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A FIXED POINT AND ITS ASYMPTOTIC STABILITY OF THE SOLUTION OF A DIFFERENTIAL EQUATION ON THE REAL HALF-LINE

AHMED M. A. EL-SAYED[1], MALAK M. S. BA-ALI[2,*], EMAN M. A. HAMDALLAH[1]

[1]Faculty of Science, Alexandria University, Alexandria, Egypt

[2]Faculty of Science, Princess Nourah Bint Abdul Rahman University, Riyadh 11671, Saudi Arabia

Abstract. This research study aims to analyze the solvability of a differential equation in two ways. The first approach involves by applying of Darbo's fixed point Theorem and the measure of noncompactness (MNC) technique, the second approach by using some fixed point theories within the space $BC(R_+)$. Moreover, we establish the asymptotic stability of the solution and dependency on the initial data and on the some functions. Additionally, we delve into the study of Hyers-Ulam stability. Finally, some examples are provided to verify our investigation.

Keywords: differential equation; Schauder's fixed point theorem; measure of noncompactness; existence of solutions; asymptotic stability and dependency; Hyers–Ulam stability.

2020 AMS Subject Classification: 34L30; 34K06.

1. INTRODUCTION

The study of differential equations has received much attention over the last 30 years or so. For papers studying such kind of problems (see [1, 2, 3, 4]) and the references therein.

It is known that the nonlinear initial value problems create an important branch of nonlinear analysis and have numerous applications in describing of miscellaneous real world problems.

*Corresponding author

E-mail address: Malak.mohamed_pg@alexu.edu.eg

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Such kind of these equations have been considered in numerous papers see [5] and references therein.

The technique associated with MNC in the Banach space $BC(R_+)$ (of all bounded and continuous functions on R_+) have been successfully used by J. Banas (see see [5, 6, 7]) to prove the existence of asymptotically stable solutions for some functional equation (see [8, 9]).

The authors in [10, 11] extensively investigated its solvability, asymptotic stability and dependency of the solution on some parameters. They utilized the technique of MNC within the space $BC(R_+)$.

M. Benchohra et al. [12, 13] were concerned with the existence of solutions to some problems of differential equations on an unlimited field, where researchers relied on Schoder's fixed point Theorem [14] combined with the diagonalization process. Let us mention that this method was widely used for differential equations; see for instance [15, 16].

Here we are concerning with the initial value problem of the differential equation,

$$\frac{dx}{dt} = f(t,x(t)),\ t \in (0,\infty), \tag{1}$$

with the nonlocal integral condition

$$x(\tau) + \int_0^\tau g(s,x(s))ds = x_0,\ \ \tau \geq 0. \tag{2}$$

Our aim here is to establish the solvability of the solution $x \in BC(R+)$ of the problem (1)-(2). The main tools in our study is applying Darbo's fixed point Theorem [17] and MNC technique and using Schauder's fixed point Theorem [18]. Furthermore, the asymptotic stability and dependency of $x \in BC(R_+)$ on the initial data x_0 has been studied. The Hyers – Ulam stability of the problem (1)-(2) will be studied. Finally, we give an examples illustrate our results.

The first main tool in our work are the measure of MNC and Darbo fixed point Theorem [17]. Let $BC(R_+)$ be the class of all bounded and continuous functions in R_+, with the standard norm

$$\|x\| = \sup_{t\in R_+} |x(t)|.$$

Now, let $x \in X \subseteq BC(R_+)$ and $\varepsilon \geq 0$ be given, denote by $\omega^T(x,\varepsilon)$, $T \geq 0$, the modulus of continuity of the function x on the interval $[0,T]$

$$\omega^T(x,\varepsilon) = sup\ [|x(t) - x(s)| : t,s \in [0,T], |t-s| \leq \varepsilon]$$

and

$$\omega^T(X,\varepsilon) = sup\ [\omega^T(x,\varepsilon) : x \in X].$$

Also

$$\omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon),\ \omega_0(X) = \lim_{T\to\infty}\omega_0^T(X).$$

and

$$diam\ X(t) = sup\ \{|x(t)-y(t)|, x,y \in X\}$$

The measure of MNC on $BC(R_+)$ is given by

$$\mu(X) = \omega_0(X) + \lim_{t\to\infty} sup\ diam X(t). \tag{3}$$

Finally, we state the Darbo fixed point Theorem [17].

The following Theorem will be needed.

Theorem 1. *Let Q be nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous operator such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of Q, where $k \in [0,1)$ is a constant. Then F has a fixed point in the set Q.*

The second tool in our work is an application of Schauder's Theorem [18].

The following Lemma will be needed.

Lemma 1 (9)**.** *Let $D \in BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) *D is uniformly bounded in BC.*

(b) *The functions belonging to D are almost equicontinuous on R_+, i.e. equicontinuous on every compact interval of R_+.*

(c) *The functions from D are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|u(t) - u(+\infty)| < \varepsilon$, for any $t \ge T(\varepsilon)$ and $u \in D$.*

2. Existence of Solution

Consider now the initial value problem (1) and (2) under the following assumptions:

(i) $f : R_+ \times R \to R$ is continuous in $t \in R_+$, $\forall x \in R$ and satisfies Lipschitz condition,

(4) $$|f(t,x)-f(t,y)| \leq b_2(t)|x-y| \ \forall\ t \in R_+,\ x,y \in R,$$

where

$$\lim_{t\to\infty}\int_0^t |b_2(s)|ds = 0,\ \sup_{t\in R_+}\int_0^t |b_2(s)|ds = b_2^* \ and\ \int_0^\tau |b_2(s)|ds \leq b_2.$$

(ii) $g : R_+ \times R \to R$ is continuous in $t \in R_+$, $\forall x \in R$ and satisfies Lipschitz condition,

(5) $$|g(t,x)-g(t,y)| \leq b_1(t)|x-y| \ \forall\ t \in R_+,\ x,y \in R,$$

where

$$\int_0^\tau |b_1(s)|ds \leq b_1,\ \tau \geq 0.$$

(iii) $b_1 + b_2 + b_2^* < 1$.

From equation (11), we have

$$|f(t,x)| - |f(t,0)| \leq |f(t,x)-f(t,0)| \leq b_2(t)|x|,$$

$$|f(t,x)| \leq |f(t,0)| + b_2(t)|x|$$

and

$$|f(t,x)| \leq |m(t)| + b_2(t)|x|,$$

where

$$|m(t)| = |f(t,0)| \in BC(R_+)\ < \infty,\ \lim_{t\to\infty}\int_0^t |m(s)|ds = 0,$$

$$\sup_{t\in R_+}\int_0^t |m(s)|ds \leq m^* \ and\ \int_0^\tau |m(s)|ds \leq m.$$

Also, from equation (5), we get

$$|g(t,x)| \leq |v(t)| + b_1(t)|x|,$$

where

$$\int_0^\tau |v(s)|ds \leq v,\ \tau \geq 0.$$

Now, we have the following lemma.

Lemma 2. *The problem (1) and (2) is equivalent to the functional integral equation*

$$x(t) = x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds,\ t \geq 0,\ \tau \geq 0. \tag{6}$$

Proof. Let $x \in BC(R_+)$ be a solution of the problem (1)-(2), then by integrating, we get

$$x(t) = x(0) + \int_0^t f(s,x(s))ds \tag{7}$$

for $t = \tau$, we get

$$\begin{aligned} x(\tau) &= x(0) + \int_0^\tau f(s,x(s))ds \\ x(0) &= x(\tau) - \int_0^\tau f(s,x(s))ds, \end{aligned}$$

from (2), we have

$$x(0) = x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds, \tag{8}$$

substituting by (8) in (7), we obtain (6).

Conversely, let $x \in BC(R_+)$ be a solution of (6), then by differentiation, we obtain (1).

If $t = \tau$, we obtain (2).

Now, we have the following existences theorem.

Theorem 2. *Let the assumptions $(i)-(iii)$ be satisfied, then the problem (1)-(2) has at least one solution $x \in BC(R_+)$.*

Proof. Define the set

$$Q_r = \{x \in BC(R_+) : \|x\| \leq r\}.$$

Consider the functional integral equation (6) and define the operator

$$Fx(t) = x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds.$$

Now, let $x \in Q_r$, then

$$\begin{aligned} |Fx(t)| &= \left| x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds \right| \\ &\leq |x_0| + \int_0^\tau (|v(s)| + |b_1(s)||x(s)|)ds + \int_0^\tau (|m(s)| + |b_2(s)||x(s)|)ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t (|m(s)| + |b_2(s)||x(s)|)ds \\
&\leq |x_0| + v + r\, b_1 + m + r\, b_2 + m^* + r\, b_2^*,
\end{aligned}
$$

then

$$
\|Fx\| \leq |x_0| + v + r\, b_1 + m + r\, b_2 + m^* + r\, b_2^* = r,\ r = \frac{|x_0| + v + m + m^*}{1 - (b_1 + b_2 + b_2^*)}.
$$

Hence the operator F maps the ball Q_r into itself.

Now, let $\delta > 0$ be given take $x_1, x_2 \in Q_r$, such that $\|x_2 - x_1\| \leq \delta$, then

$$
\begin{aligned}
|Fx_2(t) - Fx_1(t)| &= \Big| x_0 - \int_0^\tau g(s, x_2(s))ds - \int_0^\tau f(s, x_2(s))ds + \int_0^t f(s, x_2(s))ds \\
&- x_0 + \int_0^\tau g(s, x_1(s))ds + \int_0^\tau f(s, x_1(s))ds - \int_0^t f(s, x_1(s))ds \Big| \\
&\leq \int_0^\tau |g(s, x_2(s)) - g(s, x_1(s))|ds + \int_0^\tau |f(s, x_2(s)) - f(s, x_1(s))|ds \\
&+ \int_0^t |f(s, x_2(s)) - f(s, x_1(s))|ds \\
&\leq \int_0^\tau |b_1(s)||x_2(s) - x_1(s)|ds + \int_0^\tau |b_2(s)||x_2(s) - x_1(s)|ds \\
&+ \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds \\
&\leq \|x_2 - x_1\| \int_0^\tau |b_1(s)|ds + \|x_2 - x_1\| \int_0^\tau |b_2(s)|ds \\
&+ \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds \\
&\leq \delta\, b_1 + \delta\, b_2 + \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds
\end{aligned}
$$

(i) Choose $T > 0$ such that $t \geq T$, then we have

$$
\begin{aligned}
\|Fx_2 - Fx_1\| &\leq \delta\, b_1 + \delta\, b_2 + b_2^* \|x_2 - x_1\| \\
&\leq b^* \delta\, b_1 + \delta\, b_2 + b_2^* \delta = \varepsilon.
\end{aligned}
$$

(ii) Also, for $T > 0$ and $t \in [0, T]$, then

$$
\begin{aligned}
\|Fx_2 - Fx_1\| &\leq \delta\, b_1 + \delta\, b_2 + b_2^* \|x_2 - x_1\| \\
&\leq b^* \delta\, b_1 + \delta\, b_2 + b_2^* \delta = \varepsilon.
\end{aligned}
$$

Hence the operator F is continuous.

Now, let $T > 0$ and $\delta > 0$ be given, choose a function $x \in X$ and $t \in [0,T]$ such that $|t_2 - t_1| < \delta$, $t_1 \leq t_2$

$$\begin{aligned} |Fx(t_2) - Fx(t_1)| &= \left| x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^{t_2} f(s,x(s))ds \right. \\ &- \left. x_0 + \int_0^\tau g(s,x(s))ds + \int_0^\tau f(s,x(s))ds - \int_0^{t_1} f(s,x(s))ds \right| \\ &\leq \int_{t_1}^{t_2} |f(s,x(s))|ds. \end{aligned}$$

Now, let $t_1, t_2 \in [0,T]$, $|t_2 - t_1| < \delta$, then we deduce that

$$\begin{aligned} \omega^T(FX, \varepsilon) &\leq \int_{t_1}^{t_2} |f(s,x(s))|ds < \varepsilon \\ \omega_0^T(FX) &\leq 0 \end{aligned}$$

and as $T \to \infty$

$$\omega_0(FX) = 0. \tag{9}$$

Now, let $X \in Q_r$ be nonempty, then for any $x,\ y \in X$ and $T > 0$ such that $t \geq T$, then we get

$$\begin{aligned} & |Fx(t) - Fy(t)| \\ \leq & \int_0^\tau |b_1(s)||x(s) - y(s)|ds + \int_0^\tau |b_2(s)||x(s) - y(s)|ds + \int_0^t |b_2(s)||x(s) - y(s)|ds \\ \leq & \int_0^\tau |b_1(s)| \sup_{x,y \in X} |x(s) - y(s)|ds + \int_0^\tau |b_2(s)| \sup_{x,y \in X} |x(s) - y(s)|ds \\ + & \int_0^t |b_2(s)| \sup_{x,y \in X} |x(s) - y(s)|ds \\ \leq & \int_0^\tau |b_1(s)| \left(\lim_{s \to \infty} (\sup_{x,y \in X} |x(s) - y(s)|) + \varepsilon_1 \right) ds \\ + & \int_0^\tau |b_2(s)| \left(\lim_{s \to \infty} (\sup_{x,y \in X} |x(s) - y(s)|) + \varepsilon_2 \right) ds \\ + & \int_0^t |b_2(s)| \left(\lim_{s \to \infty} (\sup_{x,y \in X} |x(s) - y(s)|) + \varepsilon_3 \right) ds \\ \leq & \lim_{t \to \infty} \left(\sup_{x,y \in X} |x(t) - y(t)| + \varepsilon_1 \right) . \int_0^\tau |b_1(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+ \lim_{t\to\infty}\left(\sup_{x,y\in X}|x(t)-y(t)|+\varepsilon_2\right).\int_0^\tau|b_2(s)|ds \\
&+ \lim_{t\to\infty}\left(\sup_{x,y\in X}|x(t)-y(t)|+\varepsilon_3\right).\int_0^t|b_2(s)|ds \\
&\leq \lim_{t\to\infty}\left(\sup_{x,y\in X}|x(t)-y(t)|+\varepsilon_1\right).b_1+\lim_{t\to\infty}\left(\sup_{x,y\in X}|x(t)-y(t)|+\varepsilon_2\right).b_2 \\
&+ \lim_{t\to\infty}\left(\sup_{x,y\in X}|x(t)-y(t)|+\varepsilon_3\right).b_2^*,
\end{aligned}
$$

then

$$diam\ FX(t) \quad\leq\quad (b_1+b_2+b_2^*)\lim_{t\to\infty} diamX(t).$$

Hence

$$\lim_{t\to\infty} sup\ diamFX(t) \quad\leq\quad (b_1+b_2+b_2^*)\lim_{t\to\infty} sup\ diamX(t). \tag{10}$$

Now, from (9), (10) and the definition of μ in (3), we get

$$\mu(FX) \quad\leq\quad (b_1+b_2+b_2^*)\ \mu(X).$$

Since $(b_1+b_2+b_2^*)\ <1$, F is a contraction regarding MNC (μ), which implies that $x\in Q_r$ is a solution of (6). Consequently there exists at least one solution $x\in BC(R_+)$ of the problem (1)-(2).

Now, we will show the existence of solution $x\in BC(R_+)$ of the problem (1)-(2) by using Schauder's fixed point Theorem [18].

Theorem 3. *Let the assumptions $(i)-(iii)$ be satisfied, then the problem (1)-(2) has at least one solution $x\in BC(R_+)$.*

Proof. Define B_ρ by

$$B_\rho=\{x\in BC(R_+):\|x\|\leq\rho\},\ \rho=\frac{|x_0|+v+m+\ m^*}{1-(b_1+b_2+b_2^*)}$$

and the operator K by

$$Kx(t)=x_0-\int_0^\tau g(s,x(s))ds-\int_0^\tau f(s,x(s))ds+\int_0^t f(s,x(s))ds.$$

Now, let $x \in B_\rho$, then

$$\begin{aligned}
|Kx(t)| &= \left| x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds \right| \\
&\leq |x_0| + \int_0^\tau (|v(s)| + |b_1(s)||x(s)|)ds + \int_0^\tau (|m(s)| + |b_2(s)||x(s)|)ds \\
&+ \int_0^t (|m(s)| + |b_2(s)||x(s)|)ds \\
&\leq |x_0| + v + \rho\, b_1 + m + \rho\, b_2 + m^* + \rho\, b_2^*,
\end{aligned}$$

then

$$\|Kx\| \leq |x_0| + v + r\, b_1 + m + r\, b_2 + m^* + r\, b_2^* = \rho.$$

This proves that $K : B_\rho \to B_\rho$ and the class of functions $\{Kx\}$ is uniformly bounded.

Now, let $\delta > 0$ be given take $x_1, x_2 \in B_\rho$, such that $\|x_2 - x_1\| \leq \delta$, then

$$\begin{aligned}
& |Fx_2(t) - Fx_1(t)| \\
= & \left| x_0 - \int_0^\tau g(s,x_2(s))ds - \int_0^\tau f(s,x_2(s))ds + \int_0^t f(s,x_2(s))ds \right. \\
- & \left. x_0 + \int_0^\tau g(s,x_1(s))ds + \int_0^\tau f(s,x_1(s))ds - \int_0^t f(s,x_1(s))ds \right| \\
\leq & \int_0^\tau |g(s,x_2(s)) - g(s,x_1(s))|ds + \int_0^\tau |f(s,x_2(s)) - f(s,x_1(s))|ds \\
+ & \int_0^t |f(s,x_2(s)) - f(s,x_1(s))|ds \\
\leq & \int_0^\tau |b_1(s)||x_2(s) - x_1(s)|ds + \int_0^\tau |b_2(s)||x_2(s) - x_1(s)|ds \\
+ & \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds \\
\leq & \|x_2 - x_1\| \int_0^\tau |b_1(s)|ds + \|x_2 - x_1\| \int_0^\tau |b_2(s)|ds + \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds \\
\leq & \delta\, b_1 + \delta\, b_2 + \int_0^t |b_2(s)||x_2(s) - x_1(s)|ds
\end{aligned}$$

(i) Choose $T > 0$ such that $t \geq T$, then we have

$$\begin{aligned}
\|Fx_2 - Fx_1\| &\leq \delta\, b_1 + \delta\, b_2 + b_2^* \|x_2 - x_1\| \\
&\leq b^* \delta\, b_1 + \delta\, b_2 + b_2^* \delta = \varepsilon.
\end{aligned}$$

(ii) Also, for $T>0$ and $t\in[0,T]$, then

$$\begin{aligned}\|Fx_2-Fx_1\| &\leq \delta\, b_1\,+\,\delta\, b_2\,+\,b_2^*\|x_2-x_1\|\\ &\leq b^*\delta\, b_1\,+\,\delta\, b_2\,+\,b_2^*\delta=\varepsilon.\end{aligned}$$

Hence the operator K is continuous.

Now, let $x\in B_\rho$ and $t_1,t_2\in I$ such that $t_2>t_1$ and $|\,t_1-t_2\,|\leq\delta$, then we have

$$\begin{aligned}|Kx(t_2)-Kx(t_1)| &= \Big|x_0-\int_0^\tau g(s,x(s))ds-\int_0^\tau f(s,x(s))ds+\int_0^{t_2} f(s,x(s))ds\\ &- x_0+\int_0^\tau g(s,x(s))ds+\int_0^\tau f(s,x(s))ds-\int_0^{t_1} f(s,x(s))ds\Big|\\ &\leq \int_{t_1}^{t_2}|f(s,x(s))|ds.\end{aligned}$$

This means that the class of functions $\{Kx\}$ is equicontinuous on every compact interval I of R_+. Next, let $t\in R_+$ and $x\in B_\rho$, then we have

$\forall\ \varepsilon>0,\ \exists\ T(\varepsilon)$ such that $t>\ T(\varepsilon)$ implies

$$\int_0^t|m(s)-0|ds<\frac{\varepsilon}{2}\ and\ \int_0^t|b_2(s)-0|ds\quad<\quad\frac{\varepsilon}{2},$$

then

$$\begin{aligned}|\int_0^t f(s,x(s))ds-0| &\leq |\int_0^t m(s)-0\,ds+r\int_0^t b_2(s)-0ds|\\ &\leq |\int_0^t m(s)-0\,ds|+\,r|\int_0^t b_2(s)-0|\\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

then

$$\int_0^t f(s,x(s))ds\longrightarrow 0$$

and

$$|x(t)-x(+\infty)|=\int_0^t|m(s)|ds+\|x\|\int_0^t|b_2(s)|ds\to 0.$$

Hence

$$|x(t)-x(+\infty)|\to 0,\ as\ t\to+\infty.$$

Then the class of functions $\{Kx\}$ is equiconvergent.

Then from Lemma 1, we can conclude that $K : B_\rho \to B_\rho$ is continuous and compact.

Now, by Schauder's fixed point Theorem [18] there exists at least one fixed point $x \in B_\rho \subset BC(R_+)$ of the functional integral equation (6). Consequently there exists at least one solution $x \in BC(R_+)$ of the problem (1)-(2).

Corollary 1. *Let the assumptions of theorem 2 be satisfied, then the solution of the problem (1)-(2) is unique.*

Proof. Let x_1, x_2 be two solutions of (6), then

$$
\begin{aligned}
|x_2(t) - x_1(t)| &\leq \Big| x_0 - \int_0^\tau g(s, x_2(s))ds - \int_0^\tau f(s, x_2(s))ds + \int_0^t f(s, x_2(s))ds \\
&- x_0 + \int_0^\tau g(s, x_1(s))ds + \int_0^\tau f(s, x_1(s))ds - \int_0^t f(s, x_1(s))ds \Big| \\
&\leq \int_0^\tau |g(s, x_2(s)) - g(s, x_1(s))| ds + \int_0^\tau |f(s, x_2(s)) - f(s, x_1(s))| ds \\
&+ \int_0^t |f(s, x_2(s)) - f(s, x_1(s))| ds \\
&\leq \int_0^\tau |b_1(s)||x_2(s) - x_1(s)| ds + \int_0^\tau |b_2(s)||x_2(s) - x_1(s)| ds \\
&+ \int_0^t |b_2(s)||x_2(s) - x_1(s)| ds \\
&\leq \|x_2 - x_1\| \, b_1 + \|x_2 - x_1\| \, b_2 + \|x_2 - x_1\| \, b_2^*,
\end{aligned}
$$

then

$$
\|x_2 - x_1\| \leq (b_1 + b_2 + b_2^*) \|x_2 - x_1\|.
$$

Hence

$$
\|x_2 - x_1\| (1 - (b_1 + b_2 + b_2^*)) \leq 0,
$$

then the problem (1)-(2) is unique.

3. Asymptotic Stability

Theorem 4. *The solution $x \in BC(R_+)$ of (6) is asymptotically stable in the sense that for any $\varepsilon > 0$, there exist $T(\varepsilon) > 0$ and $r > 0$. Moreover, for $x, \bar{x} \in Q_r$ any two solutions, then $|x(t) - \bar{x}(t)| \leq \varepsilon$ for $t \geq T(\varepsilon)$.*

Proof. Take $x, \bar{x} \in Q_r$ any two solutions of (6), then for every $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ such that $t \geq T(\varepsilon)$, then

$$\begin{aligned}
|x(t) - \bar{x}(t)| &= \bigg| x_0 - \int_0^\tau g(s, x(s))ds - \int_0^\tau f(s, x(s))ds + \int_0^t f(s, x(s))ds \\
&- x_0 + \int_0^\tau g(s, \bar{x}(s))ds + \int_0^\tau f(s, \bar{x}(s))ds - \int_0^t f(s, \bar{x}(s))ds \bigg| \\
&\leq \int_0^\tau |g(s, x(s)) - g(s, \bar{x}(s))|ds + \int_0^\tau |f(s, x(s)) - f(s, \bar{x}(s))|ds \\
&+ \int_0^t |f(s, x(s)) - f(s, \bar{x}(s))|ds \\
&\leq \|x - \bar{x}\| \, b_1 \; + \; \|x - \bar{x}\| \, b_2 + \; 2\int_0^t |m(s)|ds + 2 \, r \int_0^t |b_2(s)|ds \\
&\leq \|x - \bar{x}\| \, b_1 \; + \; \|x - \bar{x}\| \, b_2 + 2 \, \varepsilon_4 + 2 \, r \, \varepsilon_5,
\end{aligned}$$

then

$$\|x - \bar{x}\| \quad \leq \quad \frac{2 \, \varepsilon_4 + 2 \, r \, \varepsilon_5}{1 - (b_1 \; + \; b_2)}.$$

That is

$$|x(t) - \bar{x}(t)| \; \leq \; \|x - \bar{x}\| \; \leq \; \varepsilon.$$

Consequently, $x \in BC(R+)$ is asymptotically stable of the problem (1)-(2).

4. Dependency

Consider the following assumption:

$(i)^*$ $\; f : R_+ \times R \to R$ is continuous in $t \in R_+$, $\forall x \in R$ and satisfies,

$$|f(t,x) - f^*(t,y)| \leq b_2(t) \; \forall \; t \in R_+, \; x, y \in R, \tag{11}$$

where

$$\lim_{t\to\infty}\int_0^t |b_2(s)|ds = 0,\ t \geq 0.$$

4.1. Dependency on the initial data x_0.

Theorem 5. *Let the assumptions of Theorems 2 and* $(i)^*$ *be satisfies, then the solution of (6) asymptotically dependency on the initial data* x_0 *if*
$\forall\ \varepsilon\ >\ 0,\ \exists\ \delta(\varepsilon)$ *such that* $|x_0 - x_0^*|\ <\delta\ \Rightarrow\ \|x-x^*\|\ <\ \varepsilon,\ t\ >\ T(\varepsilon)$
where x^* *is the solution of*

$$x^*(t) = x_0^* - \int_0^\tau g(s,x^*(s))ds - \int_0^\tau f(s,x^*(s))ds + \int_0^t f(s,x^*(s))ds.$$

Proof.

$$\begin{aligned}
|x(t)-x^*(t)| &= \Big| x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds \\
&- x_0^* - \int_0^\tau g(s,x^*(s))ds - \int_0^\tau f(s,x^*(s))ds + \int_0^t f(s,x^*(s))ds \Big| \\
&\leq |x_0 - x_0^*| + \int_0^\tau |g(s,x(s)) - g(s,x^*(s))|ds + \int_0^\tau |f(s,x(s)) - f(s,x^*(s))|ds \\
&+ \int_0^t |f(s,x(s)) - f(s,x^*(s))|ds \\
&\leq \delta + \int_0^\tau |b_1(s)||x(s)-x^*(s)|ds + \int_0^\tau |b_2(s)||x(s)-x^*(s)|ds \\
&+ \int_0^t |b_2(s)||x(s)-x^*(s)|ds.
\end{aligned}$$

(i) Choose $t \in [0,\ T]$, then we get

$$\|x-x^*\| \quad \leq \quad \delta + b_1\|x-x^*\| + b_2\|x-x^*\| + b_2^*\|x-x^*\|.$$

Hence

$$\|x-x^*\| \quad \leq \quad \frac{\delta}{1-(b_1+b_2+b_2^*)} = \varepsilon.$$

(ii) Choose $t > T(\varepsilon)$ and $\tau > 0$, then we have

$$\begin{aligned}
\|x-x^*\| &\leq \delta + b_1\ \|x-x^*\| +\ b_2\ \|x-x^*\| +\ \|x-x^*\|\ \int_0^t |b_2(s)|ds \\
&\leq \delta + b_1\ \|x-x^*\| +\ b_2\ \|x-x^*\| + \|x-x^*\|\ \varepsilon_1
\end{aligned}$$

$$\leq \quad \delta + b_1\ \|x - x^*\| + \ b_2\ \|x - x^*\| + \ \varepsilon_1^*.$$

Hence

$$\|x - x^*\| \quad \leq \quad \frac{\delta + \ \varepsilon_1^*}{1 - (b_1 + b_2)} = \varepsilon.$$

4.2. Dependency on the function g.

Theorem 6. *Let the assumptions of Theorems 2 and* $(i)^*$ *be satisfies, then the solution of (6) asymptotically dependency on the function* g *if*

$\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon)\ > 0$ *and* $T(\varepsilon) >\ 0$ *such that*

$$\int_0^\tau |g(s,x) - g^*(s,x)|ds\ <\ \delta,\ then\ for\ t > T(\varepsilon) \Rightarrow \|x - x^*\| < \varepsilon,$$

where x^* *is the solution of*

$$x^*(t) = x_0 - \int_0^\tau g^*(s, x^*(s))ds - \int_0^\tau f(s, x^*(s))ds + \int_0^t f(s, x^*(s))ds.$$

Proof.

$$\begin{aligned}
|x(t) - x^*(t)| \quad &= \quad \Big|x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds \\
&- \quad x_0 + \int_0^\tau g^*(s,x^*(s))ds + \int_0^\tau f(s,x^*(s))ds - \int_0^t f(s,x^*(s))ds\Big| \\
&\leq \quad \int_0^\tau \Big|g(s,x(s)) - g^*(s,x(s))\Big|ds + \int_0^\tau \Big|g^*(s,x(s)) - g^*(s,x^*(s))\Big|ds \\
&+ \quad \int_0^\tau \Big|f(s,x(s)) - f(s,x^*(s))\Big|ds + \int_0^t \Big|f(s,x(s)) - f(s,x^*(s))\Big|ds.
\end{aligned}$$

(i) Choose $t \in [0,\ T]$, then we get

$$\|x - x^*\| \quad \leq \quad \delta + b_1\|x - x^*\| + b_2\|x - x^*\| + b_2^*\|x - x^*\|.$$

Hence

$$\|x - x^*\| \quad \leq \quad \frac{\delta}{1 - (b_1 + b_2 + b_2^*)} = \varepsilon.$$

(ii) Choose $t > T(\varepsilon)$ and $\tau > 0$, then we have

$$\begin{aligned}
\|x - x^*\| \quad &\leq \quad \delta + b_1\|x - x^*\| + b_2\ \|x - x^*\| + \|x - x^*\|\ \int_0^t |b_2(s)|ds \\
&\leq \quad \delta + b_1\|x - x^*\| + b_2\ \|x - x^*\| + \|x - x^*\|\ \varepsilon_1
\end{aligned}$$

$$\leq \quad \delta + b_1\|x-x^*\| + b_2\ \|x-x^*\| + \ \varepsilon_1^*.$$

Hence

$$\|x-x^*\| \quad \leq \quad \frac{\delta + \ \varepsilon_1^*}{1-(b_1+b_2)} = \varepsilon.$$

4.3. Dependency on the function f.

Theorem 7. *Let the assumptions of Theorems 2 and $(i)^*$ be satisfies, then the solution of (6) asymptotically dependency on the function f if*
$\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0$ *and* $T(\varepsilon) >\ 0$ *such that*

$$\int_0^t |f(s,x) - f^*(s,x)|ds \ <\ \delta,\ then\ for\ t > T(\varepsilon) \Rightarrow \|x-x^*\| < \varepsilon,$$

where x^* is the solution of

$$x^*(t) = x_0 - \int_0^\tau g(s,x^*(s))ds - \int_0^\tau f^*(s,x^*(s))ds + \int_0^t f^*(s,x^*(s))ds.$$

Proof.

$$\begin{aligned} |x(t)-x^*(t)| \quad &= \quad \Big| x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds \\ &- \quad x_0 + \int_0^\tau g(s,x^*(s))ds + \int_0^\tau f^*(s,x^*(s))ds - \int_0^t f^*(s,x^*(s))ds \Big| \\ &\leq \quad \int_0^\tau \Big| g(s,x(s)) - g(s,x^*(s)) \Big| ds \ + \ \int_0^\tau \Big| f(s,x(s)) - f^*(s,x^*(s)) \Big| ds \\ &+ \quad \int_0^t \Big| f(s,x(s)) - f^*(s,x^*(s)) \Big| ds \\ &\leq \quad b_1\|x-x^*\| + \int_0^\tau \Big| f(s,x(s)) - f^*(s,x(s)) \Big| ds + \int_0^\tau \Big| f^*(s,x(s)) - f^*(s,x^*(s)) \\ &+ \quad \int_0^t \Big| f(s,x(s)) - f^*(s,x(s)) \Big| ds + \int_0^t \Big| f^*(s,x(s)) - f^*(s,x^*(s)) \Big| ds. \end{aligned}$$

(i) Choose $t \in [0,\ T]$, then we get

$$\|x-x^*\| \quad \leq \quad b_1\|x-x^*\| + \delta + b_2\|x-x^*\| + \delta + b_2^*\|x-x^*\|.$$

Hence

$$\|x-x^*\| \quad \leq \quad \frac{2\ \delta}{1-(b_1+b_2+b_2^*)} = \varepsilon.$$

(ii) Choose $t > T(\varepsilon)$ and $\tau > 0$, then we have

$$\begin{aligned}\|x-x^*\| &\leq b_1\|x-x^*\| + \delta + \|x-x^*\| \int_0^\tau |b_2(s)|ds + \delta + \|x-x^*\| \int_0^t |b_2(s)|ds \\ &\leq b_1\|x-x^*\| + \delta + \|x-x^*\|\, b_2 + \delta + \|x-x^*\|\, \varepsilon_1 \\ &\leq b_1\|x-x^*\| + \delta + \|x-x^*\|\, b_2 + \delta + \varepsilon_1^*.\end{aligned}$$

Hence

$$\|x-x^*\| \leq \frac{2\,\delta + \varepsilon_1^*}{1-(b_1+b_2)} = \varepsilon.$$

5. Hyers - Ulam stability

Definition 1. [11, 19, 20] *The problem (1)-(2) is Hyers - Ulam stable if* $\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0$ *and* $T(\varepsilon) > 0$ *such that* $t > T(\varepsilon)$ *and for any* $\delta -$ *approximate solution* x_s *, satisfies,*

$$(12)\quad \left|\frac{dx_s}{dt} - f(t,x_s(t))\right| < \delta\, a(t) \textit{ implies } \|x-x_s\| < \varepsilon, \textit{ where } \int_0^t a(s)\, ds \leq k,\ t > T(\varepsilon).$$

Theorem 8. *Let the assumptions of Theorem 2 and* $(i)^*$ *be satisfied, then the problem (1)-(2) is Hyers - Ulam stable.*

Proof. From (12), we have

$$\begin{aligned}-\delta\, a(t) &\leq \frac{dx_s}{dt} - f(t,x_s(t)) \leq \delta\, a(t) \\ -\delta^* &= -\delta \int_0^t a(s)ds \leq x_s(t) - x_s(0) - \int_0^t f(s,x_s(s))ds \leq \delta \int_0^t a(s)ds = \delta^* \\ -\delta^* &\leq x_s(t) - x_0 + \int_0^\tau g(s,x_s(s))ds + \int_0^\tau f(s,x_s(s))ds - \int_0^t f(s,x_s(s))ds \leq \delta^*\end{aligned}$$

Now,

$$\begin{aligned}|x(t)-x_s(t)| &= \left|x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds - x_s(t)\right| \\ &\leq \left|x_0 - \int_0^\tau g(s,x(s))ds - \int_0^\tau f(s,x(s))ds + \int_0^t f(s,x(s))ds\right. \\ &- \left. x_0 + \int_0^\tau g(s,x_s(s))ds + \int_0^\tau f(s,x_s(s))ds - \int_0^t f(s,x_s(s))ds\right| \\ &+ \left|x_s(t) - x_0 + \int_0^\tau g(s,x_s(s))ds + \int_0^\tau f(s,x_s(s))ds - \int_0^t f(s,x_s(s))ds\right|\end{aligned}$$

$$\leq \int_0^\tau |g(s,x(s))-g(s,x_s(s))|ds+\int_0^\tau |f(s,x(s))-f(s,x_s(s))|ds$$
$$+ \int_0^t |f(s,x(s))-f(s,x_s(s))|ds+\delta^*.$$

(i) Choose $t\in[0,\ T]$, then we get

$$\|x-x_s\| \leq b_1\|x-x_s\|+b_2\|x-x_s\|+b_2^*\|x-x_s\|+\delta^*.$$

Hence

$$\|x-x_s\| \leq \frac{\delta^*}{1-(b_1+b_2+b_2^*)}=\varepsilon.$$

(ii) Choose $t>T(\varepsilon)$ and $\tau>0$, then we have

$$\begin{aligned}\|x-x_s\| &\leq b_1\ \|x-x_s\|+\ b_2\ \|x-x_s\|+\ \|x-x_s\|\int_0^t |b_2(s)|ds+\delta^*\\ &\leq b_1\ \|x-x_s\|+\ b_2\ \|x-x_s\|+\|x-x_s\|\ \varepsilon_1+\delta^*\\ &\leq b_1\ \|x-x_s\|+\ b_2\ \|x-x_s\|+\ \varepsilon_1^*+\delta^*.\end{aligned}$$

Hence

$$\|x-x^*\| \leq \frac{\delta^*+\ \varepsilon_1^*}{1-(b_1+b_2)}=\varepsilon.$$

Example.

Taking into account the equation

$$\frac{dx}{dt}=\frac{t\ e^{-t}}{3}+\frac{(t\ e^{-t}-e^{-t})|x(t)|}{8},\ t\in(0,\infty), \tag{13}$$

with the nonlocal integral condition

$$x(1)+\int_0^1 (e^{-ln(s+1)}+\frac{e^s|x(s)|}{5})ds=x_0 \tag{14}$$

Set

$$f(t,x)=\frac{t\ e^{-t}}{3}+\frac{(t\ e^{-t}-e^{-t})|x(t)|}{8},$$
$$g(t,x)=e^{-ln(t+1)}+\frac{e^t|x(t)|}{5}.$$

Putting

$$v = ln(2) = 0.69314718,\ b_1\ = \frac{e}{5} = 0.5436563657,$$
$$b_2^* = b_2 = \frac{1}{8},\ m^* = m = \frac{1}{3}.$$

we can find that

$$b_1 + b_2 + b_2^*\ =\ 0.7936563657 < 1.$$

then the problem (13)-(14) has at least one solution $x \in BC(R_+)$.

6. Conclusions

In this investigation, the asymptotic stability and dependency of the solutions for differential equation have been established on R_+. Firstly, we discussed two cases for study investigated the solvability of the problem (1)-(2): In the first case, we studied the existences of solutions $x \in BC(R_+)$ of the problem (1)-(2), by applying the technique associated with the MNC in the Banach space $BC(R_+)$. In the second case, we used Schauder's fixed point Theorem. Next, we studied the asymptotic stability and dependency of the solution $x \in BC(R_+)$ on the initial data x_0 and on the functions f and g. Moreover, we studied the Hyers-Ulam stability. Finally, we discussed the examples to illustrate our results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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