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ON SOME QUADRUPLE FIXED POINTS OF COVARIANT MAPPINGS IN BIPOLAR METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove that common quadruple fixed point solutions of covariant mappings in complete bipolar metric spaces exist and are unique. Additionally, we discussed an example that shows how the obtained results are applied, as well as applications to integral equations and homotopy theory.

Keywords: bipolar metric space; ω-compatible mappings; completeness; common quadruple fixed point.

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1. Introduction

In nonlinear analysis, fixed-point theory is a well-known field. It has been demonstrated that the study of many equation forms that occur in the fields of physical, biological, social, engineering, and other science and technology has essential importance. It is frequently used to examine the conditions under which solutions to single or multivalued mappings exist.

Recently, Mutlu and Gürdal [1] proposed the idea of bipolar metric spaces and gave coupled fixed point solutions for covariant and contravariant contractive mappings ([2]-[7]).

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Using the concept of quadruple fixed point, E. Karapinar [8] recently demonstrated certain quadruple fixed results in partially ordered metric spaces. Then, in various metric spaces, several researchers ([9]-[15]) developed quadruple fixed theorems.

In this work we investigated the quadruple fixed point solutions of two covariant mappings in complete bipolar metric spaces. and we have shown an example which support the our main result, also we have discussed an applications to integral equation and to homotopy theory.

2. PRELIMINARIES

Definition 2.1 ([1]). *A Bipolar-metric on a pair of non-empty sets* $(\mathfrak{A},\mathfrak{B})$ *is defined as the mapping* $d:\mathfrak{A}\times\mathfrak{B}\rightarrow[0,\infty)$.*If, for any* $æ, æ_1, æ_2 \in \mathfrak{A}$ *and* $œ, œ_1, œ_2 \in \mathfrak{B}$.

- (B_1) $d(æ, œ) = 0$ *implies that* $æ = œ$;
- (B_2) $æ = œ$ *implies that* $d(æ, œ) = 0$;
- (B_3) *if* $(æ, œ) \in (\mathfrak{A},\mathfrak{B})$, *then* $d(æ, œ) = d(œ, æ)$;
- (B_4) $d(æ_1, œ_2) \leq d(æ_1, œ_1) + d(æ_2, œ_1) + d(æ_2, œ_2)$.

And the triple $(\mathfrak{A},\mathfrak{B},d)$ *is Bipolar-metric space.*

Example 2.2 ([1]). *Assume that* $\mathfrak{B} = [-1,1]$ *and* $\mathfrak{A} = (1,\infty)$. *Define a mapping* $d:\mathfrak{A}\times\mathfrak{B}\rightarrow[0,+\infty)$ *such that, for every* $(\eta,\theta)\in(\mathfrak{A},\mathfrak{B})$, $d(\eta,\theta) = |\eta^2-\theta^2|$. *A Bipolar-metric space is then the triple* $(\mathfrak{A},\mathfrak{B},d)$.

Example 2.3 ([1]). *For all* $(\psi,a)\in(\mathfrak{A},\mathfrak{B})$, *let* $d:\mathfrak{A}\times\mathfrak{B}\rightarrow[0,+\infty)$ *be defined as* $d(\psi,a) = \psi(a)$. *The set of all functions is* $\mathfrak{A} = \{\psi/\psi:\mathbb{R}\rightarrow[1,3]\}$, *and* $\mathfrak{B} = \mathbb{R}$. *Then, a disjoint Bipolar-metric space is the triple* $(\mathfrak{A},\mathfrak{B},d)$.

Definition 2.4 ([1]). *A function defined on two pairs of sets,* $(\mathfrak{D}_1,\mathfrak{E}_1)$ *and* $(\mathfrak{D}_2,\mathfrak{E}_2)$, *is said to be* $\Omega:\mathfrak{D}_1\cup\mathfrak{E}_1\rightarrow\mathfrak{D}_2\cup\mathfrak{E}_2$.

- (i) *covariant if* $\Omega(\mathfrak{D}_1)\subseteq\mathfrak{D}_2$ *and* $\Omega(\mathfrak{E}_1)\subseteq\mathfrak{E}_2$. *This is denoted as* $\Omega:(\mathfrak{D}_1,\mathfrak{E}_1)\rightrightarrows(\mathfrak{D}_2,\mathfrak{E}_2)$;
- (ii) *contravariant if* $\Omega(\mathfrak{D}_1)\subseteq\mathfrak{E}_2$ *and* $\Omega(\mathfrak{E}_1)\subseteq\mathfrak{D}_2$. *It is denoted as* $\Omega:(\mathfrak{D}_1,\mathfrak{E}_1)\leftrightharpoons(\mathfrak{D}_2,\mathfrak{E}_2)$.

Particularly, if d_1 is bipolar metrics on $(\mathfrak{D}_1, \mathfrak{E}_1)$ and d_2 is bipolar metrics on $(\mathfrak{D}_2, \mathfrak{E}_2)$, we often write $\Omega : (\mathfrak{D}_1, \mathfrak{E}_1, d_1) \rightrightarrows (\mathfrak{D}_2, \mathfrak{E}_2, d_2)$ *and*
$\Omega : (\mathfrak{D}_1, \mathfrak{E}_1, d_1) \leftrightharpoons (\mathfrak{D}_2, \mathfrak{E}_2, d_2)$ *respectively.*

Definition 2.5 ([1])**.** (i) *Such* æ *is a left point if* $æ \in \mathfrak{A}$*;*

(ii) *Such* æ *is a right point if* $æ \in \mathfrak{B}$*;*

(iii) *Such* æ *is a central point if it is both left and right.*

$\{æ_i\}$ *and* $\{œ_i\}$ *are both convergent, then* $(\{æ_i\}, \{œ_i\})$ *is convergent.*

The bi-sequence $(\{æ_i\}, \{œ_i\})$ *is a Cauchy bisequence if* $\lim_{i,j\to\infty} d(æ_i, œ_j) = 0$.

Every convergent Cauchy bisequence is biconvergent, as you can see. If every Cauchy bisequence is convergent, then the bipolar metric space is complete (and so it is biconvergent).

The reader go through ([1], [2]) for more characteristics of a bipolar metric.

3. Main Results

Definition 3.1. *Let* $\Omega : (\mathfrak{A}^4, \mathfrak{B}^4) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *be a covariant mapping. Let* $(\mathfrak{A}, \mathfrak{B}, d)$ *be a bipolar metric space. If* $\Omega(æ, œ, ß, Æ) = æ$, $\Omega(œ, ß, Æ, æ) = œ$, $\Omega(ß, Æ, æ, œ) = ß$ *and* $\Omega(Æ, æ, œ, ß) = Æ$*, for* $æ, œ, ß, Æ \in \mathfrak{A} \cup \mathfrak{B}$*, then* $(æ, œ, ß, Æ)$ *is referred to as a quadruple fixed point of* Ω*.*

Definition 3.2. $\Omega : (\mathfrak{A}^4, \mathfrak{B}^4) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *and* $\tau : (\mathfrak{A}, \mathfrak{B}) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *are two covariant mappings. Let* $(\mathfrak{A}, \mathfrak{B}, d)$ *be a bipolar metric space.* $(æ, œ, ß, Æ)$ *is an element that quaruple coincide of* Ω *and* τ *,if* $\Omega(æ, œ, ß, Æ) = \tau æ$, $\Omega(œ, ß, Æ, æ) = \tau œ$, $\Omega(ß, Æ, æ, œ) = \tau ß$, *and* $\Omega(Æ, æ, œ, ß) = \tau Æ$.

Definition 3.3. $\Omega : (\mathfrak{A}^4, \mathfrak{B}^4) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *and* $\tau : (\mathfrak{A}, \mathfrak{B}) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *are two covariant mappings, let* $(\mathfrak{A}, \mathfrak{B}, d)$ *be a bipolar metric space. An element* $(æ, œ, ß, Æ)$ *is considered to be* Ω *and* τ*'s quadruple fixed point.If for* $\Omega(æ, œ, ß, Æ) = \tau æ = æ$, $\Omega(œ, ß, Æ, æ) = \tau œ = œ$, $\Omega(ß, Æ, æ, œ) = \tau ß = ß$ *and* $\Omega(Æ, æ, œ, ß) = \tau Æ = Æ$.

Definition 3.4. *Let* $\Omega : (\mathfrak{A}^4, \mathfrak{B}^4)$ *and* $\tau : (\mathfrak{A}, \mathfrak{B}) \rightrightarrows (\mathfrak{A}, \mathfrak{B})$ *be two covariant mappings are called* ω*-compatible, if* $\tau(\Omega(æ, œ, ß, Æ)) = \Omega(\tau æ, \tau œ, \tau ß, \tau Æ)$, $\tau(\Omega(œ, ß, Æ, æ)) = \Omega(\tau œ, \tau ß, \tau Æ, \tau æ)$, $\tau(\Omega(ß, Æ, æ, œ)) = \Omega(\tau ß, \tau Æ, \tau æ, \tau œ)$ *and*

$\tau(\Omega(Æ,æ,œ,ß)) = \Omega(\tau Æ,\tau æ,\tau œ,\tau ß)$ *whenever* $\Omega(æ,œ,ß,Æ) = \tau æ$, $\Omega(œ,ß,Æ,æ) = \tau œ$, $\Omega(ß,Æ,æ,œ) = \tau ß$ *and* $\Omega(Æ,æ,œ,ß) = \tau Æ$.

Theorem 3.5. *Let* $(\mathfrak{A},\mathfrak{B},d)$ *is bipolar metric space. Suppose* $\Omega : (\mathfrak{A}^4,\mathfrak{B}^4) \to (\mathfrak{A},\mathfrak{B})$ *and* $\tau : (\mathfrak{A},\mathfrak{B}) \to (\mathfrak{A},\mathfrak{B})$ *be a two covirant mappings satisfying*

$$d(\Omega(æ,œ,ß,Æ),\Omega(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w}) \leq \theta \max\left\{ d(\tau æ,\tau\mathfrak{x}), d(\tau œ,\tau\mathfrak{y}), d(\tau ß,\tau\mathfrak{z}), d(\tau Æ,\tau\mathfrak{w}) \right\} \tag{3.1}$$

for all $æ,œ,ß,Æ \in \mathfrak{A}$, $\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w} \in \mathfrak{B}$, $\theta \in (0,1)$ *and*

a) $\Omega(\mathfrak{A}^4 \cup \mathfrak{B}^4) \subseteq \tau(\mathfrak{A}\cup\mathfrak{B})$ *and* $\tau(\mathfrak{A}\cup\mathfrak{B})$ *is complete,*

b) pair (Ω,τ) *is* ω*-compatible.*

Then there is a unique common quadruple fixed point of Ω,τ *in* $\mathfrak{A}\cup\mathfrak{B}$.

Proof. Let $æ_0,œ_0,ß_0,Æ_0 \in \mathfrak{A}$ and $\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0 \in \mathfrak{B}$ be arbitrary, and from (a), we can construct the bisequences $(\{\alpha_p\},\{\zeta_p\}),(\{\beta_p\},\{\eta_p\})$, $(\{\gamma_p\},\{\chi_p\}),(\{\kappa_p\},\{\nu_p\})$ in $(\mathfrak{A},\mathfrak{B})$ as

$$\begin{aligned}
&\Omega(æ_p,œ_p,ß_p,Æ_p) = \tau æ_{p+1} = \alpha_p, \quad \Omega(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p) = \tau\mathfrak{x}_{p+1} = \zeta_p\\
&\Omega(œ_p,ß_p,Æ_p,æ_p) = \tau œ_{p+1} = \beta_p, \quad \Omega(\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p) = \tau\mathfrak{y}_{p+1} = \eta_p\\
&\Omega(ß_p,Æ_p,æ_p,œ_p) = \tau ß_{p+1} = \gamma_p, \quad \Omega(\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p) = \tau\mathfrak{z}_{p+1} = \chi_p\\
&\Omega(Æ_p,æ_p,œ_p,ß_p,) = \tau Æ_{p+1} = \kappa_p, \quad \Omega(\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p) = \tau\mathfrak{w}_{p+1} = \nu_p
\end{aligned}$$

where $p = 0,1,2,\ldots.$

From eqn (3.1) we have

$$\begin{aligned}
d(\alpha_p,\zeta_{p+1}) &= d(\Omega(æ_p,œ_p,ß_p,Æ_p),\Omega(\mathfrak{x}_{p+1},\mathfrak{y}_{p+1},\mathfrak{z}_{p+1},\mathfrak{w}_{p+1}))\\
&\leq \theta\max\left\{ d(\tau æ_p,\tau\mathfrak{x}_{p+1}), d(\tau œ_p,\tau\mathfrak{y}_{p+1}), d(\tau ß_p,\tau\mathfrak{z}_{p+1}), d(\tau Æ_p,\tau\mathfrak{w}_{p+1}) \right\}\\
&\leq \theta\max\left\{ d(\alpha_{p-1},\zeta_p), d(\beta_{p-1},\eta_p), d(\gamma_{p-1},\chi_p), d(\kappa_{p-1},\nu_p) \right\} \qquad (3.2)
\end{aligned}$$

Similarly,

$$d(\beta_p,\eta_{p+1}) \leq \theta\max(d(\alpha_{p-1},\zeta_p), d(\beta_{p-1},\eta_p), d(\gamma_{p-1},\chi_p), d(\kappa_{p-1},\nu_p)) \tag{3.3}$$

and

$$d(\gamma_p,\chi_{p+1}) \leq \theta\max(d(\alpha_{p-1},\zeta_p), d(\beta_{p-1},\eta_p), d(\gamma_{p-1},\chi_p), d(\kappa_{p-1},\nu_p)) \tag{3.4}$$

also

$$d\left(\kappa_p, \nu_{p+1}\right) \leq \theta \max\left(d(\alpha_{p-1}, \zeta_p), d(\beta_{p-1}, \eta_p), d(\gamma_{p-1}, \chi_p) + d(\kappa_{p-1}, \nu_p)\right). \tag{3.5}$$

From eqns (3.2)-(3.5), we conclude that

$$\begin{aligned}
\max\left\{\begin{array}{c} d\left(\alpha_p, \zeta_{p+1}\right), \\ d\left(\beta_p, \eta_{p+1}\right), \\ d\left(\gamma_p, \chi_{p+1}\right), \\ d\left(\kappa_p, \nu_{p+1}\right) \end{array}\right\} &\leq \theta \max\left\{\begin{array}{c} d(\alpha_{p-1}, \zeta_p), \\ d(\beta_{p-1}, \eta_p), \\ d(\gamma_{p-1}, \chi_p), \\ d(\kappa_{p-1}, \nu_p) \end{array}\right\} \\
&\leq \theta^2 \max\left\{\begin{array}{c} d(\alpha_{p-2}, \zeta_{p-1}), \\ d(\beta_{p-2}, \eta_{p-1}), \\ d(\gamma_{p-2}, \chi_{p-1}), \\ d(\kappa_{p-2}, \nu_{p-1}) \end{array}\right\} \\
&\vdots \\
&\leq \theta^p \max\left\{\begin{array}{c} d(\alpha_0, \zeta_1), \\ d(\beta_0, \eta_1), \\ d(\gamma_0, \chi_1), \\ d(\kappa_0, \nu_1) \end{array}\right\}
\end{aligned}$$

Which means that

$$d\left(\alpha_p, \zeta_{p+1}\right) \leq \theta^p \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1), d(\gamma_0, \chi_1), d(\kappa_0, \nu_1) \right\}$$

and

$$d\left(\beta_p, \eta_{p+1}\right) \leq \theta^p \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1), d(\gamma_0, \chi_1), d(\kappa_0, \nu_1) \right\}$$

and

$$d\left(\gamma_p, \chi_{p+1}\right) \leq \theta^p \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1), d(\gamma_0, \chi_1), d(\kappa_0, \nu_1) \right\}$$

also

$$d\left(\kappa_p, \nu_{p+1}\right) \leq \theta^p \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1), d(\gamma_0, \chi_1), d(\kappa_0, \nu_1) \right\}$$

On the other hand

$$
\begin{aligned}
d\left(\alpha_{p+1},\zeta_p\right) &= d\left(\Omega\left(\text{æ}_{p+1},\text{œ}_{p+1},\text{ß}_{p+1},\text{Æ}_{p+1}\right),\Omega\left(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p\right)\right)\\
&\leq \theta\max\left\{\; d(\tau\text{æ}_{p+1},\tau\mathfrak{x}_p), d(\tau\text{œ}_{p+1},\tau\mathfrak{y}_p), d(\tau\text{ß}_{p+1},\tau\mathfrak{z}_p), d(\tau\text{Æ}_{p+1},\tau\mathfrak{w}_p)\;\right\}\\
&\leq \theta\max\left\{\; d(\alpha_p,\zeta_{p-1}), d(\beta_p,\eta_{p-1}), d(\gamma_p,\chi_{p-1}), d(\kappa_p,\nu_{p-1})\;\right\} \qquad (3.6)
\end{aligned}
$$

Similarly we can prove that

$$d\left(\beta_{p+1},\eta_p\right)\leq\theta\max\left(d(\alpha_p,\zeta_{p-1}),d(\beta_p,\eta_{p-1}),d(\gamma_p,\chi_{p-1}),d(\kappa_p,\nu_{p-1})\right) \qquad (3.7)$$

and

$$d\left(\gamma_{p+1},\chi_p\right)\leq\theta\max\left(d(\alpha_p,\zeta_{p-1}),d(\beta_p,\eta_{p-1}),d(\gamma_p,\chi_{p-1}),d(\kappa_p,\nu_{p-1})\right) \qquad (3.8)$$

also

$$d\left(\kappa_{p+1},\nu_p\right)\leq\theta\max\left(d(\alpha_p,\zeta_{p-1}),d(\beta_p,\eta_{p-1}),d(\gamma_p,\chi_{p-1}),d(\kappa_p,\nu_{p-1})\right). \qquad (3.9)$$

From eqns (3.6)-(3.9), we conclude that

$$
\begin{aligned}
\max\left\{\begin{array}{c} d\left(\alpha_{p+1},\zeta_p\right),\\ d\left(\beta_{p+1},\eta_p\right),\\ d\left(\gamma_{p+1},\chi_p\right),\\ d\left(\kappa_{p+1},\nu_p\right)\end{array}\right\} &\leq \theta\max\left\{\begin{array}{c} d(\alpha_p,\zeta_{p-1}),\\ d(\beta_p,\eta_{p-1}),\\ d(\gamma_p,\chi_{p-1}),\\ d(\kappa_p,\nu_{p-1})\end{array}\right\}\\
&\leq \theta^2\max\left\{\begin{array}{c} d(\alpha_{p-1},\zeta_{p-2}),\\ d(\beta_{p-1},\eta_{p-2}),\\ d(\gamma_{p-1},\chi_{p-2}),\\ d(\kappa_{p-1},\nu_{p-2})\end{array}\right\}\\
&\;\vdots\\
&\leq \theta^p\max\left\{\begin{array}{c} d(\alpha_1,\zeta_0),\\ d(\beta_1,\eta_0),\\ d(\gamma_1,\chi_0),\\ d(\kappa_1,\nu_0)\end{array}\right\}
\end{aligned}
$$

Which implies that

$$d\left(\alpha_{p+1},\zeta_p\right)\leq\theta^p\max\left\{\ d(\alpha_1,\zeta_0),d(\beta_1,\eta_0),d(\gamma_1,\chi_0),d(\kappa_1,\nu_0)\ \right\}$$

and

$$d\left(\beta_{p+1},\eta_p\right)\leq\theta^p\max\left\{\ d(\alpha_1,\zeta_0),d(\beta_1,\eta_0),d(\gamma_1,\chi_0),d(\kappa_1,\nu_0)\ \right\}$$

and

$$d\left(\gamma_{p+1},\chi_p\right)\leq\theta^p\max\left\{\ d(\alpha_1,\zeta_0),d(\beta_1,\eta_0),d(\gamma_1,\chi_0),d(\kappa_1,\nu_0)\ \right\}$$

also

$$d\left(\kappa_{p+1},\nu_p\right)\leq\theta^p\max\left\{\ d(\alpha_1,\zeta_0),d(\beta_1,\eta_0),d(\gamma_1,\chi_0),d(\kappa_1,\nu_0)\ \right\}$$

Moreover

$$\begin{aligned}
&d\left(\alpha_p,\zeta_p\right)=d\left(\Omega\left(æ_p,œ_p,ß_p,Æ_p\right),\Omega\left(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p\right)\right)\\
&\leq\theta\max\left\{\ d(\tau æ_p,\tau\mathfrak{x}_p),d(\tau œ_p,\tau\mathfrak{y}_p),d(\tau ß_p,\tau\mathfrak{z}_p),d(\tau Æ_p,\tau\mathfrak{w}_p)\ \right\}\\
&\leq\theta\max\left\{\ d(\alpha_{p-1},\zeta_{p-1}),d(\beta_{p-1},\eta_{p-1}),d(\gamma_{p-1},\chi_{p-1}),d(\kappa_{p-1},\nu_{p-1})\ \right\} \qquad (3.10)
\end{aligned}$$

Similarly we can prove that

$$d\left(\beta_p,\eta_p\right)\leq\theta\max\left(d(\alpha_{p-1},\zeta_{p-1}),d(\beta_{p-1},\eta_{p-1}),d(\gamma_{p-1},\chi_{p-1}),d(\kappa_{p-1},\nu_{p-1})\right) \qquad (3.11)$$

and

$$d\left(\gamma_p,\chi_p\right)\leq\theta\max\left(d(\alpha_{p-1},\zeta_{p-1}),d(\beta_{p-1},\eta_{p-1}),d(\gamma_{p-1},\chi_{p-1}),d(\kappa_{p-1},\nu_{p-1})\right) \qquad (3.12)$$

also

$$d\left(\kappa_p,\nu_p\right)\leq\theta\max\left(d(\alpha_{p-1},\zeta_{p-1}),d(\beta_{p-1},\eta_{p-1}),d(\gamma_{p-1},\chi_{p-1}),d(\kappa_{p-1},\nu_{p-1})\right). \qquad (3.13)$$

From eqns (3.10)-(3.13), we conclude that

$$\max\left\{\begin{array}{c} d\left(\alpha_p,\zeta_p\right),\\ d\left(\beta_p,\eta_p\right),\\ d\left(\gamma_p,\chi_p\right),\\ d\left(\kappa_p,\nu_p\right)\end{array}\right\}\quad\leq\quad\theta\max\left\{\begin{array}{c} d(\alpha_{p-1},\zeta_{p-1}),\\ d(\beta_{p-1},\eta_{p-1}),\\ d(\gamma_{p-1},\chi_{p-1}),\\ d(\kappa_{p-1},\nu_{p-1})\end{array}\right\}$$

$$\leq \theta^2 \max \left\{ \begin{array}{c} d(\alpha_{p-2}, \zeta_{p-2}), \\ d(\beta_{p-2}, \eta_{p-2}), \\ d(\gamma_{p-2}, \chi_{p-2}), \\ d(\kappa_{p-2}, \nu_{p-2}) \end{array} \right\}$$

$$\vdots$$

$$\leq \theta^p \max \left\{ \begin{array}{c} d(\alpha_0, \zeta_0), \\ d(\beta_0, \eta_0), \\ d(\gamma_0, \chi_0), \\ d(\kappa_0, \nu_0) \end{array} \right\}$$

Which implies that

$$d(\alpha_p, \zeta_p) \leq \theta^p \max \left\{ d(\alpha_0, \zeta_0), d(\beta_0, \eta_0), d(\gamma_0, \chi_0), d(\kappa_0, \nu_0) \right\}$$

and

$$d(\beta_p, \eta_p) \leq \theta^p \max \left\{ d(\alpha_0, \zeta_0), d(\beta_0, \eta_0), d(\gamma_0, \chi_0), d(\kappa_0, \nu_0) \right\}$$

and

$$d(\gamma_p, \chi_p) \leq \theta^p \max \left\{ d(\alpha_0, \zeta_0), d(\beta_0, \eta_0), d(\gamma_0, \chi_0), d(\kappa_0, \nu_0) \right\}$$

also

$$d(\kappa_p, \nu_p) \leq \theta^p \max \left\{ d(\alpha_0, \zeta_0), d(\beta_0, \eta_0), d(\gamma_0, \chi_0), d(\kappa_0, \nu_0) \right\}$$

Using the property (B_4), we obtain

$$d(\alpha_n, \zeta_m) \leq d(\alpha_n, \zeta_{n+1}) + d(\alpha_{n+1}, \zeta_{n+1}) + ... + d(\alpha_{m-1}, \zeta_{m-1}) + d(\alpha_{m-1}, \zeta_m)$$

$$d(\beta_n, \eta_m) \leq d(\beta_n, \eta_{n+1}) + d(\beta_{n+1}, \eta_{n+1}) + ... + d(\beta_{m-1}, \eta_{m-1}) + d(\beta_{m-1}, \eta_m)$$

$$d(\gamma_n, \chi_m) \leq d(\gamma_n, \chi_{n+1}) + d(\gamma_{n+1}, \chi_{n+1}) + ... + d(\gamma_{m-1}, \chi_{m-1}) + d(\gamma_{m-1}, \chi_m)$$

$$d(\kappa_n, \nu_m) \leq d(\kappa_n, \nu_{n+1}) + d(\kappa_{n+1}, \nu_{n+1}) + ... + d(\kappa_{m-1}, \nu_{m-1}) + d(\kappa_{m-1}, \nu_m)$$

and

$$d(\alpha_m, \zeta_n) \leq d(\alpha_m, \zeta_{m-1}) + d(\alpha_{m-1}, \zeta_{m-1}) + ... + d(\alpha_{n+1}, \zeta_{n+1}) + d(\alpha_{n+1}, \zeta_n)$$

$$d(\beta_m,\eta_n) \le d(\beta_m,\eta_{m-1}) + d(\beta_{m-1},\eta_{m-1}) + ... + d(\beta_{n+1},\eta_{n+1}) + d(\beta_{n+1},\eta_n)$$

$$d(\gamma_m,\chi_n) \le d(\gamma_m,\chi_{m-1}) + d(\gamma_{m-1},\chi_{m-1}) + ... + d(\gamma_{n+1},\chi_{n+1}) + d(\gamma_{n+1},\chi_n)$$

$$d(\kappa_m,\nu_n) \le d(\kappa_m,\nu_{m-1}) + d(\kappa_{m-1},\nu_{m-1}) + ... + d(\kappa_{n+1},\nu_{n+1}) + d(\kappa_{n+1},\nu_n)$$

Now for each $n,m \in \mathbb{N}$ with $n < m$.Then from we have

$$\begin{aligned}
&(d(\alpha_n,\zeta_m) + d(\beta_n,\eta_m) + d(\gamma_n,\chi_m) + d(\kappa_n,\nu_m))\\
&\le (d(\alpha_n,\zeta_{n+1}) + d(\beta_n,\eta_{n+1}) + d(\gamma_n,\chi_{n+1}) + d(\kappa_n,\nu_{n+1}))\\
&(d(\alpha_{n+1},\zeta_{n+1}) + d(\beta_{n+1},\eta_{n+1}) + d(\gamma_{n+1},\chi_{n+1}) + d(\kappa_{n+1},\nu_{n+1})) + ...+\\
&(d(\alpha_{m-1},\zeta_{m-1}) + d(\beta_{m-1},\eta_{m-1}) + d(\gamma_{m-1},\chi_{m-1}) + d(\kappa_{m-1},\nu_{m-1}))+\\
&(d(\alpha_{m-1},\zeta_m) + d(\beta_{m-1},\eta_m) + d(\gamma_{m-1},\chi_m) + d(\kappa_{m-1},\nu_m))\\
&\le (4\theta^n + 4\theta^{n+1} + ... + \theta^{m-1}) \max\left\{\begin{array}{c} d(\alpha_0,\zeta_1),\\ d(\beta_0,\eta_1),\\ d(\gamma_0,\chi_1),\\ d(\kappa_0,\nu_1)\end{array}\right\} + (4\theta^{n+1} + 4\theta^{n+2} + ... +\\
&4\theta^{m-1} \max\left\{\begin{array}{c} d(\alpha_0,\zeta_0),\\ d(\beta_0,\eta_0),\\ d(\gamma_0,\chi_0),\\ d(\kappa_0,\nu_0)\end{array}\right\}\\
&\le 4\frac{\theta^n}{1-\theta} \max\left\{\begin{array}{c} d(\alpha_0,\zeta_1),\\ d(\beta_0,\eta_1),\\ d(\gamma_0,\chi_1),\\ d(\kappa_0,\nu_1)\end{array}\right\} + 4\frac{\theta^{n+1}}{1-\theta} \max\left\{\begin{array}{c} d(\alpha_0,\zeta_0),\\ d(\beta_0,\eta_0),\\ d(\gamma_0,\chi_0),\\ d(\kappa_0,\nu_0)\end{array}\right\} \to 0 \ as \ n \to \infty.
\end{aligned}$$

Which means that

$(d(\alpha_n,\zeta_m) + d(\beta_n,\eta_m) + d(\gamma_n,\chi_m) + d(\kappa_n,\nu_m)) \to 0 \ as \ n,m \to \infty$. Similarly we can prove that $(d(\alpha_m,\zeta_n) + d(\beta_m,\eta_n) + d(\gamma_m,\chi_n) + d(\kappa_m,\nu_n)) \to 0 \ as \ n,m \to \infty$.

This shows that $(\alpha_p,\zeta_p),(\beta_p,\eta_p),(\gamma_p,\chi_p),(\kappa_p,\nu_p)$ are Cauchy bisequences in $(\mathfrak{A},\mathfrak{B})$ Since $\tau(\mathfrak{A}\cup\mathfrak{B})$ is complete subspace of $(\mathfrak{A},\mathfrak{B},d)$, then the sequences $\{\alpha_p\},\{\beta_p\},\{\gamma_p\},\{\kappa_p\}$ and $\{\zeta_p\},\{\eta_p\},\{\chi_p\},\{\nu_p\} \subseteq \tau(\mathfrak{A}\cup\mathfrak{B})$ are convergence in complete bipolar metric spaces $(\tau(\mathfrak{A}),\tau(\mathfrak{B}),d)$. Therefore, there exist $\upsilon,\varkappa,\varsigma,\vartheta \in \tau(\mathfrak{A})$ and $\ell,\wp,\xi,\mu \in \tau(\mathfrak{B})$ such that

$$\begin{aligned}
&\lim_{p\to\infty}\alpha_p=\ell \quad \lim_{p\to\infty}\beta_p=\wp \quad \lim_{p\to\infty}\gamma_p=\xi \quad \lim_{p\to\infty}\kappa_p=\mu\\
&\lim_{p\to\infty}\zeta_p=\upsilon \quad \lim_{p\to\infty}\eta_p=\varkappa \quad \lim_{p\to\infty}\chi_p=\varsigma \quad \lim_{p\to\infty}\nu_p=\vartheta.
\end{aligned}\tag{3.14}$$

Since $\tau:\mathfrak{A}\cup\mathfrak{B}\to\mathfrak{A}\cup\mathfrak{B}$ and $\upsilon,\varkappa,\varsigma,\vartheta\in\tau(\mathfrak{A})$ and $\ell,\wp,\xi,\mu\in\tau(\mathfrak{B})$, there exist $\imath,\partial,\aleph,\varpi\in\mathfrak{A}$ and $\eth,\mho,\rho,\varphi\in\mathfrak{B}$ such that $\tau\imath=\upsilon,\tau\partial=\varkappa,\tau\aleph=\varsigma,\tau\varpi=\vartheta$ and $\tau\eth=\ell,\tau\mho=\wp,\tau\rho=\xi,\tau\varphi=\vartheta$. Hence

$$\begin{aligned}
&\lim_{p\to\infty}\alpha_p=\ell=\tau\eth \quad \lim_{p\to\infty}\beta_p=\wp=\tau\mho \quad \lim_{p\to\infty}\gamma_p=\xi=\tau\rho \quad \lim_{p\to\infty}\kappa_p=\mu=\tau\varphi\\
&\lim_{p\to\infty}\zeta_p=\upsilon=\tau\imath \quad \lim_{p\to\infty}\eta_p=\varkappa=\tau\partial \quad \lim_{p\to\infty}\chi_p=\varsigma=\tau\aleph \quad \lim_{p\to\infty}\nu_p=\vartheta=\tau\varpi.
\end{aligned}\tag{3.15}$$

Now claim that

$$\Omega(\imath,\partial,\aleph,\varpi)=\ell,\Omega(\partial,\aleph,\varpi,\imath)=\wp,\Omega(\aleph,\varpi,\imath,\partial)=\xi,\Omega(\varpi,\imath,\partial,\aleph)=\mu$$

$$\Omega(\eth,\mho,\rho,\varphi)=\upsilon,\Omega(\mho,\rho,\varphi,\eth)=\varkappa,\Omega(\rho,\varphi,\eth,\mho)=\varsigma,\Omega(\varphi,\eth,\mho,\rho)=\vartheta.$$

Consider,

$$\begin{aligned}
&d\left(\Omega(\imath,\partial,\aleph,\varpi),\ell\right)\\
&\le d\left(\Omega\left(\imath,\partial,\aleph,\varpi\right),\zeta_{p+1}\right)+d\left(\alpha_{p+1},\zeta_{p+1}\right)+d\left(\alpha_{p+1},\ell\right)\\
&\le d\left(\Omega\left(\imath,\partial,\aleph,\varpi\right),\Omega(p_{p+1},q_{p+1},r_{p+1},s_{p+1})\right)+d\left(\alpha_{p+1},\zeta_{p+1}\right)+,d\left(\alpha_{p+1},\ell\right)\\
&\le \max\left\{\ (d(\tau\imath,\tau p_{p+1}),d(\tau\partial,\tau q_{p+1}),d(\tau\aleph,\tau r_{p+1}),d(\tau\varpi,\tau s_{p+1}))\ \right\}\\
&\quad +d\left(\alpha_{p+1},\zeta_{p+1}\right)+d\left(\alpha_{p+1},\ell\right)\\
&\le \max\left\{\ (d(\tau\imath,\zeta_p),d(\tau\partial,\eta_p),d(\tau\aleph,\chi_p),d(\tau\varpi,\nu_p))\ \right\}\\
&\quad +d\left(\alpha_{p+1},\zeta_{p+1}\right)+d\left(\alpha_{p+1},\ell\right)
\end{aligned}$$

Taking the limit as $p\to\infty$ in the above inequality, we obtain
$d\left(\Omega(\imath,\partial,\aleph,\varpi),\ell\right)=0$ which implies $\Omega(\imath,\partial,\aleph,\varpi)=\ell$.
Similarly, we can prove that $\Omega(\partial,\aleph,\varpi,\imath)=\wp,\Omega(\aleph,\varpi,\imath,\partial)=\xi,\Omega(\varpi,\imath,\partial,\aleph)=\mu$ and $\Omega(\eth,\mho,\rho,\varphi)=\upsilon,\Omega(\mho,\rho,\varphi,\eth)=\varkappa,\Omega(\rho,\varphi,\eth,\mho)=\varsigma,\Omega(\varphi,\eth,\mho,\rho)=\vartheta$.
Therefore, it follows that

$\Omega(\iota,\partial,\aleph,\varpi)=\ell=\tau\eth, \Omega(\partial,\aleph,\varpi,\iota)=\wp=\tau\mho$, $\Omega(\aleph,\varpi,\iota,\partial)=\xi=\tau\rho$, $\Omega(\varpi,\iota,\partial,\aleph)=\mu=\tau\vartheta$ and $\Omega(\eth,\mho,\rho,\varphi)=\upsilon=\tau\iota, \Omega(\mho,\rho,\varphi,\eth)=\varkappa=\tau\partial$, $\Omega(\rho,\varphi,\eth,\mho)=\varsigma=\tau\aleph$, $\Omega(\varphi,\eth,\mho,\rho)=\vartheta=\tau\varpi$. Since $\{\Omega,\tau\}$ is ω-compatible pair, we have $\Omega(\ell,\wp,\xi,\mu)=\tau\ell$, $\Omega(\wp,\xi,\mu,\ell)=\tau\xi$, $\Omega(\xi,\mu,\ell,\wp)=\tau\xi$ and $\Omega(\mu,\ell,\wp,\ell)=\tau\mu$. And $\Omega(\upsilon,\varkappa,\varsigma,\vartheta)=\tau\upsilon$, $\Omega(\varkappa,\varsigma,\vartheta,\upsilon)=\tau\varkappa$, $\Omega(\varsigma,\vartheta,\upsilon,\varkappa)=\tau\varsigma$, $\Omega(\vartheta,\upsilon,\varkappa,\varsigma)=\tau\vartheta$. Now we prove that $\tau\ell=\ell$, $\tau\wp=\wp$, $\tau\xi=\xi$, $\tau\mu=\mu$ and $\tau\upsilon=\upsilon$, $\tau\varkappa=\varkappa$, $\tau\varsigma=\varsigma$, $\tau\vartheta=\vartheta$. we have

$$
\begin{aligned}
(3.16)\qquad & d(\tau\upsilon,\zeta_p)=d(\Omega(\upsilon,\varkappa,\varsigma,\vartheta),\Omega(p_p,q_p,r_p,s_p))\\
&\leq\theta\max d(\tau\upsilon,\tau p_p),d(\tau\varkappa,\tau q_p),d(\tau\varsigma,\tau r_p),d(\tau\vartheta,\tau s_p)\\
&\leq\theta\max d(\tau\upsilon,\zeta_{p-1}),d(\tau\varkappa,\eta_{p-1}),d(\tau\varsigma,\chi_{p-1}),d(\tau\vartheta,\nu_{p-1})\\
&as\ p\to\infty,\ d(\tau\upsilon,\upsilon)\leq\theta\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))\\
&similarly\ we\ get,\ d(\tau\varkappa,\varkappa)\leq\theta\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))\\
&d(\tau\varsigma,\varsigma)\leq\theta\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))\\
&d(\tau\vartheta,\vartheta)\leq\theta\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))
\end{aligned}
$$

Therefore,

$\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))\leq\theta\max(d(\tau\upsilon,\upsilon),d(\tau\varkappa,\varkappa),d(\tau\varsigma,\varsigma),d(\tau\vartheta,\vartheta))$ which holds only $d(\tau\upsilon,\upsilon)=0$, $d(\tau\varkappa,\varkappa)=0$, $d(\tau\varsigma,\varsigma)=0$ and $d(\tau\vartheta,\vartheta)=0$ which implies that $\tau\upsilon=\upsilon$, $\tau\varkappa=\varkappa$, $\tau\varsigma=\varsigma$ and $\tau\vartheta=\vartheta$. Therefore, $\Omega(\upsilon,\varkappa,\varsigma,\vartheta)=\tau\upsilon=\upsilon$, $\Omega(\varkappa,\varsigma,\vartheta,\upsilon)=\tau\varkappa=\varkappa$, $\Omega(\varsigma,\vartheta,\upsilon,\varkappa)=\tau\varsigma=\varsigma$, $\Omega(\vartheta,\upsilon,\varkappa,\varsigma)=\tau\vartheta=\vartheta$. Similarly, we can prove $\Omega(\ell,\wp,\xi,\mu)=\tau\ell=\ell$, $\Omega(\wp,\xi,\mu,\ell)=\tau\wp=\wp$, $\Omega(\xi,\mu,\ell,\wp)=\tau\xi=\xi$ and $\Omega(\mu,\ell,\wp,\xi)=\tau\mu=\mu$.

Therefore,

$$
\begin{aligned}
&\Omega(\eth,\mho,\rho,\varphi)=\tau\iota=\upsilon=\tau\upsilon=\Omega(\upsilon,\varkappa,\varsigma,\vartheta)\Omega(\iota,\partial,\aleph,\varpi)=\tau\eth=\ell=\tau\ell=\Omega(\ell,\wp,\xi,\mu)\\
&\Omega(\mho,\rho,\varphi,\eth)=\tau\partial=\varkappa=\tau\varkappa=\Omega(\varkappa,\varsigma,\vartheta,\upsilon)\Omega(\partial,\aleph,\varpi,\iota)=\tau\mho=\wp=\tau\wp=\Omega(\wp,\xi,\mu,\ell)\\
&\Omega(\rho,\varphi,\eth,\mho)=\tau\aleph=\varsigma=\tau\varsigma=\Omega(\varsigma,\vartheta,\upsilon,\varkappa)\Omega(\aleph,\varpi,\iota,\partial)=\tau\rho=\xi=\tau\xi=\Omega(\xi,\mu,\ell,\wp)\\
&\Omega(\varphi,\eth,\mho,\rho)=\tau\varpi=\vartheta=\tau\vartheta=\Omega(\vartheta,\upsilon,\varkappa,\varsigma)\Omega(\varpi,\iota,\partial,\aleph)=\tau\mu=\mu=\tau\mu=\Omega(\mu,\ell,\wp,\xi)
\end{aligned}
$$

Now we will prove that, $\upsilon=\ell, \varkappa=\wp, \varsigma=\xi, \vartheta=\mu$. Now consider

$d(\upsilon,\ell)=d(\Omega(\upsilon,\varkappa,\varsigma,\vartheta),\Omega(\ell,\wp,\xi,\mu))$

$\leq \theta(d(\tau\upsilon,\tau\ell),d(\tau\varkappa,\tau\wp),d(\tau\varsigma,\tau\xi),d(\tau\vartheta,\tau\mu))$

$\leq \theta(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu))$

Similarly we can prove that

$d(\varkappa,\wp)\leq \theta(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu))$,

$d(\varsigma,\xi)\leq \theta(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu))$,

$d(\vartheta,\mu)\leq \theta(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu))$.

From above we can write

$\max(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu)) \leq \theta(d(\upsilon,\ell),d(\varkappa,\wp),d(\varsigma,\xi),d(\vartheta,\mu))$ which holds $\upsilon=\ell, \varkappa=\wp, \varsigma=\xi$ and $\vartheta=\mu$ Therefore, $(\upsilon,\varkappa,\varsigma,\vartheta)\in \mathfrak{S}^4\cap\mathfrak{T}^4$ is a common quadruple fixed point of Ω and τ. In the following we will show the uniqueness. Assume that there is another quadruple fixed point $(\upsilon',\varkappa',\varsigma',\vartheta')$ of Ω,τ. Then

$$\begin{aligned} d\left(\upsilon,\upsilon'\right) &= d\left(\Omega(\upsilon,\varkappa,\varsigma,\vartheta),\Omega(\upsilon',\varkappa',\varsigma',\vartheta')\right) \\ &\leq \theta\max(d(\tau\upsilon,\tau\upsilon'),d(\tau\varkappa,\tau\varkappa'),d(\tau\varsigma,\tau\varsigma'),d(\tau\vartheta,\tau\vartheta')) \\ &\leq \theta\max d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta'). \end{aligned}$$

Similarly we get $d\left(\varkappa,\varkappa'\right)\leq\theta\max\left(d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta')\right)$,

$d\left(\varsigma,\varsigma'\right)\leq\theta\max\left(d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta')\right)$,

$d\left(\vartheta,\vartheta'\right)\leq\theta\max\left(d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta')\right)$.

Thus, $\max(d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta'))\leq(d(\upsilon,\upsilon'),d(\varkappa,\varkappa'),d(\varsigma,\varsigma'),d(\vartheta,\vartheta'))$.

Hence, we get $\upsilon=\upsilon',\varkappa,\varkappa',\varsigma=\varsigma'$ and ϑ,ϑ'. Therefore, $(\upsilon,\varkappa,\varsigma,\vartheta)$ is a unique common quadruple fixed point of Ω and τ.

Finally we will show $\upsilon=\varkappa=\varsigma=\vartheta$.

$$\begin{aligned} d\left(\upsilon,\varkappa\right) &= d\left(\Omega(\upsilon,\varkappa,\varsigma,\vartheta),\Omega(\varkappa,\varsigma,\vartheta,\upsilon)\right) \\ &\leq \theta(d(\tau\upsilon,\tau\varkappa),d(\tau\varkappa,\tau\varsigma),d(\tau\varsigma,\tau\vartheta),d(\tau\vartheta,\tau\upsilon)) \\ &\leq \max\left(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon)\right). \end{aligned}$$

Similarly we get $d(\varkappa,\varsigma)\leq\theta\max(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon))$,
$d(\varsigma,\vartheta)\leq\theta\max(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon))$,
$d(\vartheta,\upsilon)\leq\theta\max(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon))$.
Thus, $\max(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon))\leq(d(\upsilon,\varkappa),d(\varkappa,\varsigma),d(\varsigma,\vartheta),d(\vartheta,\upsilon))$
hence, we get $\upsilon=\varkappa,\varkappa=\varsigma,\varsigma=\vartheta$ and $\vartheta=\upsilon$. Therefore, $(\upsilon,\upsilon,\upsilon,\upsilon)$ is a unique common quadruple fixed point of Ω and τ.

□

Corollary 3.6. $(\mathfrak{A},\mathfrak{B},d)$ *be a bipolar metric space and* $\Omega:\left(\mathfrak{A}^4,\mathfrak{B}^4\right)\rightrightarrows(\mathfrak{A},\mathfrak{B})$ *be covariant mapping. Such that* $d(\Omega(\text{æ},\text{œ},\text{ß},\text{Æ}),\Omega(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w})\leq\theta\max\left\{d(\text{æ},\mathfrak{x}),d(\text{œ},\mathfrak{y}),d(\text{ß},\mathfrak{z}),d(\text{Æ},\mathfrak{w})\right\}$ *for all* $\text{æ},\text{œ},\text{ß},\text{Æ}\in\mathfrak{A}$, $\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w}\in\mathfrak{B}$, $\theta\in(0,1)$ *and Then there is a unique quadruple fixed point of* Ω *in* $\mathfrak{A}\cup\mathfrak{B}$.

Example 3.7. *Let* $\mathscr{U}_m(\mathbb{R})$ *and* $\mathscr{L}_m(\mathbb{R})$ *be the set of all* $m\times m$ *upper and lower triangular matrices over* $\mathbb{R}$. *And* $d:\mathscr{U}_m(\mathbb{R})\times\mathscr{L}_m(\mathbb{R})\to[0,\infty)$ *as* $d(\mathscr{U},\mathscr{V})=\sum_{i,j=1}^{m}|u_{ij}-v_{ij}|$. *for all* $\mathscr{U}=(u_{ij})_{m\times m}\in\mathscr{U}_m(\mathbb{R})$ *and* $\mathscr{V}=(v_{ij})_{m\times m}\in\mathscr{L}_m(\mathbb{R})$. *Then* $(\mathscr{U}_m(\mathbb{R}),\mathscr{L}_m(\mathbb{R}),d)$ *is a bipolar metric space. Now define* Ω *as* $\Omega:\left(\mathfrak{A}^4,\mathfrak{B}^4\right)\to(\mathfrak{A},\mathfrak{B})$ *as* $\Omega(A,B,C,D)=(\frac{a_{ij}}{15}+\frac{b_{ij}}{15}+\frac{c_{ij}}{15}+\frac{d_{ij}}{15})_{m\times m}$ *where* $(A=(a_{ij})_{m\times m},B=(b_{ij})_{m\times m},C=(c_{ij})_{m\times m},D=(d_{ij}))_{m\times m}\in\mathscr{U}_m(\mathbb{R})^4\cup\mathscr{L}_m(\mathbb{R})^4$, *and define* $\tau:(\mathfrak{A},\mathfrak{B})\to(\mathfrak{A},\mathfrak{B})$ *as* $\tau(A)=(3a_{ij})_{m\times m}$, *where* $A=(a_{ij})_{m\times m}\in\mathscr{U}_m(\mathbb{R})\cup\mathscr{L}_m(\mathbb{R})$. *Now consider,*

$$
\begin{aligned}
(3.0)\qquad & d(\Omega(A,B,C,D),\Omega(P,Q,R,S)) \\
= \ & d((\frac{a_{ij}}{15}+\frac{b_{ij}}{15}+\frac{c_{ij}}{15}+\frac{d_{ij}}{15})_{m\times m},(\frac{p_{ij}}{15}+\frac{q_{ij}}{15}+\frac{r_{ij}}{15}+\frac{s_{ij}}{15})_{m\times m}) \\
= \ & \sum_{i,j=1}^{m}|(\frac{a_{ij}}{15}+\frac{b_{ij}}{15}+\frac{c_{ij}}{15}+\frac{d_{ij}}{15})-(\frac{p_{ij}}{15}+\frac{q_{ij}}{15}+\frac{r_{ij}}{15}+\frac{s_{ij}}{15})| \\
\leq \ & \sum_{i,j=1}^{m}|\frac{a_{ij}}{15}-\frac{p_{ij}}{15}|+|\frac{b_{ij}}{15}-\frac{q_{ij}}{15}|+|\frac{c_{ij}}{15}-\frac{r_{ij}}{15}|+|\frac{d_{ij}}{15}-\frac{s_{ij}}{15}| \\
\leq \ & \sum_{i,j=1}^{m}\frac{1}{45}|3a_{ij}-3p_{ij}|+|3b_{ij}-3q_{ij}|+|3c_{ij}-3r_{ij}|+|3d_{ij}-3s_{ij}| \\
\leq \ & \frac{1}{45}\max(d(\tau A,\tau P),d(\tau B,\tau Q),d(\tau C,\tau R),d(\tau D,\tau S))
\end{aligned}
$$

Then from Theorem 3.5 we can conclude that $(O_{m\times m}, O_{m\times m}, O_{m\times m}, O_{m\times m})$ *is unique common quadruple fixed point of* Ω *and* τ.

4. Application to Integral Equations

As an application of Corollary (3.6), we investigate the existence of unique solution to IVP.

$$(4.1)\quad \mathfrak{x}'(t) = \Omega\left(t, \mathfrak{x}(t), \mathfrak{y}(t), \mathfrak{z}(t), \mathfrak{w}(t)\right), t \in I = [0,1], (\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w})(0) = (\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{w}_0).$$

Where $\Omega : I \times (E_1^4 \cup E_2^4) \to \mathbb{R}$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{w}_0 \in E_1 \cup E_2$, where $E_1 \cup E_2$ is a Lebesgue measurable set with $m(E_1 \cup E_2) < \infty$ with

$$\int_0^t \Omega(\ell, \mathfrak{x}(\ell), \mathfrak{y}(\ell), \mathfrak{z}(\ell), \mathfrak{w}(\ell))d\ell = \max\left\{\begin{array}{c} \int_0^t \Omega(\ell, \mathfrak{x}(\ell), \mathfrak{x}(\ell), \mathfrak{x}(\ell), \mathfrak{x}(\ell))d\ell, \\ \int_0^t \Omega(\ell, \mathfrak{y}(\ell), \mathfrak{y}(\ell), \mathfrak{y}(\ell), \mathfrak{y}(\ell))d\ell, \\ \int_0^t \Omega(\ell, \mathfrak{z}(\ell), \mathfrak{z}(\ell), \mathfrak{z}(\ell), \mathfrak{z}(\ell))d\ell, \\ \int_0^t \Omega(\ell, \mathfrak{w}(\ell), \mathfrak{w}(\ell), \mathfrak{w}(\ell), \mathfrak{w}(\ell))d\ell \end{array}\right\}.$$

Then there exists a unique solution in $C\left(I, L^\infty(E_1) \cup L^\infty(E_2)\right)$.

Proof. The integral equation for IVP is

$\mathfrak{x}(t) = \mathfrak{x}_0 + 4 \int_{E_1\cup E_2} \Omega(\ell, \mathfrak{x}(\ell), \mathfrak{y}(\ell), \mathfrak{z}(\ell), \mathfrak{w}(\ell))d\ell.$

Let $\mathfrak{A} = C\left(I, L^\infty(E_1)\right), \mathfrak{B} = C\left(I, L^\infty(E_2)\right)$ and $d(\wp, \varpi) = ||\wp - \varpi||$ for all $\wp, \varpi \in \mathfrak{A} \cup \mathfrak{B}$ and $\tau(\ell) = \ell$, for all $\ell \in [0, \infty)$. Define $R : \mathfrak{A}^4 \cup \mathfrak{B}^4 \to \mathfrak{A} \cup \mathfrak{B}$ by

$$R(\alpha, \beta, \gamma, \delta)(t) \quad = \quad \frac{\mathfrak{x}_0}{4} + \int_{E_1\cup E_2} \Omega(\ell, \mathfrak{x}(\ell), \mathfrak{y}(\ell), \mathfrak{z}(\ell), \mathfrak{w}(\ell))d\ell.$$

Now $d(R(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w})(t), R(\rho, \rho, \sigma, \varsigma)(t)) = ||R(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w})(t) - R(\rho, \rho, \sigma, \varsigma)(t)||$

$$||\tfrac{\alpha_0}{4} + \int_{E_1\cup E_2} \Omega(\ell, (\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w})(\ell))d\ell - \tfrac{\rho_0}{4} - \int_{E_1\cup E_2} \Omega(\ell, (\rho, \rho, \sigma, \varsigma)(\ell))d\ell||$$

$$\leq \frac{1}{4}\max\left\{\begin{array}{c} ||\mathfrak{x}(t) - \rho(t)||, \\ ||\mathfrak{y}(t) - \rho(t)||, \\ ||\mathfrak{z}(t) - \sigma(t)||, \\ ||\mathfrak{w}(t) - \varsigma(t)|| \end{array}\right\}$$

$$\leq \frac{1}{4}\max\left\{d(\mathfrak{x}, \rho), d(\mathfrak{y}, \rho), d(\mathfrak{z}, \sigma), d(\mathfrak{w}, \varsigma)\right\}$$

$$\leq \theta \max\left\{d(\mathfrak{x}, \rho), d(\mathfrak{y}, \rho), d(\mathfrak{z}, \sigma), d(\mathfrak{w}, \varsigma)\right\}.$$

Since by Corollary we can say that R has a unique solution in $\mathfrak{A} \cup \mathfrak{B}$. □

5. Application to Homotopy

In this section we examine a unique solution to Homotopy theory.

Theorem 5.1. *Let $(\mathfrak{X},\mathfrak{Y})$ and $(\overline{\mathfrak{X}},\overline{\mathfrak{Y}})$ be an open and closed subset of $(\mathfrak{A},\mathfrak{B})$ such that $(\mathfrak{X},\mathfrak{Y}) \subseteq (\overline{\mathfrak{X}},\overline{\mathfrak{Y}})$. Let $(\mathfrak{A},\mathfrak{B},d)$ be the complete bipolar metric space. Assume that the operator $\mathfrak{H}_b : (\overline{\mathfrak{X}^4} \cup \overline{\mathfrak{Y}^4}) \times [0,1] \to \mathfrak{A} \cup \mathfrak{B}$ satisfies the following conditions:*

- (τ_0) $\text{æ} \neq \mathfrak{H}_b(\text{æ},\text{œ},\text{ß},\text{Æ},\varpi)$, $\text{œ} \neq \mathfrak{H}_b(\text{œ},\text{ß},\text{Æ},\varpi,\text{æ})$, $\text{ß} \neq \mathfrak{H}_b(\text{ß},\text{Æ},\varpi,\text{æ},\text{œ})$, $\text{Æ} \neq \mathfrak{H}_b(\text{Æ},\varpi,\text{æ},\text{œ},\text{ß})$, *for each* $\text{æ},\text{œ},\text{ß},\text{Æ} \in \partial\mathfrak{X} \cup \partial\mathfrak{Y}$ *and* $\varpi \in [0,1]$;
- (τ_1) $d(\mathfrak{H}_b(\text{œ},\text{œ},\text{ß},\text{Æ},\varpi),\mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\varpi)) \leq \theta \max(d(\text{æ},\mathfrak{x}),d(\text{œ},\mathfrak{y}),d(\text{ß},\mathfrak{z}),d(\text{Æ},\mathfrak{w}))$ *for all* $\text{æ},\text{œ},\text{ß},\text{Æ} \in \overline{\mathfrak{X}}$, $\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w} \in \overline{\mathfrak{Y}}$ *and* $\varpi \in [0,1]$,
- (τ_2) $\exists\, L \geq 0 \ni d(\mathfrak{H}_b(\text{œ},\text{œ},\text{ß},\text{Æ},\varpi),\mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\wp)) \leq L|\varpi - \wp|$
  for every $\text{æ},\text{œ},\text{ß},\text{Æ} \in \overline{\mathfrak{X}}$, $\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w} \in \overline{\mathfrak{Y}}$ *and* $\varpi,\wp \in [0,1]$.

Then, $\mathfrak{H}_b(.,0)$ has quadruple fixed point $\iff$ $\mathfrak{H}_b(.,1)$ has quadruple fixed point.

Proof. Consider the sets

$$\mathscr{A} = \left\{ \begin{array}{c} \varpi \in [0,1] : \mathfrak{H}_b(\text{æ},\text{œ},\text{ß},\text{Æ},\varpi) = \text{æ}, \mathfrak{H}_b(\text{œ},\text{ß},\text{Æ},\text{æ},\varpi) = \text{œ}, \\ \mathfrak{H}_b(\text{ß},\text{Æ},\text{æ},\text{œ},\varpi) = \text{ß}, \mathfrak{H}_b(\text{Æ},\text{æ},\text{œ},\text{ß},\varpi) = \text{Æ}, \text{ for some } (\text{æ},\text{œ},\text{ß},\text{Æ},\varpi) \in \mathfrak{X}^4 \cup \mathfrak{Y}^4 \end{array} \right\}$$

$$\mathscr{B} = \left\{ \begin{array}{c} \sigma \in [0,1] : \mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\sigma) = \mathfrak{x}, \mathfrak{H}_b(\mathfrak{y},\mathfrak{z},\mathfrak{w},\mathfrak{x},\sigma) = \mathfrak{y}, \\ \mathfrak{H}_b(\mathfrak{z},\mathfrak{w},\mathfrak{x},\mathfrak{y},\sigma) = \mathfrak{z}, \mathfrak{H}_b(\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z},\sigma) = \mathfrak{w}, \text{ for some } \mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w} \in \mathfrak{X}^4 \cup \mathfrak{Y}^4 \end{array} \right\}$$

Let $\mathfrak{H}_b(.,0)$ has quadruple fixed point in $\mathfrak{X}^4 \cup \mathfrak{Y}^4$, then $(0,0,0,0) \in \mathscr{A}^4 \cap \mathscr{B}^4$. Consequently, $\mathscr{A}^4 \cap \mathscr{B}^4 \neq \phi$. Using the connectedness $\mathscr{A} = \mathscr{B} = [0,1]$, we now demonstrate that $\mathscr{A} \cap \mathscr{B}$ is both closed and open in $[0,1]$.

Let $(\{\varpi_p\}_{p=1}^{\infty},\{\sigma_p\}_{p=1}^{\infty}) \subseteq (\mathscr{A},\mathscr{B})$ with $(\varpi_p,\sigma_p) \to (\varpi,\sigma) \in [0,1]$ as $p \to \infty$. We must demonstrate that $\varpi = \sigma \in \mathscr{A} \cap \mathscr{B}$.

Since $(\varpi_p,\sigma_p) \in (\mathscr{A},\mathscr{B})$ for $p = 0,1,2,3,\cdots$, there exist bisequences $(\text{æ}_p,\mathfrak{x}_p),(\text{œ}_p,\mathfrak{y}_p)$, $(\text{ß}_p,\mathfrak{z}_p),(\text{Æ}_p,\mathfrak{w}_p)$ with $\text{æ}_{p+1} = \mathfrak{H}_b(\text{æ}_p,\text{œ}_p,\text{ß}_p,\text{Æ}_p,\varpi_p)$, $\text{œ}_{p+1} = \mathfrak{H}_b(\text{œ}_p,\text{ß}_p,\text{Æ}_p,\text{æ}_p,\varpi_p)$, $\text{ß}_{p+1} = \mathfrak{H}_b(\text{ß}_p,\text{Æ}_p,\text{æ}_p,\text{œ}_p,\varpi_p)$, $\text{Æ}_{p+1} = \mathfrak{H}_b(\text{Æ}_p,\text{æ}_p,\text{œ}_p,\text{ß}_p,\varpi_p)$ and $\mathfrak{x}_{p+1} = \mathfrak{H}_b(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\sigma_p)$, $\mathfrak{y}_{p+1} = \mathfrak{H}_b(\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p,\sigma_p)$, $\mathfrak{z}_{p+1} = \mathfrak{H}_b(\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p,\sigma_p)$, $\mathfrak{w}_{p+1} = \mathfrak{H}_b(\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\sigma_p)$.

Consider

$$\begin{aligned} d(\text{æ}_p,\mathfrak{x}_{p+1}) &= d(\mathfrak{H}_b(\text{æ}_{p-1},\text{œ}_{p-1},\text{ß}_{p-1},\text{Æ}_{p-1},\varpi_{p-1}),\mathfrak{H}_b(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\sigma_p)) \\ &\leq \theta \max(d(\text{æ}_{p-1},\mathfrak{x}_p),d(\text{œ}_{p-1},\mathfrak{y}_p),d(\text{ß}_{p-1},\mathfrak{z}_p),d(\text{Æ}_{p-1},\mathfrak{w}_p)). \end{aligned}$$

Similarly

$$\begin{aligned} d(\text{œ}_p,\mathfrak{y}_{p+1}) &= d(\mathfrak{H}_b(\text{œ}_{p-1},\text{ß}_{p-1},\text{Æ}_{p-1},\text{æ}_{p-1},\varpi_{p-1}),\mathfrak{H}_b(\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p,\sigma_p)) \\ &\leq \theta \max(d(\text{œ}_{p-1},\mathfrak{y}_p),d(\text{ß}_{p-1},\mathfrak{z}_p),d(\text{Æ}_{p-1},\mathfrak{w}_p),d(\text{æ}_{p-1},\mathfrak{x}_p)) \end{aligned}$$

$$\begin{aligned} d(\text{ß}_p,\mathfrak{z}_{p+1}) &= d(\mathfrak{H}_b(\text{ß}_{p-1},\text{Æ}_{p-1},\text{æ}_{p-1},\text{œ}_{p-1},\varpi_{p-1}),\mathfrak{H}_b(\mathfrak{z}_p,\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p,\sigma_p)) \\ &\leq \theta \max(d(\text{ß}_{p-1},\mathfrak{z}_p),d(\text{Æ}_{p-1},\mathfrak{w}_p),d(\text{æ}_{p-1},\mathfrak{x}_p),d(\text{œ}_{p-1},\mathfrak{y}_p)) \end{aligned}$$

and

$$\begin{aligned} d(\text{Æ}_p,\mathfrak{w}_{p+1}) &= d(\mathfrak{H}_b(\text{Æ}_{p-1},\text{æ}_{p-1},\text{œ}_{p-1},\text{ß}_{p-1},\varpi_{p-1}),\mathfrak{H}_b(\mathfrak{w}_p,\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\sigma_p)) \\ &\leq \theta \max(d(\text{Æ}_{p-1},\mathfrak{w}_p),d(\text{æ}_{p-1},\mathfrak{x}_p),d(\text{œ}_{p-1},\mathfrak{y}_p),d(\text{ß}_{p-1},\mathfrak{z}_p)). \end{aligned}$$

From above we can write

$$\begin{aligned} \max\left\{\begin{array}{c} d(\text{æ}_p,\mathfrak{x}_{p+1}), \\ d(\text{œ}_p,\mathfrak{y}_{p+1}), \\ d(\text{ß}_p,\mathfrak{z}_{p+1}), \\ d(\text{Æ}_p,\mathfrak{w}_{p+1}) \end{array}\right\} &\leq \theta \max\left\{\begin{array}{c} d(\text{æ}_{p-1},\mathfrak{x}_p), \\ d(\text{œ}_{p-1},\mathfrak{y}_p), \\ d(\text{ß}_{p-1},\mathfrak{z}_p), \\ d(\text{Æ}_{p-1},\mathfrak{w}_p) \end{array}\right\} \\ &\leq \theta^2 \max\left\{\begin{array}{c} d(\text{æ}_{p-2},\mathfrak{x}_{p-1}), \\ d(\text{œ}_{p-2},\mathfrak{y}_{p-1}), \\ d(\text{ß}_{p-2},\mathfrak{z}_{p-1}), \\ d(\text{Æ}_{p-2},\mathfrak{w}_{p-1}) \end{array}\right\} \\ &\vdots \\ &\leq \theta^p \max\left\{\begin{array}{c} d(\text{æ}_0,\mathfrak{x}_1), \\ d(\text{œ}_0,\mathfrak{y}_1), \\ d(\text{ß}_0,\mathfrak{z}_1), \\ d(\text{Æ}_0,\mathfrak{w}_1) \end{array}\right\}. \end{aligned}$$

So, we can write

$$\begin{aligned} d(\text{æ}_p,\mathfrak{x}_{p+1}) &\leq \theta^p \max\left\{ d(\text{æ}_0,\mathfrak{x}_1), d(\text{œ}_0,\mathfrak{y}_1), d(\text{ß}_0,\mathfrak{z}_1), d(\text{Æ}_0,\mathfrak{w}_1) \right\}, \\ d(\text{œ}_p,\mathfrak{y}_{p+1}) &\leq \theta^p \max\left\{ d(\text{æ}_0,\mathfrak{x}_1), d(\text{œ}_0,\mathfrak{y}_1), d(\text{ß}_0,\mathfrak{z}_1), d(\text{Æ}_0,\mathfrak{w}_1) \right\}, \\ d(\text{ß}_p,\mathfrak{z}_{p+1}) &\leq \theta^p \max\left\{ d(\text{æ}_0,\mathfrak{x}_1), d(\text{œ}_0,\mathfrak{y}_1), d(\text{ß}_0,\mathfrak{z}_1), d(\text{Æ}_0,\mathfrak{w}_1) \right\}, \end{aligned}$$

and

$$d(\text{Æ}_p,\mathfrak{w}_{p+1}) \leq \theta^p \max\left\{ d(\text{æ}_0,\mathfrak{x}_1), d(\text{œ}_0,\mathfrak{y}_1), d(\text{ß}_0,\mathfrak{z}_1), d(\text{Æ}_0,\mathfrak{w}_1) \right\}. \tag{5.1}$$

Now consider

$$\begin{aligned} d(\text{æ}_{p+1},\mathfrak{x}_p) &= d(\mathfrak{H}_b(\text{æ}_p,\text{œ}_p,\text{ß}_p,\text{Æ}_p,\varpi_p),\mathfrak{H}_b(\mathfrak{x}_{p-1},\mathfrak{y}_{p-1},\mathfrak{z}_{p-1},\mathfrak{w}_{p-1},\sigma_{p-1})) \\ &\leq \theta \max(d(\text{æ}_p,\mathfrak{x}_{p-1}), d(\text{œ}_p,\mathfrak{y}_{p-1}), d(\text{ß}_p,\mathfrak{z}_{p-1}), d(\text{Æ}_p,\mathfrak{w}_{p-1})). \end{aligned}$$

Similarly

$$\begin{aligned} d(\text{œ}_{p+1},\mathfrak{y}_p) &= d(\mathfrak{H}_b(\text{œ}_p,\text{ß}_p,\text{Æ}_p,\text{æ}_p,\varpi_p),\mathfrak{H}_b(\mathfrak{y}_{p-1},\mathfrak{z}_{p-1},\mathfrak{w}_{p-1},\mathfrak{x}_{p-1},\sigma_{p-1})) \\ &\leq \theta \max(d(\text{œ}_p,\mathfrak{y}_{p-1}), d(\text{ß}_p,\mathfrak{z}_{p-1}), d(\text{Æ}_p,\mathfrak{w}_{p-1}), d(\text{æ}_p,\mathfrak{x}_{p-1})) \end{aligned}$$

$$\begin{aligned} d(\text{ß}_{p+1},\mathfrak{z}_p) &= d(\mathfrak{H}_b(\text{ß}_p,\text{Æ}_p,\text{æ}_p,\text{œ}_p,\varpi_p),\mathfrak{H}_b(\mathfrak{z}_{p-1},\mathfrak{w}_{p-1},\mathfrak{x}_{p-1},\mathfrak{y}_{p-1},\sigma_{p-1})) \\ &\leq \theta \max(d(\text{ß}_p,\mathfrak{z}_{p-1}), d(\text{Æ}_p,\mathfrak{w}_{p-1}), d(\text{æ}_p,\mathfrak{x}_{p-1}), d(\text{œ}_p,\mathfrak{y}_{p-1})) \end{aligned}$$

and

$$\begin{aligned} d(\text{Æ}_{p+1},\mathfrak{w}_p) &= d(\mathfrak{H}_b(\text{Æ}_p,\text{æ}_p,\text{œ}_p,\text{ß}_p,\varpi_p),\mathfrak{H}_b(\mathfrak{w}_{p-1},\mathfrak{x}_{p-1},\mathfrak{y}_{p-1},\mathfrak{z}_{p-1},\sigma_{p-1})) \\ &\leq \theta \max(d(\text{Æ}_p,\mathfrak{w}_{p-1}), d(\text{æ}_p,\mathfrak{x}_{p-1}), d(\text{œ}_p,\mathfrak{y}_{p-1}), d(\text{ß}_p,\mathfrak{z}_{p-1})). \end{aligned}$$

From above we can write

$$\max\left\{ \begin{array}{c} d(\text{æ}_{p+1},\mathfrak{x}_p), \\ d(\text{œ}_{p+1},\mathfrak{y}_p), \\ d(\text{ß}_{p+1},\mathfrak{z}_p), \\ d(\text{Æ}_{p+1},\mathfrak{w}_p) \end{array} \right\} \leq \theta \max\left\{ \begin{array}{c} d(\text{æ}_p,\mathfrak{x}_{p_1}), \\ d(\text{œ}_p,\mathfrak{y}_{p_1}), \\ d(\text{ß}_p,\mathfrak{z}_{p-1}), \\ d(\text{Æ}_p,\mathfrak{w}_{p_1}) \end{array} \right\}$$

$$
\begin{aligned}
&\leq \theta^2 \max \left\{ \begin{array}{c} d(\text{æ}_{p-1}, \mathfrak{x}_{p-2}), \\ d(\text{œ}_{p-1}, \mathfrak{y}_{p-2}), \\ d(\text{ß}_{p-1}, \mathfrak{z}_{p-2}), \\ d(\text{Æ}_{p-1}, \mathfrak{w}_{p-2}) \end{array} \right\} \\
&\vdots \\
&\leq \theta^p \max \left\{ \begin{array}{c} d(\text{æ}_1, \mathfrak{x}_0), \\ d(\text{œ}_1, \mathfrak{y}_0), \\ d(\text{ß}_1, \mathfrak{z}_0), \\ d(\text{Æ}_1, \mathfrak{w}_0) \end{array} \right\}.
\end{aligned}
$$

So, we can write

$$
\begin{aligned}
d(\text{æ}_{p+1}, \mathfrak{x}_p) &\leq \theta^p \max \left\{ d(\text{æ}_1, \mathfrak{x}_0), d(\text{œ}_1, \mathfrak{y}_0), d(\text{ß}_1, \mathfrak{z}_0), d(\text{Æ}_1, \mathfrak{w}_0) \right\}, \\
d(\text{œ}_{p+1}, \mathfrak{y}_p) &\leq \theta^p \max \left\{ d(\text{æ}_1, \mathfrak{x}_0), d(\text{œ}_1, \mathfrak{y}_0), d(\text{ß}_1, \mathfrak{z}_0), d(\text{Æ}_1, \mathfrak{w}_0) \right\}, \\
d(\text{ß}_{p+1}, \mathfrak{z}_p) &\leq \theta^p \max \left\{ d(\text{æ}_1, \mathfrak{x}_0), d(\text{œ}_1, \mathfrak{y}_0), d(\text{ß}_1, \mathfrak{z}_0), d(\text{Æ}_1, \mathfrak{w}_0) \right\},
\end{aligned}
$$

and

$$
d(\text{Æ}_{p+1}, \mathfrak{w}_p) \leq \theta^p \max \left\{ d(\text{æ}_1, \mathfrak{x}_0), d(\text{œ}_1, \mathfrak{y}_0), d(\text{ß}_1, \mathfrak{z}_0), d(\text{Æ}_1, \mathfrak{w}_0) \right\}. \tag{5.2}
$$

Now again consider

$$
\begin{aligned}
d(\text{æ}_p, \mathfrak{x}_p) &= d(\mathfrak{H}_b(\text{æ}_{p-1}, \text{œ}_{p-1}, \text{ß}_{p-1}, \text{Æ}_{p-1}, \varpi_{p-1}), \mathfrak{H}_b(\mathfrak{x}_{p-1}, \mathfrak{y}_{p-1}, \mathfrak{z}_{p-1}, \mathfrak{w}_{p-1}, \sigma_{p-1})) \\
&\leq \theta \max(d(\text{æ}_{p-1}, \mathfrak{x}_{p-1}), d(\text{œ}_{p-1}, \mathfrak{y}_{p-1}), d(\text{ß}_{p-1}, \mathfrak{z}_{p-1}), d(\text{Æ}_{p-1}, \mathfrak{w}_{p-1})).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
d(\text{œ}_p, \mathfrak{y}_p) &= d(\mathfrak{H}_b(\text{œ}_{p-1}, \text{ß}_{p-1}, \text{Æ}_{p-1}, \text{æ}_{p-1}, \varpi_{p-1}), \mathfrak{H}_b(\mathfrak{y}_{p-1}, \mathfrak{z}_{p-1}, \mathfrak{w}_{p-1}, \mathfrak{x}_{p-1}, \sigma_{p-1})) \\
&\leq \theta \max(d(\text{œ}_{p-1}, \mathfrak{y}_{p-1}), d(\text{ß}_{p-1}, \mathfrak{z}_{p-1}), d(\text{Æ}_{p-1}, \mathfrak{w}_{p-1}), d(\text{æ}_{p-1}, \mathfrak{x}_{p-1}))
\end{aligned}
$$

$$
\begin{aligned}
d(\text{ß}_p, \mathfrak{z}_p) &= d(\mathfrak{H}_b(\text{ß}_p, \text{Æ}_{p-1}, \text{æ}_{p-1}, \text{œ}_{p-1}, \varpi_{p-1}), \mathfrak{H}_b(\mathfrak{z}_{p-1}, \mathfrak{w}_{p-1}, \mathfrak{x}_{p-1}, \mathfrak{y}_{p-1}, \sigma_{p-1})) \\
&\leq \theta \max(d(\text{ß}_{p-1}, \mathfrak{z}_{p-1}), d(\text{Æ}_{p-1}, \mathfrak{w}_{p-1}), d(\text{æ}_{p-1}, \mathfrak{x}_{p-1}), d(\text{œ}_{p-1}, \mathfrak{y}_{p-1}))
\end{aligned}
$$

and

$$\begin{aligned} d(Æ_p, \mathfrak{w}_p) &= d(\mathfrak{H}_b(Æ_{p-1}, æ_{p-1}, œ_{p-1}, ß_{p-1}, \varpi_p), \mathfrak{H}_b(\mathfrak{w}_{p-1}, \mathfrak{x}_{p-1}, \mathfrak{y}_{p-1}, \mathfrak{z}_{p-1}, \sigma_{p-1})) \\ &\leq \theta \max(d(Æ_{p-1}, \mathfrak{w}_{p-1}), d(æ_{p-1}, \mathfrak{x}_{p-1}), d(œ_{p-1}, \mathfrak{y}_{p-1}), d(ß_{p-1}, \mathfrak{z}_{p-1})). \end{aligned}$$

From above we can write

$$\begin{aligned} \max\left\{\begin{array}{c} d(æ_p, \mathfrak{x}_p), \\ d(œ_p, \mathfrak{y}_p), \\ d(ß_p, \mathfrak{z}_p), \\ d(Æ_p, \mathfrak{w}_p) \end{array}\right\} &\leq \theta \max\left\{\begin{array}{c} d(æ_{p-1}, \mathfrak{x}_{p_1}), \\ d(œ_{p-1}, \mathfrak{y}_{p_1}), \\ d(ß_{p-1}, \mathfrak{z}_{p-1}), \\ d(Æ_{p-1}, \mathfrak{w}_{p_1}) \end{array}\right\} \\ &\leq \theta^2 \max\left\{\begin{array}{c} d(æ_{p-2}, \mathfrak{x}_{p-2}), \\ d(œ_{p-2}, \mathfrak{y}_{p-2}), \\ d(ß_{p-2}, \mathfrak{z}_{p-2}), \\ d(Æ_{p-2}, \mathfrak{w}_{p-2}) \end{array}\right\} \\ &\vdots \\ &\leq \theta^p \max\left\{\begin{array}{c} d(æ_0, \mathfrak{x}_0), \\ d(œ_0, \mathfrak{y}_0), \\ d(ß_0, \mathfrak{z}_0), \\ d(Æ_0, \mathfrak{w}_0) \end{array}\right\}. \end{aligned}$$

So we can write

$$\begin{aligned} d(æ_p, \mathfrak{x}_p) &\leq \theta^p \max\left\{ d(æ_0, \mathfrak{x}_0), d(œ_0, \mathfrak{y}_0), d(ß_0, \mathfrak{z}_0), d(Æ_0, \mathfrak{w}_0) \right\}, \\ d(œ_p, \mathfrak{y}_p) &\leq \theta^p \max\left\{ d(æ_0, \mathfrak{x}_0), d(œ_0, \mathfrak{y}_0), d(ß_0, \mathfrak{z}_0), d(Æ_0, \mathfrak{w}_0) \right\}, \\ d(ß_p, \mathfrak{z}_p) &\leq \theta^p \max\left\{ d(æ_0, \mathfrak{x}_0), d(œ_0, \mathfrak{y}_0), d(ß_0, \mathfrak{z}_0), d(Æ_0, \mathfrak{w}_0) \right\}, \end{aligned}$$

and

$$d(Æ_p, \mathfrak{w}_p) \leq \theta^p \max\left\{ d(æ_0, \mathfrak{x}_0), d(œ_0, \mathfrak{y}_0), d(ß_0, \mathfrak{z}_0), d(Æ_0, \mathfrak{w}_0) \right\}. \tag{5.3}$$

For each $n, m \in \mathbb{N}$ with $n < m$. Using (B_4), equations (5.1), (5.2) and (5.3) we have
$d(æ_n, \mathfrak{x}_m) + d(œ_n, \mathfrak{y}_m) + d(ß_n, \mathfrak{z}_m) + d(Æ_n, \mathfrak{w}_m)$
$\leq (d(æ_n, \mathfrak{x}_{n+1}) + d(œ_n, \mathfrak{y}_{n+1}) + d(ß_n, \mathfrak{z}_{n+1}) + d(Æ_n, \mathfrak{w}_{n+1}))$

$+\left(d(æ_{n+1},\mathfrak{x}_{n+1})+d(œ_{n+1},\mathfrak{y}_{n+1})+d(ß_{n+1},\mathfrak{z}_{n+1})+d(Æ_{n+1},\mathfrak{w}_{n+1})\right)$

$\cdots$

$+\left(d(æ_{m-1},\mathfrak{x}_{m-1})+d(œ_{m-1},\mathfrak{y}_{m-1})+d(ß_{m-1},\mathfrak{z}_{m-1})+d(Æ_{m-1},\mathfrak{w}_{m-1})\right)$

$+\left(d(æ_{m-1},\mathfrak{x}_{m})+d(œ_{m-1},\mathfrak{y}_{m})+d(ß_{m-1},\mathfrak{z}_{m})+d(Æ_{m-1},\mathfrak{w}_{m})\right)$

$$\leq 4\theta^p \max\left\{\begin{array}{c} d(æ_0,\mathfrak{x}_1),\\ d(œ_0,\mathfrak{y}_1),\\ d(ß_0,\mathfrak{z}_1),\\ d(Æ_0,\mathfrak{w}_1)\end{array}\right\}+L|\varpi_{n+1}-\wp_{n+1}|+....+$$

$$|\varpi_{m-1}-\wp_{m-1}|+4\theta^p \max\left\{\begin{array}{c} d(æ_0,\mathfrak{x}_0),\\ d(œ_0,\mathfrak{y}_0),\\ d(ß_0,\mathfrak{z}_0),\\ d(Æ_0,\mathfrak{w}_0)\end{array}\right\}\to 0 \text{ as } n,m\to\infty.$$

It means that $\lim_{n,m\to\infty} d\left(æ_n,\mathfrak{x}_m\right)+d(œ_n,\mathfrak{y}_m)+d(ß_n,\mathfrak{z}_m)+d(Æ_n,\mathfrak{w}_m)=0.$

Similarly we can prove that

$\lim_{m,n\to\infty} d\left(æ_m,\mathfrak{x}_n\right)+d(œ_m,\mathfrak{y}_n)+d(ß_m,\mathfrak{z}_n)+d(Æ_m,\mathfrak{w}_n)=0$. Which implies that

$(æ_p,\mathfrak{x}_p),(œ_p,\mathfrak{y}_p),(ß_p,\mathfrak{z}_p),(Æ_p,\mathfrak{w}_p)$ are Cauchy bisequences in $(\mathfrak{X},\mathfrak{Y})$.

By the completeness property there exists, λ,μ,ν,ξ and $\rho,\upsilon,\rho,\varsigma$ in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, with

$$\begin{gathered}\lim_{p\to\infty} æ_p=\rho,\quad \lim_{p\to\infty} œ_p=\upsilon,\quad \lim_{p\to\infty} ß_p=\rho,\quad \lim_{p\to\infty} Æ_p=\varsigma\\ \lim_{p\to\infty}\mathfrak{x}_p=\lambda,\quad \lim_{p\to\infty}\mathfrak{y}_p=\mu,\quad \lim_{p\to\infty}\mathfrak{z}_p=\nu,\quad \lim_{p\to\infty}\mathfrak{w}_p=\xi.\end{gathered}\tag{5.4}$$

Now consider

$$\begin{aligned} &d\left(\left(\mathfrak{H}_b(\lambda,\mu,\nu,\xi,\varpi)\right),\rho\right) \\ \leq\;& d(\mathfrak{H}_b(\lambda,\mu,\nu,\xi,\wp),\mathfrak{x}_{p+1})+d(æ_{p+1},\mathfrak{x}_{p+1})+d(æ_{p+1},\rho)\\ \leq\;& d(\mathfrak{H}_b(\lambda,\mu,\nu,\xi,\wp),\mathfrak{H}_b(\mathfrak{x}_p,\mathfrak{y}_p,\mathfrak{z}_p,\mathfrak{w}_p,\sigma_p))+d(æ_{p+1},\mathfrak{x}_{p+1})+d(æ_{p+1},\rho)\\ \leq\;& \theta\max(d(æ_p,\mathfrak{x}_p),d(œ_p,\mathfrak{y}_p),d(ß_p,\mathfrak{z}_p),d(Æ_p,\mathfrak{w}_p))+L|\wp_p-\sigma_p|+d(æ_{p+1},\rho).\end{aligned}\tag{5.5}$$

which is $\to 0$ as $p\to\infty$.

That is $d\left(\left(\mathfrak{H}_b(\lambda,\mu,\nu,\xi,\varpi)\right),\rho\right)=0 \Rightarrow d\left(\left(\mathfrak{H}_b(\lambda,\mu,\nu,\xi,\varpi)\right)\right)=\rho.$

Similarly $d((\mathfrak{H}_b(\mu,\nu,\xi,\lambda,\varpi))=\upsilon, d((\mathfrak{H}_b(\nu,\xi,\lambda,\mu,\varpi))=\rho, d((\mathfrak{H}_b(\xi,\lambda,\mu,\nu,\varpi))=\varsigma$ and $d((\mathfrak{H}_b(\rho,\upsilon,\rho,\varsigma,\sigma))=\lambda, d((\mathfrak{H}_b(\upsilon,\rho,\varsigma,\rho,\sigma))=\mu, d((\mathfrak{H}_b(\rho,\varsigma,\rho,\upsilon,\sigma))=\nu$, $d((\mathfrak{H}_b(\varsigma,\rho,\upsilon,\rho,\sigma))=\xi$. On the other hand from eqn (5.4),

$d(\lambda,\rho)=d(\lim_{p\to\infty}\mathfrak{x}_p,\lim_{p\to\infty}æ_p)=\lim_{p\to\infty}d(æ_p,\mathfrak{x}_p)=0$ that implies $\lambda=\rho$

Therefore $\mu=\upsilon,\nu=\rho$ and $\xi=\varsigma$.And hence $\varpi=\sigma$. Thus $\varpi=\sigma\in\mathscr{A}\cap\mathscr{B}$. Clearly $\mathscr{A}\cap\mathscr{B}$ closed in [0,1].

Let $(\varpi_0,\sigma_0)\in\mathscr{A}\cap\mathscr{B}$,then there exists bisequences $(æ_0,\mathfrak{x}_0),(œ_0,\mathfrak{y}_0),(ß_0,\mathfrak{z}_0),(Æ_0,\mathfrak{w}_0)$ with $æ_0=\mathfrak{H}_b(æ_0,œ_0,ß_0,Æ_0,\varpi_0), œ_0=\mathfrak{H}_b(œ_0,ß_0,Æ_0,æ_0,\varpi_0), ß_0=\mathfrak{H}_b(ß_0,Æ_0,æ_0,œ_0,\varpi_0), Æ_0=\mathfrak{H}_b(Æ_0,æ_0,œ_0,ß_0,\varpi_0)$ and $\mathfrak{x}_0=\mathfrak{H}_b(\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0,\sigma_0), \mathfrak{y}_0=\mathfrak{H}_b(\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0,\mathfrak{x}_0,\sigma_0), \mathfrak{z}_0=\mathfrak{H}_b(\mathfrak{z}_0,\mathfrak{w}_0,\mathfrak{x}_0,\mathfrak{y}_0,\sigma_0)$,

$\mathfrak{w}_0=\mathfrak{H}_b(\mathfrak{w}_0,\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\sigma_0)$.

Since $\mathscr{A}\cup\mathscr{B}$ is open, then there exists $\omega>0$ such that $B_d(æ_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}$,

$B_d(œ_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}, B_d(ß_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}, B_d(Æ_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}$

and $B_d(\mathfrak{x}_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}, B_d(\mathfrak{y}_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}, B_d(\mathfrak{z}_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}, B_d(\mathfrak{w}_0,\omega)\subseteq\mathscr{A}\cup\mathscr{B}$.

Choose $\varpi\in(\sigma_0-\varepsilon,\sigma_0+\varepsilon),\sigma\in(\varpi_0-\varepsilon,\varpi_0+\varepsilon)$ such that $|\varpi-\sigma_0|<\frac{1}{L^P}<\frac{\varepsilon}{2}$,

$|\sigma-\varpi_0|<\frac{1}{L^P}<\frac{\varepsilon}{2}$ and $|\varpi_0-\sigma_0|<\frac{1}{L^P}<\frac{\varepsilon}{2}$.

Then for each $\mathfrak{x}\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(æ_0,\omega)=\{\mathfrak{x},\mathfrak{x}_0\in\mathfrak{Y}/d(æ_0,\mathfrak{x})\leq\omega+d(æ_0,\mathfrak{x}_0)\}$,

$\mathfrak{y}\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(œ_0,\omega)=\{\mathfrak{y},\mathfrak{y}_0\in\mathfrak{Y}/d(œ_0,\mathfrak{y})\leq\omega+d(œ_0,\mathfrak{y}_0)\}$,

$\mathfrak{z}\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(ß_0,\omega)=\{\mathfrak{z},\mathfrak{z}_0\in\mathfrak{Y}/d(ß_0,\mathfrak{z})\leq\omega+d(ß_0,\mathfrak{z}_0)\}$,

$\mathfrak{w}\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(Æ_0,\omega)=\{\mathfrak{w},\mathfrak{w}_0\in\mathfrak{Y}/d(Æ_0,\mathfrak{w})\leq\omega+d(Æ_0,\mathfrak{w}_0)\}$,

$æ\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\omega,\mathfrak{x}_0)=\{æ,æ_0\in\mathfrak{X}/d(æ,\mathfrak{x}_0)\leq\omega+d(æ_0,\mathfrak{x}_0)\}$,

$œ\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\omega,\mathfrak{y}_0)=\{œ,œ_0\in\mathfrak{X}/d(œ,\mathfrak{y}_0)\leq\omega+d(œ_0,\mathfrak{y}_0)\}$,

$ß\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\omega,\mathfrak{z}_0)=\{ß,ß_0\in\mathfrak{X}/d(ß,\mathfrak{z}_0)\leq\omega+d(ß_0,\mathfrak{z}_0)\}$,

and $Æ\in\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\omega,\mathfrak{w}_0)=\{Æ,Æ_0\in\mathfrak{X}/d(Æ,\mathfrak{w}_0)\leq\omega+d(Æ_0,\mathfrak{w}_0)\}$.

Also $d(\mathfrak{H}_b(æ,œ,ß,Æ,\varpi),\mathfrak{x}_0)=d(\mathfrak{H}_b(æ,œ,ß,Æ,\varpi),\mathfrak{H}_b(\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0),\sigma_0)$

$\leq d(\mathfrak{H}_b(æ,œ,ß,Æ,\varpi),\mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\sigma_0))+d(\mathfrak{H}_b(æ_0,œ_0,ß_0,Æ_0,\varpi),\mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\sigma_0))+$

$d(\mathfrak{H}_b(æ_0,œ_0,ß_0,Æ_0,\varpi),\mathfrak{H}_b(\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0,\sigma_0)$

$\leq\frac{2}{L^{p-1}}+\theta\max(d(æ_0,\mathfrak{x}),d(œ_0,\mathfrak{y}),d(ß_0,\mathfrak{z}),d(Æ_0,\mathfrak{w}))$ as $p\to\infty$ we have

$d(\mathfrak{H}_b(æ,œ,ß,Æ,\varpi),\mathfrak{x}_0)\leq\theta\max(d(æ_0,\mathfrak{x}),d(œ_0,\mathfrak{y}),d(ß_0,\mathfrak{z}),d(Æ_0,\mathfrak{w}))$.

Similarly we have $d(\mathfrak{H}_b(\oe,\ss,\AE,\ae,\varpi),\mathfrak{y}_0) \le \theta \max(d(\oe_0,\mathfrak{y}),d(\ss_0,\mathfrak{z}),d(\AE_0,\mathfrak{w}),d(\ae_0,\mathfrak{x}))$

$d(\mathfrak{H}_b(\ss,\AE,\ae,\oe,\varpi),\mathfrak{z}_0) \le \theta \max(d(\ss_0,\mathfrak{z}),d(\AE_0,\mathfrak{w}),d(\ae_0,\mathfrak{x}),d(\oe_0,\mathfrak{y}))$

$d(\mathfrak{H}_b(\AE,\ae,\oe,\ss,\varpi),\mathfrak{w}_0) \le \theta \max(d(\AE_0,\mathfrak{w}),d(\ae_0,\mathfrak{x}),d(\oe_0,\mathfrak{y}),d(\ss_0,\mathfrak{z}))$.

$$
\begin{aligned}
\max\left\{\begin{array}{c} d(\mathfrak{H}_b(\ae,\oe,\ss,\AE,\varpi),\mathfrak{x}_0),\\ d(\mathfrak{H}_b(\oe,\ss,\AE,\ae,\varpi),\mathfrak{y}_0),\\ d(\mathfrak{H}_b(\ss,\AE,\ae,\oe,\varpi),\mathfrak{z}_0),\\ d(\mathfrak{H}_b(\AE,\ae,\oe,\ss,\varpi),\mathfrak{w}_0) \end{array}\right\} &\le \theta\max\left\{\begin{array}{c} d(\ae_0,\mathfrak{x}),\\ d(\oe_0,\mathfrak{y}),\\ d(\ss_0,\mathfrak{z}),\\ d(\AE_0,\mathfrak{w}) \end{array}\right\}\\
&< \max\left\{\begin{array}{c} d(\ae_0,\mathfrak{x}),\\ d(\oe_0,\mathfrak{y}),\\ d(\ss_0,\mathfrak{z}),\\ d(\AE_0,\mathfrak{w}) \end{array}\right\}\\
&\le \max\left\{\begin{array}{c} d(\ae_0,\mathfrak{x}_0)+\omega,\\ d(\oe_0,\mathfrak{y}_0)+\omega,\\ d(\ss_0,\mathfrak{z}_0)+\omega,\\ d(\AE_0,\mathfrak{w}_0)+\omega \end{array}\right\}. \qquad (5.6)
\end{aligned}
$$

which gives

$d(\mathfrak{H}_b(\ae,\oe,\ss,\AE,\varpi),\mathfrak{x}_0) \le d(\ae_0,\mathfrak{x}_0)+\omega, d(\mathfrak{H}_b(\oe,\ss,\AE,\ae,\varpi),\mathfrak{y}_0) \le d(\oe_0,\mathfrak{y}_0)+\omega,$

$d(\mathfrak{H}_b(\ss,\AE,\ae,\oe,\varpi),\mathfrak{z}_0) \le d(\ss_0,\mathfrak{z}_0)+\omega, d(\mathfrak{H}_b(\AE,\ae,\oe,\ss,\varpi),\mathfrak{w}_0) \le d(\AE_0,\mathfrak{w}_0)+\omega$. Similarly we can write

$d(\ae_0,\mathfrak{H}_b(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w},\sigma)) \le d(\ae,\mathfrak{x}_0) \le d(\ae_0,\mathfrak{x}_0)+\omega,$

$d(\oe_0,\mathfrak{H}_b(\mathfrak{y},\mathfrak{z},\mathfrak{w},\mathfrak{x},\sigma)) \le d(\oe,\mathfrak{y}_0) \le d(\oe_0,\mathfrak{y}_0)+\omega,$

$d(\ss_0,\mathfrak{H}_b(\mathfrak{z},\mathfrak{w},\mathfrak{x},\mathfrak{y},\sigma)) \le d(\ss,\mathfrak{z}_0) \le d(\ss_0,\mathfrak{z}_0)+\omega,$

$d(\AE_0,\mathfrak{H}_b(\mathfrak{w},\mathfrak{x},\mathfrak{y},\mathfrak{z},\sigma)) \le d(\AE,\mathfrak{w}_0) \le d(\AE_0,\mathfrak{w}_0)+\omega.$

Now $d(\ae_0,\mathfrak{x}_0) = d(d(\mathfrak{H}_b(\ae_0,\oe_0,\ss_0,\AE_0,\varpi_0),\mathfrak{H}_b(\mathfrak{x}_0,\mathfrak{y}_0,\mathfrak{z}_0,\mathfrak{w}_0,\sigma_0)) \le L|\varpi_0-\sigma_0|$

$\le L\frac{1}{L^p} \le \frac{1}{L^{p-1}} \to 0$ as $p\to\infty \Rightarrow d(\ae_0,\mathfrak{x}_0)=0 \Rightarrow \ae_0=\mathfrak{x}_0$. Similarly we get

$\oe_0=\mathfrak{y}_0, \ss_0=\mathfrak{z}_0, \AE_0=\mathfrak{w}_0$. Hence $\varpi=\sigma$.

Thus for each fixed $\varpi\in(\varpi_0-\varepsilon,\varpi_0+\varepsilon)$, $\mathfrak{H}_b(.,\varpi):\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\ae_0,\omega)\to\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\ae_0,\omega)$, $\mathfrak{H}_b(.,\varpi):$ $\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\oe_0,\omega)\to\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\oe_0,\omega)$, $\mathfrak{H}_b(.,\varpi):\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\ss_0,\omega)\to\overline{B}_{\mathscr{A}\cup\mathscr{B}}(\ss_0,\omega)$ and $\mathfrak{H}_b(.,\varpi):$

$\overline{B}_{\mathscr{A}\cup\mathscr{B}}(Æ_0,\omega) \to \overline{B}_{\mathscr{A}\cup\mathscr{B}}(Æ_0,\omega)$. Hence from the main theorem is satisfied in all respects. As a result, we draw the conclusion that $\mathfrak{H}_b(.,\varpi)$ has a quadruple fixed point in $\overline{\mathfrak{X}}\cap\overline{\mathfrak{Y}}$. However, this has to be in $\mathfrak{X}\cap\mathfrak{Y}$. Because condition (τ_1) is true. Therefore, for $\varpi \in (\varpi_0-\varepsilon,\varpi_0+\varepsilon)$, $\varpi \in \mathscr{A}\cap\mathscr{B}$ and hence, $(\varpi_0-\varepsilon,\varpi_0+\varepsilon) \subseteq \mathscr{A}\cap\mathscr{B}$. Then it is evident that $[0,1]$ is open for $\mathscr{A}\cap\mathscr{B}$.

We can employ the same procedure to demonstrate the opposite. □

6. CONCLUSION

We presence the uniqueness of a common quadruple fixed point for two mappings in the class of bipolar metric spaces, with an example,also applications to integral eqution and Homotopy theory.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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