mathfrak{z}\imath})=\natural_{\mathfrak{z}\imath}(\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{-})), & \mathfrak{z}\imath=1,2,3,\cdots,\dot{\varsigma},\\ {}^{\mathscr{L}\mathscr{C}}D^{\Omega}\breve{\intercal}(0)={}^{\mathscr{L}\mathscr{C}}D^{\Omega}\breve{\intercal}(\wp)=0, & \breve{\intercal}(0)=\psi_1\int_0^{\wp}\breve{\intercal}(\top)d\top+\psi_2, \end{cases}$$

for all $\psi_1,\psi_1\in\mathbb{R}$, where D^{τ} be a fractional derivative of Riemann-Liouville of order τ and ${}^{\mathscr{L}\mathscr{C}}D^{\Omega}$ be a Liouville-Caputo fractional derivative of order Ω. $1<\tau<2,\ 0<\Omega<1,\ \mathscr{I}^{\ell}$ is fractional integral order $\ell\in(0,1)$ by Riemann-Liouville, and $\mathsf{q}:[0,\wp]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a nonlinear continuous function.

$\Delta\breve{\intercal}(\sigma_{\mathfrak{z}\imath})=\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{+})-\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{-})$ means the jump of $\breve{\intercal}$ at $\sigma=\sigma_{\mathfrak{z}\imath}$, $\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{+})$ and $\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{-})$ represent the right and left limits of $\breve{\intercal}(\sigma)$ at $\sigma=\sigma_{\mathfrak{z}\imath}$ respectively, $\mathfrak{z}\imath=1,2,\cdots,\dot{\varsigma}$.

We will now provide some details and results related to fractional calculus. Consider the Banach space

$$\mathscr{P}\mathscr{C}(\Phi,\dashv)=\{\breve{\intercal}:\Phi\to\dashv,\ \breve{\intercal}\in\mathscr{C}(\sigma_{\mathfrak{z}\imath},\sigma_{\mathfrak{z}\imath+1}),\dashv\},\ \mathfrak{z}\imath=0,1,2,\cdots,\dot{\varsigma}$$

and there exist $\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{-})$ and $\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{+})$, $\mathfrak{z}\imath=0,1,2,\cdots,\dot{\varsigma}$ with $\breve{\intercal}(\sigma_{\mathfrak{z}\imath}^{-})=\breve{\intercal}(\sigma_{\mathfrak{z}\imath})$ with the norm

$$\|\breve{\intercal}\|_{\mathscr{P}\mathscr{C}}=\sup\{\|\breve{\intercal}(\sigma)\|^2:\sigma\in\Phi\}.$$

Definition 4.1. *A function $\mathscr{K}:\Phi\to\mathbb{R}$ of order $\Omega>0$ has a fractional integral that is determined by*

$$\mathscr{I}^{\Omega}\mathscr{K}(\sigma)=\frac{1}{\Gamma(\Omega)}\int_0^{\sigma}(\sigma-\top)^{\Omega-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top,$$

provided the integral exists.

Definition 4.2. *A function $\mathscr{K}:\Phi\to\mathbb{R}$ of order $\Omega>0$ has a Liouville-Caputo fractional derivative specified by*

$${}^{\mathscr{L}\mathscr{C}}D^{\Omega}\mathscr{K}(\sigma)=D^{\Omega}\left[\mathscr{K}(\sigma)-\sum_{\mathfrak{z}\imath=0}^{\dot{\varsigma}-1}\frac{\mathscr{K}^{\mathfrak{z}\imath}(0)}{\mathfrak{z}!}\sigma^{\mathfrak{z}\imath}\right],$$

where $\dot{\varsigma} = [\Omega] + 1$ for $\Omega \notin \mathbb{N}_0$, $\dot{\varsigma} = \Omega$ for $\Omega \in \mathbb{N}_0$ and $D^{\Omega}_{0^+}$ is a fractional derivative of order Ω in Riemann-Liouville sense, obtained by

$$D^{\Omega}\mathscr{K}(\sigma) = D^{\dot{\varsigma}}\mathscr{I}^{\dot{\varsigma}-\Omega}\mathscr{K}(\sigma) = \frac{1}{\Gamma(\sigma-\Omega)}\frac{d^{\dot{\varsigma}}}{d^{\dot{\varsigma}}_{\sigma}}\int_0^{\sigma}(\sigma-\top)^{\dot{\varsigma}-\Omega-1}\mathscr{K}(\top)d\top.$$

For any $\breve{\top}$ belonging to $\mathscr{AC}^{\dot{\varsigma}}(\Phi)$, there exists a fractional Liouville-Caputo derivative ${}^{\mathscr{LC}}D^{\Omega}_{0^+}$. Here, it is established that

$${}^{\mathscr{LC}}D^{\Omega}\mathscr{K}(\sigma) = \mathscr{I}^{\dot{\varsigma}-\Omega}\breve{\top}^{\dot{\varsigma}}(\sigma) = \frac{1}{\Gamma(\zeta-\Omega)}\int_0^{\sigma}(\sigma-\top)^{\dot{\varsigma}-\Omega-1}\mathscr{K}^{\dot{\varsigma}}(\top)d\top.$$

Remark 4.1. *When $\Omega = \dot{\varsigma}$, we obtain ${}^{\mathscr{LC}}D^{\Omega}\mathscr{K}(\sigma) = \mathscr{K}^{\dot{\varsigma}}(\sigma)$.*

To ensure our findings, a particular lemma is important.

Lemma 4.2. *For any $\mathscr{K} \in \mathscr{C}(\Phi)$, the following equation*

$$\begin{cases} D^{\tau}\,{}^{\mathscr{LC}}D^{\Omega}\breve{\top}(\sigma) + \aleph_1\breve{\top}(\sigma) = \mathscr{K}(\sigma), & \sigma \neq \sigma_{\mathfrak{z}\imath}, \sigma \in (0,\wp),\ \aleph_1 \in \mathbb{R}, \\ \Delta\breve{\top}(\sigma_{\mathfrak{z}\imath}) = \natural_{\mathfrak{z}\imath}(\breve{\top}(\sigma^-_{\mathfrak{z}\imath})), & \mathfrak{z}\imath = 1,2,3,\cdots,\dot{\varsigma}, \\ {}^{\mathscr{LC}}D^{\Omega}\breve{\top}(0) = {}^{\mathscr{LC}}D^{\Omega}\breve{\top}(\wp) = 0, & \breve{\top}(0) = \psi_1\int_0^{\wp}\breve{\top}(\top)d\top + \psi_2,\ \psi_1,\psi_2 \in \mathbb{R}, \end{cases}$$

corresponds to the integral equation

$$\breve{\top}(\sigma) = \mathscr{I}^{\Omega+\tau}\mathscr{K}(\sigma) - \aleph_1\mathscr{I}^{\Omega+\tau}\breve{\top}(\sigma) - \frac{\sigma^{\tau+\Omega-1}}{\wp^{\tau-1}\Gamma(\tau+\Omega)}(\mathscr{I}^{\tau}\mathscr{K}(\wp)) - \aleph_1\mathscr{I}^{\tau}\breve{\top}(\wp)$$
$$+\psi_1\int_0^{\wp}\breve{\top}(\top)\mathrm{d}_{\mathfrak{bm}}\top + \psi_2 + \sum_{\mathfrak{z}\imath=1}^{\dot{\varsigma}}\natural_{\mathfrak{z}\imath}(\breve{\top}(\sigma\mathfrak{z}\imath)).$$

The following uses a few f.p. theorems to demonstrate the existence and uniqueness of the solution to the equation (25). To obtain our results, it is necessary to accept the given conditions:

(A1) There exists $\sigma_1, \sigma_2 > 0$ such that

$$|\mathsf{q}(\sigma,\breve{\top}_1,\tilde{\mathsf{O}}_1) - \mathsf{q}(\sigma,\breve{\top}_2,\tilde{\mathsf{O}}_2)| \leq \sigma_1|\breve{\top}_1 - \breve{\top}_2| + \sigma_2|\tilde{\mathsf{O}}_1 - \tilde{\mathsf{O}}_2|,$$

for any $\sigma \in \Phi$ and each $\breve{\top}_{\mathfrak{i}}, \tilde{\mathsf{O}}_{\mathfrak{i}} \in \mathbb{R}$, $\mathfrak{i} = 1,2$.

(A2) There exists $\rho > 0$ that says

$$|\natural_{\mathfrak{z}\imath}(\breve{\top}) - \natural_{\mathfrak{z}\imath}(\tilde{\mathsf{O}})| \leq \rho|\breve{\top} - \tilde{\mathsf{O}}|,\ \forall\ \breve{\top},\tilde{\mathsf{O}} \in \daleth \text{ with } \mathfrak{z}\imath = 1,2,\cdots,\dot{\varsigma}.$$

Theorem 4.3. *Let* $(A1)$ *satisfied. If*

(26)
$$\varphi = \left(\frac{(\dot{\varsigma}+1)\wp^{\Omega+\tau}}{\Gamma(\Omega+\tau+1)} + \frac{\wp^{2\tau+\Omega-1}}{\tau\wp^{\tau-1}\Gamma(\Omega+\tau)}\right)\Big(\sigma_1+\sigma_2\frac{\wp^{\eta}}{\Gamma(\eta+1)}+|\aleph_1|\Big)+|\psi_1|\wp+\dot{\varsigma}\rho<1,$$

then, the equation (25) *has a unique solution.*

Proof. Define the orthogonal relation $\perp$ on $\daleth$ by

$$\check{\mathsf{T}}\perp\tilde{\mathsf{O}} \iff \check{\mathsf{T}}(\sigma)\tilde{\mathsf{O}}(\sigma)\geq\check{\mathsf{T}}(\sigma) \quad or \quad \check{\mathsf{T}}(\sigma)\tilde{\mathsf{O}}(\sigma)\geq\tilde{\mathsf{O}}(\sigma),\ \forall\ \sigma\in\Phi.$$

Define a function $\mathsf{d}_{\mathfrak{bm}}:\daleth\times\daleth\to[0,\infty)$ by

$$\mathsf{d}_{\mathfrak{bm}}(\check{\mathsf{T}},\tilde{\mathsf{O}}) = ||\check{\mathsf{T}}-\tilde{\mathsf{O}}||_\infty = \sup\{|\check{\mathsf{T}}(\sigma)-\tilde{\mathsf{O}}(\sigma)|^2\},\ \forall\ \check{\mathsf{T}},\tilde{\mathsf{O}}\in\daleth.$$

Clearly, $(\daleth,\perp,\mathsf{d}_{\mathfrak{bm}})$ is an O-complete $\mathscr{P}_b\mathscr{M}\mathscr{S}$.

Define a map $\mathtt{A}:\daleth\to\daleth$, as follows

$$\begin{aligned}(\mathtt{A}\check{\mathsf{T}})(\sigma) = {} & \frac{1}{\Gamma(\Omega+\tau)}\sum_{0<\sigma_{\mathfrak{z}\imath}<\sigma}\Big(\int_{\sigma_{\mathfrak{z}\imath}-1}^{\sigma_{\mathfrak{z}\imath}}(\sigma_{\mathfrak{z}\imath}-\top)^{\Omega+\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top \\ & -\aleph_1\int_{\sigma_{\mathfrak{z}\imath}-1}^{\sigma_{\mathfrak{z}\imath}}(\sigma_{\mathfrak{z}\imath}-\top)^{\Omega+\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big) \\ & +\frac{1}{\Gamma(\Omega+\tau)}\Big(\int_{\sigma_{\dot{\varsigma}}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top-\aleph_1\int_{\sigma_{\dot{\varsigma}}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big) \\ & -\frac{\sigma^{\tau+\Omega-1}}{\wp^{\tau-1}\Gamma(\tau+\Omega)}\Big(\int_0^{\wp}(\wp-\top)^{\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top-\aleph_1\int_0^{\wp}(\wp-\top)^{\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big) \\ & +\psi_1\int_0^{\wp}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top+\psi_2+\sum_{0<\sigma_{\mathfrak{z}\imath}<\sigma}\natural_{\mathfrak{z}\imath}(\check{\mathsf{T}}(\sigma_{\mathfrak{z}\imath}^-)).\end{aligned}$$

Now, we prove that $\mathtt{A}$ is $\perp$-preserving. For every $\check{\mathsf{T}},\tilde{\mathsf{O}}\in\daleth$ with $\check{\mathsf{T}}\perp\tilde{\mathsf{O}}$ and $\sigma\in\Phi$, we get

$$\begin{aligned}(\mathtt{A}\check{\mathsf{T}})(\sigma) = {} & \frac{1}{\Gamma(\Omega+\tau)}\sum_{0<\sigma_{\mathfrak{z}\imath}<\sigma}\Big(\int_{\sigma_{\mathfrak{z}\imath}-1}^{\sigma_{\mathfrak{z}\imath}}(\sigma_{\mathfrak{z}\imath}-\top)^{\Omega+\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top \\ & -\aleph_1\int_{\sigma_{\mathfrak{z}\imath}-1}^{\sigma_{\mathfrak{z}\imath}}(\sigma_{\mathfrak{z}\imath}-\top)^{\Omega+\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big) \\ & +\frac{1}{\Gamma(\Omega+\tau)}\Big(\int_{\sigma_{\dot{\varsigma}}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top-\aleph_1\int_{\sigma_{\dot{\varsigma}}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big) \\ & -\frac{\sigma^{\tau+\Omega-1}}{\wp^{\tau-1}\Gamma(\tau+\Omega)}\Big(\int_0^{\wp}(\wp-\top)^{\tau-1}\mathscr{K}(\top)\mathsf{d}_{\mathfrak{bm}}\top-\aleph_1\int_0^{\wp}(\wp-\top)^{\tau-1}\check{\mathsf{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top\Big)\end{aligned}$$

$$+\psi_1\int_0^{\wp}\breve{\mathtt{T}}(\top)\mathsf{d}_{\mathfrak{bm}}\top+\psi_2+\sum_{0<\sigma_{\mathfrak{z}1}<\sigma}\natural_{\mathfrak{z}1}(\breve{\mathtt{T}}(\sigma_{\mathfrak{z}1}^{-}))\geq 1.$$

It follows that $[(\mathtt{A}\breve{\mathtt{T}})(\sigma)][(\mathtt{A}\tilde{\mathtt{O}})(\sigma)]\geq(\mathtt{A}\tilde{\mathtt{O}})(\sigma)$ and so $(\mathtt{A}\breve{\mathtt{T}})(\sigma)\perp(\mathtt{A}\tilde{\mathtt{O}})(\sigma)$.

Then, $\mathtt{A}$ is $\perp$-preserving.

Let $\breve{\mathtt{T}},\tilde{\mathtt{O}}\in\daleth$ with $\breve{\mathtt{T}}\perp\tilde{\mathtt{O}}$. Assume that $\breve{\mathtt{T}}\sigma\neq\tilde{\mathtt{O}}\sigma$, for every $\breve{\mathtt{T}},\tilde{\mathtt{O}}\in\daleth$ and $\sigma\in\Phi$, we obtain

$$\begin{aligned}
&\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))=|\mathtt{A}(\breve{\mathtt{T}})-\mathtt{A}(\tilde{\mathtt{O}})|^2\\
&\leq\frac{1}{\Gamma(\Omega+\tau)}\sum_{0<\sigma_{\mathfrak{z}1}<\sigma}\Big(\int_{\sigma_{\mathfrak{z}1}-1}^{\sigma_{\mathfrak{z}1}}(\sigma_{\mathfrak{z}1}-\top)^{\Omega+\tau-1}\\
&\quad|\mathtt{A}(\top,\breve{\mathtt{T}}(\top),\mathscr{I}^{\eta}\breve{\mathtt{T}}(\top))-\mathtt{A}(\top,\tilde{\mathtt{O}}(\top),\mathscr{I}^{\eta}\tilde{\mathtt{O}}(\top))|^2\mathsf{d}_{\mathfrak{bm}}\top\Big)\\
&\quad+\frac{1}{\Gamma(\Omega+\tau)}\Big(\int_{\sigma_{\dot\varsigma}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}|\mathtt{A}(\top,\breve{\mathtt{T}}(\top),\mathscr{I}^{\eta}\breve{\mathtt{T}}(\top))-\mathtt{A}(\top,\tilde{\mathtt{O}}(\top),\mathscr{I}^{\eta}\tilde{\mathtt{O}}(\top))|^2\mathsf{d}_{\mathfrak{bm}}\top\Big)\\
&\quad+\frac{|\aleph_1|}{\Gamma(\Omega+\tau)}\sum_{0<\sigma_{\mathfrak{z}1}<\sigma}\int_{\sigma_{\mathfrak{z}1}-1}^{\sigma_{\mathfrak{z}1}}(\sigma_{\mathfrak{z}1}-\top)^{\Omega+\tau-1}|\breve{\mathtt{T}}(\top)-\tilde{\mathtt{O}}(\top)|^2\mathsf{d}_{\mathfrak{bm}}\top\\
&\quad+\frac{|\aleph_1|}{\Gamma(\Omega+\tau)}\int_{\sigma_{\dot\varsigma}}^{\sigma}(\sigma-\top)^{\Omega+\tau-1}|\breve{\mathtt{T}}(\top)-\tilde{\mathtt{O}}(\top)|^2\mathsf{d}_{\mathfrak{bm}}\top\\
&\quad+\frac{\sigma^{\tau+\Omega-1}}{\wp^{\tau-1}\Gamma(\tau+\Omega)}\Bigg(\int_0^{\wp}(\wp-\top)^{\tau-1}|\mathtt{A}(\top,\breve{\mathtt{T}}(\top),\mathscr{I}^{\eta}\breve{\mathtt{T}}(\top))-\mathtt{A}(\top,\tilde{\mathtt{O}}(\top),\mathscr{I}^{\eta}\tilde{\mathtt{O}}(\top))|^2\mathsf{d}_{\mathfrak{bm}}\top\\
&\quad-|\aleph_1|\int_0^{\wp}(\wp-\top)^{\tau-1}|\breve{\mathtt{T}}(\top)-\tilde{\mathtt{O}}(\top)|^2\mathsf{d}_{\mathfrak{bm}}\top\Bigg)+\psi_1\int_0^{\wp}|\breve{\mathtt{T}}(\top)-\tilde{\mathtt{O}}(\top)|^2\mathsf{d}_{\mathfrak{bm}}\top\\
&\quad+\sum_{\mathfrak{z}1=1}|\natural_{\mathfrak{z}1}(\breve{\mathtt{T}}(\sigma_{\mathfrak{z}1}^{-}))-\natural_{\mathfrak{z}1}(\tilde{\mathtt{O}}(\sigma_{\mathfrak{z}1}^{-}))|^2\\
&\leq\Bigg[\Bigg(\frac{(\dot\varsigma+1)\wp^{\Omega+\tau}}{\Gamma(\Omega+\tau+1)}+\frac{\wp^{2\tau+\Omega-1}}{\tau\wp^{\tau-1}\Gamma(\Omega+\tau)}\Bigg)\Big(\sigma_1+\sigma_2\frac{\wp^{\eta}}{\Gamma(\eta+1)}+|\aleph_1|\Big)+|\psi_1|\wp+\dot\varsigma\rho\Bigg]|\breve{\mathtt{T}}-\tilde{\mathtt{O}}|^2.
\end{aligned}$$

Thus we obtain

$$\|\mathtt{A}(\breve{\mathtt{T}})-\mathtt{A}(\tilde{\mathtt{O}})\|\leq\varphi\|\breve{\mathtt{T}}-\tilde{\mathtt{O}}\|. \tag{27}$$

We find that $\mathtt{A}$ is a contraction by using (27), we get

$$\mathsf{d}_{\mathfrak{bm}}(\mathtt{A}(\breve{\mathtt{T}}),\mathtt{A}(\tilde{\mathtt{O}}))\leq\varphi(\mathsf{d}_{\mathfrak{bm}}(\breve{\mathtt{T}},\tilde{\mathtt{O}})).$$

Therefore, all the conditions are fulfilled on Corollary 3. As a result, $\mathtt{A}$ is a unique solution to the equation (25) on Φ. $\square$

Example 4.4. *Consider the fractional relaxation impulsive integro-differential equation as follows*

$$(28)\qquad \begin{cases} D^{\frac{3}{2}}\ {}^{\mathscr{LC}}D^{\frac{1}{2}}\breve{\mathtt{T}}(\sigma)+\frac{1}{4}\breve{\mathtt{T}}(\sigma)=\mathtt{q}((\sigma),\breve{\mathtt{T}}(\sigma),\mathscr{I}^{\frac{1}{3}}\breve{\mathtt{T}}(\sigma)),\ \sigma\neq\sigma_{\mathfrak{z}\mathtt{1}},\sigma\in(0,1),\\ \Delta\breve{\mathtt{T}}(\sigma_{\mathfrak{z}\mathtt{1}})=\natural_{\mathfrak{z}\mathtt{1}}(\breve{\mathtt{T}}(\sigma_{\mathfrak{z}\mathtt{1}}^{-})),\ \mathfrak{z}\mathtt{1}=1,2,3,\cdots,\dot{\varsigma},\\ {}^{\mathscr{LC}}D^{\frac{1}{2}}\breve{\mathtt{T}}(0)={}^{\mathscr{LC}}D^{\frac{1}{2}}\breve{\mathtt{T}}(1)=0,\ \breve{\mathtt{T}}(0)=\frac{1}{10}\int_0^1\breve{\mathtt{T}}(\top)d\top+2. \end{cases}$$

Here $\Omega=\frac{1}{2},\ \tau=\frac{3}{2},\ \eta=\frac{1}{3},\ \aleph_1=\frac{1}{4},\ \psi_1=\frac{1}{10}$, *and* $\psi_2=2$. *Set*

$$\mathtt{A}((\sigma),\breve{\mathtt{T}}(\sigma),\mathscr{I}^{\frac{1}{3}}\breve{\mathtt{T}}(\sigma))=\frac{\sin(\sigma)}{\exp(\sigma^2)+7}\left(\frac{|\breve{\mathtt{T}}(\sigma)|}{|\breve{\mathtt{T}}(\sigma)|+1}+\frac{|\mathscr{I}^{\frac{1}{3}}\breve{\mathtt{T}}(\sigma)|}{1+|\mathscr{I}^{\frac{1}{3}}\breve{\mathtt{T}}(\sigma)|}\right).$$

For $\breve{\mathtt{T}}_{\mathtt{i}},\tilde{\mathtt{O}}_{\mathtt{i}}\in\mathbb{R},\ \mathtt{i}=1,2$ *we get*

$$\begin{aligned} &|\mathtt{A}(\sigma,\breve{\mathtt{T}}_1,\breve{\mathtt{T}}_2)-\mathtt{A}(\sigma,\tilde{\mathtt{O}}_1,\tilde{\mathtt{O}}_2)|\\ &\qquad=\Big|\frac{\sin(\sigma)}{\exp(\sigma^2)+7}\Big(\Big(\frac{|\breve{\mathtt{T}}_1|}{|\breve{\mathtt{T}}_1|+1}-\frac{|\tilde{\mathtt{O}}_1|}{|\tilde{\mathtt{O}}_1|+1}\Big)+\Big(\frac{|\breve{\mathtt{T}}_2|}{|\breve{\mathtt{T}}_2|+1}-\frac{|\tilde{\mathtt{O}}_2|}{|\tilde{\mathtt{O}}_2|+1}\Big)\Big)\Big|\\ &\qquad\leq\frac{1}{\exp(\sigma^2)+7}\Big(\frac{|\breve{\mathtt{T}}_1-\tilde{\mathtt{O}}_1|}{(|\breve{\mathtt{T}}_1|+1)(|\tilde{\mathtt{O}}_1|+1)}+\frac{|\breve{\mathtt{T}}_2-\tilde{\mathtt{O}}_2|}{(|\breve{\mathtt{T}}_2|+1)(|\tilde{\mathtt{O}}_2|+1)}\Big)\\ &\qquad\leq\frac{1}{8}(|\breve{\mathtt{T}}_1-\tilde{\mathtt{O}}_1|+|\breve{\mathtt{T}}_2-\tilde{\mathtt{O}}_2|). \end{aligned}$$

Thus the assumption $(A1)$ *is satisfied with* $\sigma_1=\sigma_2=\frac{1}{8},\ \wp=1,\ \rho=\frac{1}{7}$ *and* $\dot{\varsigma}=1$. *We will confirm that* (26) *is fulfilled. In fact*

$$\begin{aligned} \varphi&=\left(\frac{(\dot{\varsigma}+1)\wp^{\Omega+\tau}}{\Gamma(\Omega+\tau+1)}+\frac{\wp^{2\tau+\Omega-1}}{\tau\wp^{\tau-1}\Gamma(\Omega+\tau)}\right)\left(\sigma_1+\sigma_2\frac{\wp^{\eta}}{\Gamma(\eta+1)}+|\aleph_1|\right)+|\psi_1|\wp+\dot{\varsigma}\rho\\ &=\Big(\frac{1}{\Gamma(3)}+\frac{2}{3\Gamma(2)}\Big)\Big(\frac{1}{8}+\frac{1}{8}\frac{1}{\Gamma(\frac{1}{3}+1)}+\frac{1}{4}\Big)+\frac{1}{10}+\frac{1}{7}\\ &\simeq 0.842<1. \end{aligned}$$

Thus, the problem (28) *has a unique solution on* $[0,1]$ *based on the Theorem 4.3.*

5. Conclusions

In this paper, we investigate the existence and uniqueness of a f.p. of almost $\mathscr{Z}$-contractions via simulation function in complete $\mathscr{P}_bMS$ using $(\hat{\propto},\hat{\beta})$-admissibility. Additionally, we provide illustrative examples to corroborate the findings. Furthermore, an application to the integro-differential domain is presented. Our findings expand and generalize numerous results previously documented in the literature.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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