Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:35
https://doi.org/10.28919/afpt/8625
ISSN: 1927-6303

CONVERGENCE OF PROXIMAL GRADIENT METHOD WITH ALTERNATED INERTIAL STEP FOR MINIMIZATION PROBLEM

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Abstract. This paper proposes an alternating inertial-type extrapolation technique for solving the convex minimization problem. We provide a self-adaptive proximal gradient technique using an inertial step. Under certain conditions, the strong convergence theorem is established. The numerical results illustrate the performances of our algorithm.

Keywords: convex optimization; inertial acceleration; proximal gradient algorithm.

2020 AMS Subject Classification: 47H09, 47H10.

1. Introduction

The approximation gradient approach is one of the most popular approaches for calculating the minimum value of a non-smooth function. It employs many gradient steps in the first function before conducting the proximity function in the second. Inertia has long been known to improve both theoretical and real convergence rates for this method. It is also known as Neserov's acceleration [1].

Let H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ provided $B : H \to 2^H$ is the maximal monotone operator, and $A : H \to H$ Lipschitz approached the problem as

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Received May 02, 2024

follows: find $v \in H$, which

$$0 \in (A+B)v. \tag{1.1}$$

The solution set of the problem (1.1) is represented by

$$(A+B)^{-1}(0). \tag{1.2}$$

The problem (1.1) consists of special cases: Linear inversion problems include convex constraints, discrete likelihood problems, minimization problems, fixed point problems, and other problems. The problem (1.1) involves optimizing the sum of two functions, as shown below:

$$\min_{v \in H}(g(v)+h(v)), \tag{1.3}$$

where H is a real Hilbert space, $g,h : H \to \mathbb{R} \cup \{+\infty\}$ are two proper, lower semi-continuous and convex and g is differentiable with the Lipschitz continuous gradient denoted by ∇g. It is known that v^* is a minimizer of $g+h$ if and only if

$$0 \in (\partial h + \nabla g)(v^*), \tag{1.4}$$

where ∂h denotes the subdifferential of h.

The most popular strategy for solving the convex minimization problem is the so-called forward-backward algorithm, which creates from a starting point $v_1 \in H$ and

$$v_{i+1} = \text{prox}_{\rho h}(v_i - \rho \nabla g(v_i)), \quad i \geq 1, \tag{1.5}$$

where prox_h is the proximal operator of h and the step size $\rho \in (0, 2/L)$, L is the Lipschitz constant of ∇g.

Tseng [2] proposed the modified forward-backward algorithm described below, which employs the stepsize with linesearch technique. Given $\alpha > 0, \gamma \in (0,1)$, $\mu \in (0,1)$, and $v_1 \in H$. Compute

$$\begin{aligned} z_i &= \text{prox}_{\rho_i h}(v_i - \rho_i \nabla g(v_i)), \\ v_{i+1} &= \text{prox}_{\rho_i h}(z_i - \rho_i(\nabla g(z_i) - \nabla g(v_i))), \quad \forall i \geq 1, \end{aligned} \tag{1.6}$$

where ρ_i is the largest $\rho \in \{\alpha, \alpha\gamma, \alpha\gamma^2, \dots\}$ satisfying

$$\rho \|\nabla g(z_i) - \nabla g(v_i)\| \leq \mu \|z_i - v_i\|.$$

Malitsky and Tam [3] proposed the forward-reflected-backward algorithm. Given $\rho_0, \rho_1 > 0, \gamma \in \{1, \beta^{-1}\}$, $\beta \in (0,1)$, $\delta \in (0,1)$ and v_0, $v_1 \in H$. Compute

$$v_{i+1} = \text{prox}_{\rho_i h}(v_i - \rho_i \nabla g(v_i) - \rho_{i-1}(\nabla g(v_i) - \nabla g(v_{i-1}))), \tag{1.7}$$

where the stepsize $\rho_i = \gamma \rho_{i-1} \beta^n$ with n being the smallest nonnegative integer satisfying

$$\rho_i \|\nabla g(v_{i+1}) - \nabla g(v_i)\| \leq \frac{\delta}{2} \|v_{i+1} - v_i\|.$$

Very recently, Hieu et al. [4] proposed the modified forward-reflected-backward method with adaptive stepsize. Given ρ_0, $\rho_1 > 0, \mu \in \left(0, \frac{1}{2}\right)$ and v_0, $v_1 \in H$. Compute

$$\begin{aligned} v_{i+1} &= \text{prox}_{\rho_i h}(v_i - \rho_i \nabla g(v_i) - \rho_{i-1}(\nabla g(v_i) - \nabla g(v_{i-1}))), \\ \rho_{i+1} &= \min\{\mu \frac{\|v_{i+1} - v_i\|}{\|\nabla g(v_{i+1}) - \nabla g(v_i)\|}, \rho_i\}. \end{aligned} \tag{1.8}$$

This stepsize enables the suggested method to solve the problem even without knowledge of the Lipschitz constant.

Inertial techniques (refer to [5]) were investigated to speed up algorithm convergence. They were employed in various numerical approaches to address optimization issues in finite-dimensional and infinite-dimensional spaces; see, for example, [6, 7, 8, 9, 10, 11, 12, 13, 14, 12, 16], and the references therein.

Mu and Peng [17] developed an alternate inertial technique to restore Fejér monotonicity to the even subsequence associated with the problem's solution set. Remember that the primary notion behind the alternated inertial approach is to add inertial effects only at odd iteration steps rather than even iteration steps, which is where the term ”alternated” comes from. Given a sequence of nonnegative parameters $\{\gamma_i\}$ and v_0, $v_1 \in H$. Compute

$$\begin{cases} v_i, & \text{if } i = \text{even}, \\ v_i + \gamma_i(v_i - v_{i-1}), & \text{if } i = \text{odd}. \end{cases} \tag{1.9}$$

Inspired and motivated by previous work, we provide an adaptive stepsize-based alternated inertial proximal gradient approach for convex minimization problems. We establish weak convergence of our scheme under certain assumptions.

2. Preliminaries

The symbols $\to$ and $\rightharpoonup$ mean the strong convergence and the weak convergence, respectively. It is well known that the following equation

$$\|av+(1-a)z\|^2 = a\|v\|^2+(1-a)\|z\|^2-a(1-a)\|v-z\|^2, \tag{2.1}$$

holds for all $v,z \in H$ and $a \in \mathbb{R}$.

Let $h : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. We denote the domain of h by $dom(h) = \{v \in H \,|\, h(v) < +\infty\}$. For any $v \in dom(h)$, the subdifferential of h at v is defined by

$$\partial h(v) = \{u \in H \,|\, \langle u, z-v\rangle \leq h(z)-h(v), z \in H\}.$$

Recall that the proximal operator $\text{prox}_h : dom(h) \to H$ is defined as $\text{prox}_h(v) = (I+\partial h)^{-1}(y), y \in H$. It is known that the proximal operator is single-valued. Moreover, we have

$$\frac{y-\text{prox}_{\rho h}(y)}{\rho} \in \partial h(\text{prox}_{\rho h}(y)) \quad \text{for all } y \in H, \rho > 0. \tag{2.2}$$

Definition 2.1. [18] *Let Ψ be a nonempty subset of H. A sequence $\{v_i\}$ in H is said to be quasi-Fejér convergent to Ψ if and only if for all $v \in \Psi$ there exists a positive sequence $\{\varepsilon_i\}$ such that*

Lemma 2.1. [19] *If $\{v_i\}$ is quasi-Fejér convergent to Ψ, then we have*

(i) *$\{v_i\}$ is bounded.*

(ii) *If all weak accumulation points of $\{v_i\}$ is in Ψ, then $\{v_i\}$ weakly converges to a point in Ψ.*

Lemma 2.2. [20] *The subdifferential operator ∂h is maximal monotone. Moreover, the graph of $\partial h, Gph(\partial h) = \{(v,z) \in H \times H : z \in \partial h(v)\}$ is demiclosed, i.e., if the sequence $\{(v_i,z_i)\} \subset Gph(\partial h)$ satisfies that $\{v_i\}$ converges weakly to v and $\{z_i\}$ converges strongly to z, then $(v,z) \in Gph(\partial h)$.*

Lemma 2.3. [21] *Let $\{r_i\}$, $\{p_i\}$ and $\{q_i\}$ be real positive sequences such that*

$$r_{i+1} \leq (1+q_i)r_i + p_i, \quad \textit{for all } i \geq 1.$$

If $\sum_{i=1}^{\infty} q_i < +\infty$ *and* $\sum_{i=1}^{\infty} p_i < +\infty$, *then* $\lim\limits_{i\to+\infty} r_i$ *exists.*

Lemma 2.4. [22] *Let* $\{r_i\}$ *and* $\{\gamma_i\}$ *be real positive sequences such that*

$$r_{i+1} \le (1+\gamma_i)r_i + \gamma_i r_{i-1}, \quad \textit{for all } i \ge 1.$$

Then, $r_{i+1} \le K \cdot \prod_{a=1}^{i}(1+2\gamma_a)$ *where* $K = \max\{r_1, r_2\}$. *Moreover, if* $\sum_{i=1}^{\infty} \gamma_i < +\infty$, *then* $\{r_i\}$ *is bounded.*

3. The Main Results

This section assumes that the following requirements are met for our convergence analysis:

$(C1)$ The solution set of the convex minimization problem (1.3) is nonempty, i.e.,

$$\Omega = \arg\min(h+g) \neq \emptyset.$$

$(C2)$ Let $\rho_1 > 0, \beta \in (0,1), \gamma \in (0,1), \delta \in (0,1)$, $\lim\limits_{i\to\infty} \delta_i = 0$, and $\{\sigma_i\} \subset [0,+\infty)$ such that $\sum_{i=1}^{\infty} \sigma_i < +\infty$.

$(C3)$ $g, h : H \to \mathbb{R} \cup \{+\infty\}$ are two proper, lower semicontinuous and convex functions.

$(C4)$ g is differentiable on H and ∇g is Lipschitz continuous on H with the Lipschitz constant $L > 0$.

Next, we present a new inertial forward-backward approach for solving (1.3).

Algorithm 3.1.

Initialization: *Let* $\rho_1, \beta, \gamma, \delta, \delta_i, \{\sigma_i\}$ *such that Conditions* $(C2)$ *hold. Choose initial points* $v_0 = v_1 \in H$. *Set* $i := 1$.

Step 1. *Compute*

$$z_i = \begin{cases} v_i, & \textit{if } \ i = \textit{even}, \\ v_i + \gamma(v_i - v_{i-1}), & \textit{if } \ i = \textit{odd}. \end{cases} \tag{3.1}$$

Step 2. *Compute* $s_i = \operatorname{prox}_{\rho_i h}(z_i - \rho_i \nabla g(z_i))$. *If* $s_i = z_i$ *then the iteration stops and* s_i *is the solution of* (1.3)*; otherwise, turn to* **Step 3.**

Step 3. *Compute* $w_i = s_i - \rho_i(\nabla g(s_i) - \nabla g(z_i))$.

Step 4. *Compute* $v_{i+1} = (1-\beta)z_i + \beta w_i$. *and update the next step size* ρ_{i+1} *by*

$$\rho_{i+1} = \begin{cases} \min\left\{ \frac{(\delta_i+\delta)\|z_i - s_i\|}{\|\nabla g(z_i) - \nabla g(s_i)\|}, \rho_i + \sigma_i \right\}, & \text{if } \|\nabla g(z_i) - \nabla g(s_i)\| \neq 0, \\ \rho_i + \sigma_i, & \text{otherwise} \end{cases} \tag{3.2}$$

Set $i := i+1$ *and go to* **Step 1.**

Lemma 3.1. *The sequences* $\{\rho_i\}$ *from Algorithm 3.1 is bounded and* $\rho_i \in [\min\{\frac{\delta}{L}, \rho_1\}, \rho_1 + \Lambda]$. *Furthermore there exists* $\rho \in [\min\{\frac{\delta}{L}, \rho_1\}, \rho_1 + \Lambda]$ *such that* $\lim\limits_{i \to +\infty} \rho_i = \rho$, *where* $\Lambda = \sum_{i=1}^{\infty} \sigma_i$.

Proof. By the definition of ρ_i, if $\|\nabla g(z_i) - \nabla g(s_i)\| \neq 0$, we get

$$\rho_i \geq \frac{(\delta_i + \delta)\|z_i - s_i\|}{\|\nabla g(z_i) - \nabla g(s_i)\|} \geq \frac{\delta_i + \delta}{L} \geq \frac{\delta}{L}. \tag{3.3}$$

From $\Lambda = \sum_{i=1}^{\infty} \sigma_i$, we have

$$\rho_{i+1} \leq \rho_i + \sigma_i \leq \rho_1 + \sum_{i=1}^{\infty} \sigma_i = \rho_1 + \Lambda, \tag{3.4}$$

which implies that

$$\left\{ \frac{\delta}{L}, \rho_1 \right\} \leq \rho_i \leq \rho_1 + \Lambda. \tag{3.5}$$

We have

$$\rho_{i+1} - \rho_i = [\rho_{i+1} - \rho_i]_+ - [\rho_{i+1} - \rho_i]_-, \tag{3.6}$$

where

$$[\rho_{i+1} - \rho_i]_+ = \max\{0, \rho_{i+1} - \rho_i\},$$

and

$$[\rho_{i+1} - \rho_i]_- = \max\{0, -(\rho_{i+1} - \rho_i)\}.$$

Hence,

$$\rho_{i+1} - \rho_i = \sum_{a=1}^{i} [\rho_{a+1} - \rho_a]_+ - \sum_{a=1}^{i} [\rho_{a+1} - \rho_a]_-. \tag{3.7}$$

Since $\{\rho_i\}$ is bounded and $\sum_{i=1}^{\infty} [\rho_{i+1} - \rho_i]_+ \leq \sum_{i=1}^{\infty} \sigma_i < +\infty$, we obtain $[\rho_{i+1} - \rho_i]_-$ is convergent. Therefore, there exists $\rho \in [\min\left\{\frac{\delta}{L}, \rho_1\right\}, \rho_1 + \Lambda]$ such that $\lim\limits_{i \to +\infty} \rho_i = \rho$. This proof is completed. $\square$

Lemma 3.2. *Suppose that the sequence $\{w_i\}$ is generated by Algorithm 3.1. Then*

$$\|w_i - v^*\|^2 \leq \|z_i - v^*\|^2 - \left\{1 - \frac{(\delta_i + \delta)^2 \rho_i^2}{\rho_{i+1}^2}\right\} \|s_i - z_i\|^2, \quad \forall v^* \in \Omega.$$

Proof. Let $v^* \in \Omega$. Then

$$\begin{aligned}
\|w_i - v^*\|^2 &= \|s_i - \rho_i(\nabla g(s_i) - \nabla g(z_i)) - \| \\
&= \|s_i - \|^2 + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2 \\
&\quad - 2\rho_i \langle s_i - v^*, \nabla g(s_i) - \nabla g(z_i)\rangle \\
&= \|s_i - z_i + z_i - \|^2 + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2 \\
&\quad - 2\rho_i \langle s_i - v^*, \nabla g(s_i) - \nabla g(z_i)\rangle \\
&= \|z_i - \|^2 + \|s_i - z_i\|^2 + 2\langle z_i - v^*, s_i - z_i\rangle \\
&\quad - 2\rho_i \langle s_i - v^*, \nabla g(s_i) - \nabla g(z_i)\rangle + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2 \\
&= \|z_i - \|^2 + \|s_i - z_i\|^2 + 2\langle z_i - s_i + s_i - v^*, s_i - z_i\rangle \\
&\quad - 2\rho_i \langle s_i - v^*, \nabla g(s_i) - \nabla g(z_i)\rangle + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2 \\
&= \|z_i - \|^2 + \|s_i - z_i\|^2 - 2\langle s_i - z_i, s_i - z_i\rangle \\
&\quad + 2\langle s_i - v^*, s_i - z_i\rangle - 2\rho_i \langle s_i - v^*, \nabla g(s_i) - \nabla g(z_i)\rangle \\
&\quad + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2 \\
&= \|z_i - \|^2 + \|s_i - z_i\|^2 - 2\langle s_i - v^*, s_i - z_i\rangle \\
&\quad - 2\langle s_i - v^*, \rho_i(\nabla g(s_i) - \nabla g(z_i))\rangle + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2
\end{aligned}$$

$$\begin{aligned}
&= \|z_i - v^*\|^2 - \|s_i - z_i\|^2 - 2\langle s_i - v^*, z_i - s_i + \rho_i(\nabla g(s_i) - \nabla g(z_i))\rangle \\
&\quad + \rho_i^2 \|\nabla g(s_i) - \nabla g(z_i)\|^2.
\end{aligned} \tag{3.8}$$

Note that

$$\rho_{i+1} = \min\left\{\frac{(\delta_i + \delta)\|z_i - s_i\|}{\|\nabla g(z_i) - \nabla g(s_i)\|}, \rho_i + \sigma_i\right\} \leq \frac{(\delta_i + \delta)\|z_i - s_i\|}{\|\nabla g(z_i) - \nabla g(s_i)\|}, \tag{3.9}$$

which implies that

$$\|\nabla g(z_i) - \nabla g(s_i)\| \leq \frac{\delta_i + \delta}{\rho_{i+1}} \|z_i - s_i\|. \tag{3.10}$$

Combining (3.8) and (3.10), we obtain

$$
\begin{aligned}
\|w_i - v^*\|^2 &\leq \|z_i - v^*\|^2 - \|s_i - z_i\|^2 + \frac{(\delta_i+\delta)^2\rho_i^2}{\rho_{i+1}^2}\|s_i - z_i\|^2 \\
&\quad - 2\langle s_i - v^*, z_i - s_i + \rho_i(\nabla g(s_i) - \nabla g(z_i))\rangle \\
&= \|z_i - v^*\|^2 - \left(1 - \frac{(\delta_i+\delta)^2\rho_i^2}{\rho_{i+1}^2}\right)\|s_i - z_i\|^2 \\
&\quad - 2\langle s_i - v^*, z_i - s_i + \rho_i(\nabla g(s_i) - \nabla g(z_i))\rangle.
\end{aligned} \tag{3.11}
$$

From the definition of s_i, we have $z_i - \rho_i \nabla g(z_i) \in (I + \rho_i \partial h)s_i$. Because ∂h is maximal monotone, then there is $u_i \in \partial h(s_i)$ such that

$$
z_i - \rho_i \nabla g(z_i) = s_i + \rho_i u_i. \tag{3.12}
$$

This shows that

$$
u_i = \frac{1}{\rho_i}(z_i - \rho_i \nabla g(z_i) - s_i). \tag{3.13}
$$

Because $0 \in (\nabla g + \partial h)(v^*)$ and $\nabla g(s_i) + u_i \in (\nabla g + \partial h)s_i$, we obtain

$$
\langle \nabla g(s_i) + u_i, s_i - v^*\rangle \geq 0. \tag{3.14}
$$

Substituting (3.13) into (3.14), we have

$$
\frac{1}{\rho_i}\langle z_i - \rho_i \nabla g(z_i) + s_i + \rho_i \nabla g(s_i), s_i - v^*\rangle \geq 0, \tag{3.15}
$$

which implies that $\langle z_i - \rho_i \nabla g(z_i) - s_i + \rho_i \nabla g(s_i), s_i - v^*\rangle \geq 0$. Using (3.11), we obtain

$$
\|w_i - v^*\|^2 \leq \|z_i - v^*\|^2 - \left(1 - \frac{(\delta_i+\delta)^2\rho_i^2}{\rho_{i+1}^2}\right)\|s_i - z_i\|^2. \tag{3.16}
$$

$\square$

Lemma 3.3. *Let the sequence $\{v_i\}$ be generated by Algorithm 3.1. Then the even subsequence $\{v_{2i}\}$ is bounded and it is Fejér monotone with respect to the solution set Ω. Moreover, for all $v^* \in \Omega$, $\lim_{i\to+\infty}\|v_{2i} - v^*\|^2$ exists, and $\lim_{i\to+\infty}\|v_{2i} - s_{2i}\| = 0$.*

Proof. It stems from the definition of v_{i+1} and (2.1) that

$$\|v_{i+1}-v^*\|^2 = (1-\beta)\|z_i-v^*\|^2 + \beta\|w_i-v^*\|^2 - \beta(1-\beta)\|z_i-w_i\|^2. \tag{3.17}$$

Combining (3.17) and Lemma 3.2, we obtain

$$\begin{aligned} \|v_{i+1}-v^*\|^2 &\le (1-\beta)\|z_i-v^*\|^2 + \beta\|z_i-v^*\|^2 - \beta\left(1-\frac{(\delta_i+\delta)^2\rho_i^2}{\rho_{i+1}^2}\right)\|s_i-z_i\|^2 \\ &\quad - \beta(1-\beta)\|z_i-w_i\|^2 \\ &= \|z_i-v^*\|^2 - \beta\left(1-\frac{(\delta_i+\delta)^2\rho_i^2}{\rho_{i+1}^2}\right)\|s_i-z_i\|^2 - \beta(1-\beta)\|z_i-w_i\|^2. \end{aligned} \tag{3.18}$$

Letting $i+1 := 2i+2$ in (3.18), one sees that

$$\begin{aligned} \|v_{2i+2}-v^*\|^2 &\le \|z_{2i+1}-v^*\|^2 - \beta\left(1-\frac{(\delta_{2i+1}+\delta)^2\rho_{2i+1}^2}{\rho_{2i+2}^2}\right)\|s_{2i+1}-z_{2i+1}\|^2 \\ &\quad - \beta(1-\beta)\|z_{2i+1}-w_{2i+1}\|^2. \end{aligned} \tag{3.19}$$

Putting $i+1 := 2i+1$ in (3.18) (noting that $z_{2i} = v_{2i}$), we observe that

$$\|v_{2i+1}-v^*\|^2 \le \|v_{2i}-v^*\|^2 - \beta\left(1-\frac{(\delta_{2i}+\delta)^2\rho_{2i}^2}{\rho_{2i+1}^2}\right)\|s_{2i}-v_{2i}\|^2 - \beta(1-\beta)\|v_{2i}-w_{2i}\|^2. \tag{3.20}$$

It follows from the definition of z_{2i+1} and (2.1) that

$$\begin{aligned} \|z_{2i+1}-v^*\|^2 &= \|v_{2i+1}+\gamma(v_{2i+1}-v_{2i})-v^*\|^2 \\ &= \|(1+\gamma)(v_{2i+1}-v^*)-\gamma(v_{2i}-v^*)\|^2 \\ &= (1+\gamma)\|v_{2i+1}-v^*\|^2 - \gamma\|v_{2i}-v^*\|^2 \\ &\quad + \gamma(1+\gamma)\|v_{2i+1}-v_{2i}\|^2. \end{aligned} \tag{3.21}$$

Using the definition of v_{2i+1} and noting that $z_{2i} = v_{2i}$, one obtains

$$\|v_{2i+1}-v_{2i}\|^2 = \beta^2\|w_{2i}-v_{2i}\|^2. \tag{3.22}$$

Substituting (3.20) and (3.22) into (3.21), we have

$$\begin{aligned} \|z_{2i+1}-v^*\|^2 &\le (1+\gamma)\left[\|v_{2i}-v^*\|^2 - \beta\left(1-\frac{(\delta_{2i}+\delta)^2\rho_{2i}^2}{\rho_{2i+1}^2}\right)\|s_{2i}-v_{2i}\|^2\right] \\ &\quad - (1+\gamma)\beta(1-\beta)\|v_{2i}-w_{2i}\|^2 - \gamma\|v_{2i}-v^*\|^2 + \gamma(1+\gamma)\beta^2\|w_{2i}-v_{2i}\|^2 \end{aligned} \tag{3.23}$$

$$= \|v_{2i} - v^*\|^2 - (1+\gamma)\beta \left(1 - \frac{(\delta_{2i}+\delta)^2\rho_{2i}^2}{\rho_{2i+1}^2}\right) \|s_{2i} - v_{2i}\|^2$$
$$- (1+\gamma)\beta(1-\beta-\gamma\beta)\|v_{2i} - w_{2i}\|^2.$$

Combining (3.19) and (3.23), we get

(3.24)
$$\|v_{2i+2} - v^*\|^2 \leq \|v_{2i} - v^*\|^2 - (1+\gamma)\beta \left(1 - \frac{(\delta_{2i}+\delta)^2\rho_{2i}^2}{\rho_{2i+1}^2}\right) \|s_{2i} - v_{2i}\|^2$$
$$- (1+\gamma)\beta(1-\beta-\gamma\beta)\|v_{2i} - w_{2i}\|^2$$
$$- \beta \left(1 - \frac{(\delta_{2i+1}+\delta)^2\rho_{2i+1}^2}{\rho_{2i+2}^2}\right) \|s_{2i+1} - z_{2i+1}\|^2 - \beta(1-\beta)\|z_{2i+1} - w_{2i+1}\|^2.$$

Since $\gamma \in (0,1), \beta \in (0,1), \delta \in (0,1)$ and $\delta_{2i+1}, \rho_{2i+1}, \rho_{2i+2} > 0, \forall i \geq N_0$, we have

$$(1+\gamma)\beta(1-\beta-\gamma\beta) > 0, \quad \forall i \geq N_0$$

and

$$(1+\gamma)\beta \left(1 - \frac{(\delta_{2i}+\delta)^2\rho_{2i}^2}{\rho_{2i+1}^2}\right) > 0, \quad \forall i \geq N_0.$$

So,

$$\beta \left(1 - \frac{(\delta_{2i+1}+\delta)^2\rho_{2i+1}^2}{\rho_{2i+2}^2}\right) > 0, \quad \forall i \geq N_0.$$

Thus it follows from (3.24) that

$$\|v_{2i+2} - v^*\| \leq \|v_{2i} - v^*\|, \quad \forall i \geq N_0. \tag{3.25}$$

This implies that $\{\|v_{2i} - v^*\|\}$ and $\{v_{2i}\}$ are bounded. Moreover, $\lim_{i\to+\infty} \|v_{2i} - v^*\|$ exists. Therefore, we conclude from (3.24) that

$$\lim_{i\to+\infty} \|s_{2i} - v_{2i}\| = 0 \quad \text{and} \quad \lim_{i\to+\infty} \|v_{2i} - w_{2i}\| = 0. \tag{3.26}$$

In fact that $\{v_{2i}\}$ is bounded and (3.26), we obtain that $\{s_{2i}\}$ and $\{w_{2i}\}$ are also bounded. By virtue of (3.22) and (3.26), one sees that $\lim_{i\to+\infty} \|v_{2i+1} - v_{2i}\| = 0$. From ∇g is uniformly continuous, we obtain

$$\lim_{i\to+\infty} \|\nabla g(z_{2i}) - \nabla g(s_{i2})\| = 0. \tag{3.27}$$

□

Theorem 3.1. *Suppose that the sequence $\{v_i\}$ is generated by Algorithm 3.1. Let $v^* \in H$ denote the weak limit of the subsequence $\{v_{2i_a}\}$ of $\{v_{2i}\}$. Then $v^* \in \Omega$.*

Proof. From $\{v_{2i}\}$ is bounded, there exists a subsequence $\{v_{2i_a}\}$ of $\{v_{2i}\}$ such that $v_{2i_a} \rightharpoonup v^* \in H$. By Lemma 3.3, we obtain $v_{2i_a+1} \rightharpoonup v^*$. We note that

$$s_{2i_a} = \operatorname{prox}_{\rho_{2i_a}h}(z_{2i_a} - \rho_{2i_a}\nabla g(z_{2i_a})). \tag{3.28}$$

From (2.2), we obtain

$$\frac{z_{2i_a} - \rho_{2i_a}\nabla g(z_{2i_a}) - s_{2i_a}}{\rho_{2i_a}} \in \partial h(s_{2i_a}). \tag{3.29}$$

Thus,

$$\frac{z_{2i_a} - s_{2i_a}}{\rho_{2i_a}} - \nabla g(z_{2i_a}) + \nabla g(s_{2i_a}) \in \partial h(s_{2i_a}) + \nabla g(s_{2i_a}). \tag{3.30}$$

Because, $\lim_{i\to+\infty} \|s_{2i} - v_{2i}\| = 0$, we also have $s_{2i_a} \rightharpoonup v^*$. Taking $a \to +\infty$ in (3.30) and using (3.27), by Lemma 2.2 and Lemma 3.1, we get

$$0 \in (\partial h + \nabla g)(v^*). \tag{3.31}$$

Therefore, $v^* \in \Omega$. From (3.20) we see that $\{v_{2i}\}$ is a quasi-Fejér sequence. Hence, by Lemma 2.1, we conclude that $\{v_{2i}\}$ weakly converges to a point in Ω. This completes the proof. $\square$

4. Numerical Experiments

In this section, we present various numerical experiments that demonstrate the behavior of our methods and compare them to existing method. Consider the minimization problem:

$$\min_{v\in\mathbb{R}^3} \|v\|_1 + 3\|v\|_2^2 + (-2,1,4)v + 9,$$

where $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$. Let $g(v) = 3\|v\|_2^2 + (-2,1,4)v + 9$ and $h(v) = \|v\|_1$. Thus we have $\nabla g(v) = 6v + (-2,1,4)^T$. It is easy to check that g is a convex and differentiable function and its gradient ∇g is Lipschitz continuous with $L = 6$. Moreover, we know that

$$\operatorname{prox}_{\rho_i\|\cdot\|_1}(v) = [\operatorname{prox}_{\rho_i|\cdot|}(v_1), \operatorname{prox}_{\rho_i|\cdot|}(v_2), \operatorname{prox}_{\rho_i|\cdot|}(v_3)]^T,$$

where $\text{prox}_{\rho_i|\cdot|}(v_i) = \max\{|v_i| - \rho_i, 0\}\text{sign}(v_i)$, and v_i denotes the ith element of v, $i = 1,2,3$. In this experiments, we compare our Algorithm 3.1 and Algorithm (1.8). The parameters are chosen as follows:

- Algorithm 3.1: $\rho_1 = \frac{0.6}{L}$, $\gamma = 0.9$, $\beta = 0.9$, $\delta = 0.6$, $\delta_i = \frac{1}{(1000i+2)^{10}}$, $\sigma_i = \frac{99i}{100i+1}$.
- Algorithm (1.8): $\rho_1 = \frac{0.6}{L}$, $\mu = 0.4$.

We perform the numerical experiments with four different cases of starting point v_1 and use stopping criterion $\|v_{i+1} - v_i\| \leq \varepsilon = 10^{-6}$. The numerical results are reported in Table 1.

TABLE 1. Numerical results

$v_1 = v_0$	Number of Iteration		Execution Time in Seconds	
	Algorithm (1.8)	Algorithm 3.1	Algorithm (1.8)	Algorithm 3.1
$(1,3,5)^T$	113	38	0.058383	0.044992
$(1,-6,2)^T$	107	40	0.121066	0.093168
$(-200,200,100)^T$	138	48	0.073472	0.020173
$(-1000,-5000,500)^T$	151	56	0.057175	0.033351

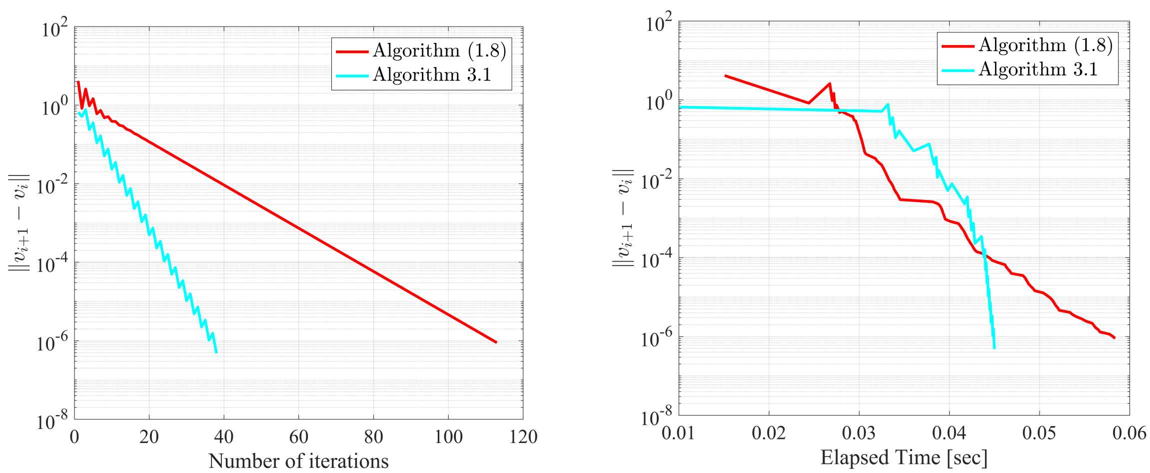

FIGURE 1. Numerical results for $v_1 = (1,3,5)^T$

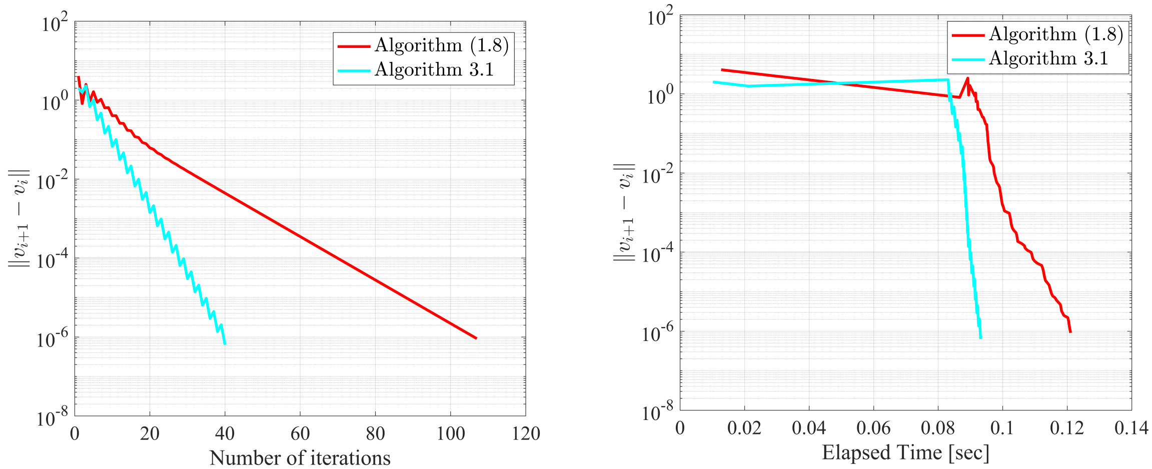

FIGURE 2. Numerical results for $v_1 = (1, -6, 2)^T$

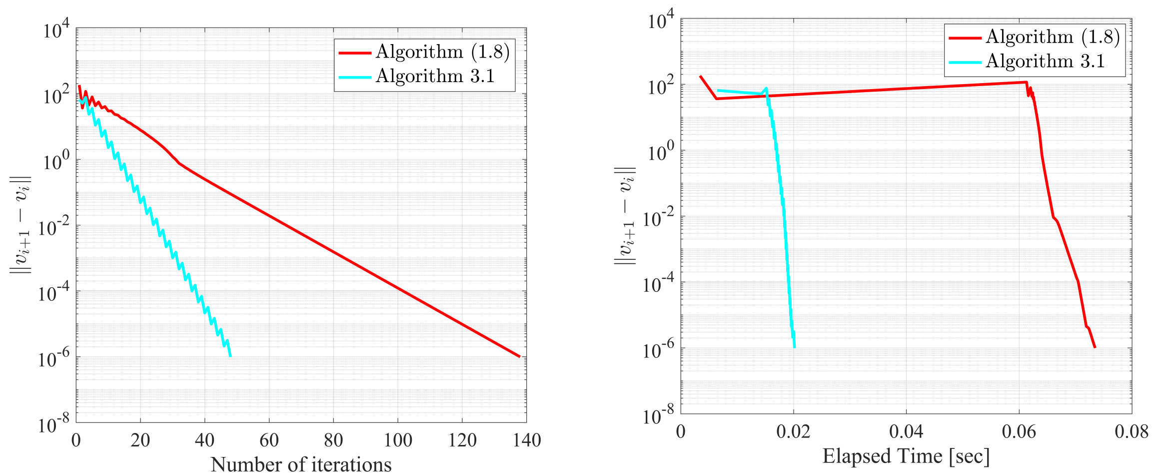

FIGURE 3. Numerical results for $v_1 = (-200, 200, 100)^T$

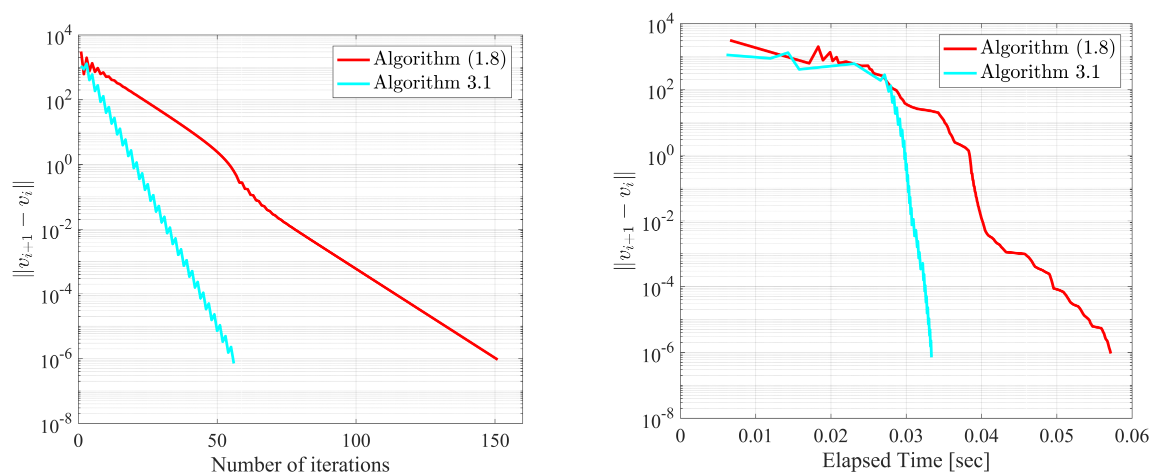

FIGURE 4. Numerical results for $v_1 = (-1000, -5000, 500)^T$

ACKNOWLEDGMENT

This project was supported by the Research and Development Institute, Rambhai Barni Rajabhat University (Grant no.2215/2566).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] Y. Nesterov, A method for solving the convex programming problem with convergence rate $o(1/k^2)$, Dokl. Akad. Nauk SSSR 269 (1983), 543–547. https://cir.nii.ac.jp/crid/1370862715914709505.

[2] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431–446. https://doi.org/10.1137/s0363012998338806.

[3] Y. Malitsky, M.K. Tam, A forward-backward splitting method for monotone inclusions without cocoercivity, SIAM J. Optim. 30 (2020), 1451–1472. https://doi.org/10.1137/18m1207260.

[4] D. Van Hieu, P.K. Anh, L.D. Muu, Modified forward-backward splitting method for variational inclusions, 4OR-Q J. Oper. Res. 19 (2020), 127–151. https://doi.org/10.1007/s10288-020-00440-3.

[5] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Comp. Math. Math. Phys. 4 (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5.

[6] D. Kitkuan, P. Kumam, J. Martínez-Moreno, et al. Inertial viscosity forward–backward splitting algorithm for monotone inclusions and its application to image restoration problems, Int. J. Computer Math. 97 (2019), 482–497. https://doi.org/10.1080/00207160.2019.1649661.

[7] A. Padcharoen, D. Kitkuan, P. Kumam, J. Rilwan, W. Kumam, Accelerated alternating minimization algorithm for Poisson noisy image recovery, Inverse Probl. Sci. Eng. 28 (2020), 1031–1056. https://doi.org/10.1080/17415977.2019.1709454.

[8] A. Adamu, D. Kitkuan, A. Padcharoen, et al. Inertial viscosity-type iterative method for solving inclusion problems with applications, Math. Computers Simul. 194 (2022), 445–459. https://doi.org/10.1016/j.matcom.2021.12.007.

[9] H.H. Bauschke, M.N. Bui, X. Wang, Applying FISTA to optimization problems (with or) without minimizers, Math. Program. 184 (2019), 349–381. https://doi.org/10.1007/s10107-019-01415-x.

[10] P. Cholamjiak, Y. Shehu, Inertial forward-backward splitting method in Banach spaces with application to compressed sensing, Appl. Math. 64 (2019), 409–435. https://doi.org/10.21136/am.2019.0323-18.

[11] K. Kankam, P. Cholamjiak, Inertial proximal gradient method using adaptive stepsize for convex minimization problems, Thai J. Math. 21 (2023), 277–287.

[12] K. Kankam, P. Cholamjiak, Double inertial proximal gradient algorithms for convex optimization problems and applications, Acta Math. Sci. 43 (2023), 1462–1476. https://doi.org/10.1007/s10473-023-0326-x.

[13] F. Cui, Y. Tang, C. Zhu, Convergence analysis of a variable metric forward-backward splitting algorithm with applications, J. Inequal. Appl. 2019 (2019), 141. https://doi.org/10.1186/s13660-019-2097-4.

[14] D. Kitkuan, P. Kumam, A. Padcharoen, et al. Algorithms for zeros of two accretive operators for solving convex minimization problems and its application to image restoration problems, J. Comput. Appl. Math. 354 (2019), 471–495. https://doi.org/10.1016/j.cam.2018.04.057.

[15] D. Kitkuan, P. Kumam, J. Martínez-Moreno, Generalized Halpern-type forward-backward splitting methods for convex minimization problems with application to image restoration problems, Optimization 69 (2019), 1557–1581. https://doi.org/10.1080/02331934.2019.1646742.

[16] D. Reem, S. Reich, A. De Pierro, A telescopic bregmanian proximal gradient method without the global Lipschitz continuity assumption, J. Optim. Theory Appl. 182 (2019), 851–884. https://doi.org/10.1007/s10957-019-01509-8.

[17] Z. Mu, Y. Peng, A note on the inertial proximal point method, Stat. Optim. Inf. Comput. 3 (2015), 241–248. https://doi.org/10.19139/soic.v3i3.124.

[18] S. Kesornprom, P. Cholamjiak, A modified inertial proximal gradient method for minimization problems and applications, AIMS Math. 7 (2022), 8147–8161. https://doi.org/10.3934/math.2022453.

[19] A.N. Iusem, B.F. Svaiter, M. Teboulle, Entropy-like proximal methods in convex programming, Math. Oper. Res. 19 (1994), 790–814. https://doi.org/10.1287/moor.19.4.790.

[20] R.S. Burachik, A.N. Iusem, Enlargements of monotone operators, in: Set-Valued Mappings and Enlargements of Monotone Operators, Springer, Boston, MA, 2008: pp. 161–220. https://doi.org/10.1007/978-0-387-69757-4_5.

[21] M. Osilike, S. Aniagbosor, G. Akuchu, Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, Panamer. Math. J. 12 (2002), 77–88.

[22] A. Hanjing, S. Suantai, A fast image restoration algorithm based on a fixed point and optimization method, Mathematics 8 (2020), 378. https://doi.org/10.3390/math8030378.