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A NEW FIXED POINT THEOREM IN CONE B-METRIC SPACES

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Abstract. The aim of this paper is to prove common fixed point theorem for quadruple mappings in cone b-metric spaces. Our result extends and improves some fixed point results in cone b-metric spaces. We illustrate our main result by an example.

Keywords: fixed point; cone metric; cone b-metric; contractive mapping; partial ordering.

2020 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The concept of b-metric space was introducd by Bakhtin [2]. He proved the principal of contraction mapping in b- metric spaces. Huang and Zhang [6] introduced the concept of cone metric space as a generalisation of metric space. In addition, certain fixed point theorems have been demonstrated for contractive mapping which extended the results of fixed point in metric spaces. Hussain and Shah [7] introduced the cone b-metric spaces as a generalization of b-metric spaces for KKM mappings. Sharma [9] used rational expression to prove fixed point

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theorem for contractive mapping without the assumption of normality in cone b-metric spaces. Similar work has been done in [1,3,4,5].

2. PRELIMINARIES

We use the following definitions for our main result:

Let E be a real Banach space and P be a subset of E. We denote the zero element of E by θ and the interior of P by $intP$. The subset P is called a cone iff :

(1) P is closed, nonempty, and $P \neq \theta$

(2) $a, b \in \mathbb{R},\ a,\ b \geq 0,\ x,\ y \in P$ *implies* $ax + by \in P$

(3) $P \cap (-P) = \theta$.

Definition 2.1. [9] *Let X be a nonempty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies:*

(1) $\theta \leq d(x,y)$ *for all* $x, y \in X$ *with* $x \neq y$ *and* $d(x,y) = \theta$ *iff* $x = y$;

(2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$*;*

(3) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) *is called a cone metric space.*

Definition 2.2. [9] *Let X be a nonempty set and s* $\geq$*1 be a given real number. A mapping* $d : X \times X \to E$ *is said to be cone b-metric iff, for all* $x, y, z \in X$*, the following conditions are satisfied:*

(1) $\theta \leq d(x,y)$ *with* $x \neq y$ *and* $d(x,y) = \theta$ *iff* $x = y$*;*

(2) $d(x,y) = d(y,x)$ *;*

(3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then pair (X,d) *is called a cone b-metric spaces.*

Definition 2.3. [9] *Let* (X,d) *be a cone b-metric space,* $x \in X$ *and* x_n *be a sequence in X. Then*

(1) x_n *converges to x whenever, for every* $c \in E$ *with* $\theta < c$*, there is a natural number* $\mathbb{N}$ *such that* $d(x_n,x) < c$ *for all* $n \geq \mathbb{N}$*. We denote this by* $x_n \to x$ *as* $(n \to \infty)$.

(2) $\{x_n\}$ *is a Cauchy sequence whenever, for every* $c \in E$ *with* $\theta < c$*, there is a natural number* $\mathbb{N}$ *such that* $d(x_n,x_m)$ $<$*c for all n, m* $\geq \mathbb{N}$.

(3) (X,d) *is a complete cone b-metric space if every Cauchy sequence in X is convergent.*

Definition 2.4. [8] *Let X be a nonempty set and $k \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric iff for each $x, y, z \in X$, following conditions are satisfied:*

(1) $d(x,y) = 0 \; iff \; x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \leq k[d(x,y) + d(y,z)]$.

Then the pair (X,d) is called a b-metric space. It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces. Indeed, a b-metric is a metric iff $k = 1$.

Definition 2.5. [8] *Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*
(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(b) Cauchy iff $d(x_n, x_m) \to 0$, as $n, m \to \infty$, where $n, m \geq N \in \mathbb{N}$.

Lemma 2.1. [8] *Let (X, d) be a b-metric space with $k \geq 1$. Suppose that x_n and y_n are b-convergent to x and y, respectively. Then, we have $\frac{1}{k^2} d(x,y) \leq \lim_{n\to\infty} \inf d(x_n, y_n) \leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq k^2 d(x,y)$.*
In particular, if x = y, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have $\frac{1}{k} d(x,z) \leq \lim_{n\to\infty} \inf d(x_n, z) \leq \lim_{n\to\infty} \sup d(x_n, z) \leq k d(x,z)$.

Lemma 2.2. [8] *Let (X, d) be a b-metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} y_n = t$.*
In this section, we present common fixed point theorem for contractive mapping in the setting of cone b-metric spaces:

3. Main Results

Theorem 3.1. *Let (X,d) be a complete cone b-metric space with the coefficient $k \geq 1$. Suppose the mappings $A, B, S, T : X \to X$ satisfy the following conditions:*

(1) $AX \subseteq TX$, $BX \subseteq SX$;

(2) The pair (A,T) and (B,S) are compatible;

(3) S and T are continuous; for all $x, y \in X$, $k \geq 1$.

(4) $d(Ax,By) \leq \frac{q}{k^4} max\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Sx,By)+d(Ax,Ty)}{2}\}$

Then S,T,A and B have a unique common fixed point in X.

Proof. Since (X,d) is a complete cone b-metric space with the coefficient $k \geq 1$, Suppose mapping $A,B,S,\ T : X \to X$ satisfy all above four conditions.

$$d(Ax,By) \leq \frac{q}{k^4} max\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Sx,By)+d(Ax,Ty)}{2}\}. \tag{3.1}$$

Since $AX \subseteq TX$ *implies* $Ax_0 = Tx_1 = y_1$ (say), $BX \subseteq SX$ *implies* $Bx_1 = Sx_2 = y_2$ (say) $Ax_{2n} = Tx_{2n+1} = y_{2n+1}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ for n = 0,1,2,....

putting $x = x_{2n},\ y = y_{2n+1}$ *and let* $\frac{q}{k^4} = \beta$ (say)

$$\begin{aligned} d(Ax_{2n},Bx_{2n+1}) &\leq \beta\ max\{d(Sx_{2n},Tx_{2n+1}), \frac{d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}{2}, \\ &\frac{d(Sx_{2n},Bx_{2n+1})+d(Ax_{2n},Tx_{2n+1})}{2}\} \\ d(y_{2n+1},y_{2n+2}) &\leq \beta\ max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n+1},y_{2n})+d(y_{2n+2},y_{2n+1})}{2}, \\ &\frac{d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}{2}\} \\ &= \beta\ max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}, \\ &\frac{d(y_{2n},y_{2n+2})+0}{2}\} \\ &= \beta\ max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}, \\ &\frac{d(y_{2n},y_{2n+2})}{2}\}. \end{aligned}$$

Case 1: If $\max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}, \frac{d(y_{2n},y_{2n+2})}{2}\} = d(y_{2n},y_{2n+1})$, then

$$d(y_{2n+1},y_{2n+2}) \leq \beta d(y_{2n},y_{2n+1}). \tag{3.2}$$

Case 2: If $\max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}, \frac{d(y_{2n},y_{2n+2})}{2}\} = \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}$, then

$$d(y_{2n+1},y_{2n+2}) \leq \beta\{\frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}\}$$

$$or\ d(y_{2n+1},y_{2n+2})\left[1-\frac{\beta}{2}\right] \leq \frac{\beta}{2} d(y_{2n},y_{2n+1})$$

$$or\ d(y_{2n+1},y_{2n+2}) \leq \frac{\frac{\beta}{2}}{[1-\frac{\beta}{2}]} d(y_{2n},y_{2n+1})$$

$$\leq k\{d(y_{2n},y_{2n+1})\},\ where\ k = \frac{\frac{\beta}{2}}{1-\frac{\beta}{2}}. \tag{3.3}$$

Case 3: If $\max\{d(y_{2n},y_{2n+1}), \frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}, \frac{d(y_{2n},y_{2n+2})}{2}\} = \frac{d(y_{2n},y_{2n+2})}{2}$

then

$$d(y_{2n+1},y_{2n+2}) \leq \frac{\beta}{2} d(y_{2n},y_{2n+2})$$

$$\leq \frac{\beta}{2}\{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})\}$$

$$or\ d(y_{2n+1},y_{2n+2})\left[1-\frac{\beta}{2}\right] \leq \frac{\beta}{2} d(y_{2n},y_{2n+1})$$

$$or\ d(y_{2n+1},y_{2n+2}) \leq \frac{\frac{\beta}{2}}{[1-\frac{\beta}{2}]} d(y_{2n},y_{2n+1})$$

$$\leq k\{d(y_{2n},y_{2n+1})\},\ where\ k = \frac{\frac{\beta}{2}}{1-\frac{\beta}{2}}. \tag{3.4}$$

Using equation (3.2), (3.3) and (3.4), we get $d(y_{2n+1},y_{2n+2}) \leq kd(y_{2n},y_{2n+1})$

Since $AX \subseteq TX$ implies that $Ax_0 = Tx_1 = y_1$ (say) $BX \subseteq SX$ implies that $Bx_1 = Sx_2 = y_2$ (say).

$Ax_{2n} = Tx_{2n+1} = y_{2n+1}$ $Bx_{2n+1} = Sx_{2n+2} = y_{2n+2}$ putting $x = x_{2n+2}, y = x_{2n+1}$

$$d(Ax_{2n+2},Bx_{2n+1}) \leq \beta\ max\{d(Sx_{2n+2},Tx_{2n+1}),$$

$$\frac{d(Ax_{2n+2},Sx_{2n+2})+d(Bx_{2n+1},Tx_{2n+1})}{2},$$

$$\frac{d(Sx_{2n+2},Bx_{2n+1})+d(Ax_{2n+2},Tx_{2n+1})}{2}\}$$

$$or\ d(y_{2n+3},y_{2n+2}) \leq \beta\ max\{d(y_{2n+2},y_{2n+1}),$$

$$\frac{d(y_{2n+3},y_{2n+2})+d(y_{2n+2},y_{2n+1})}{2},$$

$$\frac{d(y_{2n+2},y_{2n+2})+d(y_{2n+3},y_{2n+1})}{2}\}$$

$$\leq \beta\ max\{d(y_{2n+2},y_{2n+1}),$$

$$\frac{d(y_{2n+3},y_{2n+2})+d(y_{2n+2},y_{2n+1})}{2},$$
$$\frac{d(y_{2n+3},y_{2n+1})}{2}\}.$$

Case1: If $max\{d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n+3},y_{2n+2})+d(y_{2n+2},y_{2n+1})}{2},\frac{d(y_{2n+1},y_{2n+3})}{2}\}=d(y_{2n+2},y_{2n+1})$, then

$$d(y_{2n+2},y_{2n+3})\leq\beta\{d(y_{2n+1},y_{2n+2})\}. \tag{3.5}$$

Case2: If $max\{d(y_{2n},y_{2n+1}),\frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2},\frac{d(y_{2n},y_{2n+2})}{2}\}=\frac{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}{2}$, then

$$\begin{aligned}
d(y_{2n+2},y_{2n+3})&\leq\beta\{\frac{d(y_{2n+3},y_{2n+2})+d(y_{2n+2},y_{2n+1})}{2}\}\\
or\ d(y_{2n+2},y_{2n+3})\left[1-\frac{\beta}{2}\right]&\leq\frac{\beta}{2}d(y_{2n+1},y_{2n+2})\\
or\ d(y_{2n+2},y_{2n+3})&\leq\frac{\frac{\beta}{2}}{[1-\frac{\beta}{2}]}d(y_{2n+1},y_{2n+2})\\
&\leq k\{d(y_{2n+1},y_{2n+2})\},\ where\ k=\frac{\frac{\beta}{2}}{1-\frac{\beta}{2}}. \qquad (3.6)
\end{aligned}$$

Case3: If $max\{d(y_{2n+2},y_{2n+1}),\frac{d(y_{2n+3},y_{2n+2})+d(y_{2n+2},y_{2n+1})}{2},\frac{d(y_{2n+1},y_{2n+3})}{2}\}=d(y_{2n+1},y_{2n+3})$, then

$$\begin{aligned}
d(y_{2n+2},y_{2n+3})&\leq\frac{\beta}{2}\left[d(y_{2n+1},y_{2n+2})+d(y_{2n+2},y_{2n+3})\right]\\
or\ d(y_{2n+2},y_{2n+3})\left[1-\frac{\beta}{2}\right]&\leq\frac{\beta}{2}d(y_{2n+1},y_{2n+2})\\
or\ d(y_{2n+2},y_{2n+3})&\leq\frac{\frac{\beta}{2}}{\left[1-\frac{\beta}{2}\right]}d(y_{2n+1},y_{2n+2})\\
&\leq k\{d(y_{2n+1},y_{2n+2})\},\ where\ k=\frac{\frac{\beta}{2}}{1-\frac{\beta}{2}}. \qquad (3.7)
\end{aligned}$$

Using equation (3.5), (3.6) and (3.7), we get

$d(y_{2n+2},y_{2n+3})\leq kd(y_{2n+1},y_{2n+2})$. Hence $\{y_n\}$ is Cauchy sequence.

Since X is complete cone b-metric space, so there exists some y in X, such that,

$$\lim_{n\to\infty}Ax_{2n}=\lim_{n\to\infty}Tx_{2n+1}=\lim_{n\to\infty}Bx_{2n+1}=\lim_{n\to\infty}Sx_{2n+2}=y\ (say). \tag{3.8}$$

We will show that y is a common fixed point in A, T, B and S.

Since S is continuous. Therefore

$$lim_{n\to\infty} S^2 x_{2n+2} = Sy \text{ and } \lim_{n\to\infty} SAx_{2n} = Sy. \tag{3.9}$$

Since pair (A, S) is compatible. Therefore $\lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$. So,

$\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} SAx_{2n}$ and $\lim_{n\to\infty} ASx_{2n} = Sy$.

Now, put $Sx_{2n} = x \text{ and } x_{2n+1} = y$ in the inequality (3.1), we have

$$\begin{aligned} d(ASx_{2n}, Bx_{2n+1}) \leq \beta\{max\{&d(S^2x_{2n}, Tx_{2n+1}), \\ &\frac{d(ASx_{2n}, S^2x_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})}{2}, \\ &\frac{[d(S^2x_{2n}, Bx_{2n+1}) + d(ASx_{2n}, Tx_{2n+1})]}{2}\} \end{aligned} \tag{3.10}$$

Taking upper limit $\lim_{n\to\infty}$, by Lemma 2.6

$$\begin{aligned} \frac{d(Sy,y)}{k^2} &\leq \frac{q}{k^2}\{max \lim_{n\to\infty} \sup\{d(S^2x_{2n}, Tx_{2n+1}), \\ &\frac{lim_{n\to\infty} \sup d(ASx_{2n}, S^2x_{2n}) + lim_{n\to\infty} \sup d(Bx_{2n+1}, Tx_{2n+1})}{2}, \\ &\frac{lim_{n\to\infty} \sup d(ASx_{2n}, S^2x_{2n}) + lim_{n\to\infty} \sup d(Bx_{2n+1}, Tx_{2n+1})}{2} \\ &\leq \frac{q}{k^4}\{max\{k^2 d(Sy,y), 0, \frac{k^2}{2}[d(Sy,y) + d(Sy,y)]\}, \text{ which implies} \end{aligned} \tag{3.11}$$

$d(Sy,y) \leq \frac{q}{k^2}\{d(Sy,y)\} < d(sy,y)$, a contradiction. Therefore $Sy = y$, because $0 < q < 1$. Similarly, $Ay = y$ and finally $Sy = Ty = Ay = By = y$. So, A, B, S and T have common fixed point.

Uniqueness of fixed point:

Let A, B, S and T have another fixed point x(say). So,

$$\begin{aligned} d(x,y) &= d(Ax, By) \\ &\leq \frac{q}{k^4} max\{d(Sx, Ty), \frac{d(Ax,Sx) + d(By,Ty)}{2}, \\ &\quad \frac{d(Sx,By) + d(Ax,Ty)}{2}\} \\ &\leq \frac{q}{k^4} max\{d(x,y), \frac{d(x,x) + d(y,y)}{2}, \frac{d(x,y) + d(x,y)}{2}\} \\ &\leq \frac{q}{k^4} max\{d(x,y)\} \end{aligned}$$

$x = y$ because $0 < q < 1$. Fixed point is unique. This completes the proof.

We illustrate our theorem by the following example. □

Example: Let $E = \mathbb{R}^2,\ p = \{(x,y) \in E : (x,y) \geq 0\} \subset E$. X =[0,1] and $d : X \times X \to E$, such that $d(x,y) = (\alpha|x-y|^2, |x-y|^2)$, where $\alpha \geq 0$ Then (X,d) is a cone b-metric space. Now define A, B, S, T : X $\times X \to Y$ such that, $A(x) = (\frac{x}{4})^{12},\ B(x) = (\frac{x}{4})^8,\ S(x) = (\frac{x}{4})^8,\ T(x) = (\frac{x}{4})^4$

here (i) $AX \subseteq TX,\ BX \subseteq SX$;

(ii) The pair (A,T) and (B,S) are compatible;

(iii) S and T are continuous;

$$\begin{aligned}
(iv) d(Ax,By) &= (\alpha|Ax-By|^2, |Ax-By|^2)\\
&= (\alpha|(\frac{x}{4})^{12} - (\frac{x}{4})^8|^2, |(\frac{x}{4})^{12} - (\frac{x}{4})^8|^2)\\
&= (\alpha|(\frac{x}{4})^4\{(\frac{x}{4})^8 - (\frac{x}{4})^4\}|^2, |(\frac{x}{4})^4\{(\frac{x}{4})^8 - (\frac{x}{4})^4\}|^2)\\
&= (\frac{x}{4})^8(\alpha|(\frac{x}{4})^8 - (\frac{x}{4})^4|^2, |(\frac{x}{4})^8 - (\frac{x}{4})^4|^2)\\
&= |(\frac{x}{4})^8| d(Sx,Ty)\\
&\leq (\frac{1}{4})^8 d(Sx,Ty)\\
&\leq (\frac{1}{4})^8 max\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2},\\
&\quad \frac{d(Sx,By)+d(Ax,Ty)}{2}\}\\
&\leq \frac{(\frac{1}{4})^4}{4^4} max\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2},\\
&\quad \frac{d(Sx,By)+d(Ax,Ty)}{2}\}\\
&\leq (\frac{q}{k^4}) max\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2},\\
&\quad \frac{d(Sx,By)+d(Ax,Ty)}{2}\},
\end{aligned}$$

where $(\frac{1}{4})^4 \leq q < 1$ for $k = 4$. We obseerve that $x = 0$ is the unique common fixed point of A, B, S, T. This validates Theorem 3.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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