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SOME COMMON FIXED POINT THEOREMS IN B-METRIC SPACES UNDER GERAGHTY-BERINDE-SUZUKI TYPE CONTRACTION

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Abstract. This paper explores the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Geraghty-Berinde-Suzuki type contraction in complete b-metric spaces. We illustrate our findings with examples and derive a few corollaries from our findings. The significance of L in our contraction condition is also covered.

Keywords: common fixed points; complete b-metric space; weakly compatible maps; b-(E.A)-property; Geraghty-Berinde-Suzuki type contraction maps.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The expansion of contraction conditions in one direction or the expansion of the operator under consideration's ambient spaces in another direction forms the foundation for the development of fixed point theory. The Banach contraction principle is one of the most helpful findings in fixed point theory and is crucial for resolving nonlinear equations. In 1973, Geraghty [26] established a fixed point theorem that expanded upon the Banach contraction principle and moved

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the field of contraction conditions toward generality. By extending contraction map to contraction map with rational expression, Dass and Gupta [23] proved the presence of fixed points in complete metric spaces. Berinde [13] developed "weak contractions" as a generalization of contraction maps, continuing the extensions of contraction maps. In his subsequent work, Berinde reclassified "weak contractions" as "almost contractions" [15].To read more about almost contractions and their generalizations, see [2, 7, 8, 10, 11, 16]. A novel kind of generalization of the Banach contraction principle and a characterization of the metric completeness were established by Suzuki [39] in 2008. They proved two fixed point theorems.

The fundamental concept of b-metric originated from the writings of Bakhtin [12] and Bourbaki [19]. Czerwik [21] introduced the concept of b-metric space, also referred to as metric type space, as a generalization of metric space. Subsequently, a large number of authors investigated fixed point theorems in b-metric spaces for both single- and multi-valued mappings, see [3, 4, 17, 18, 22, 31, 37, 38].

The concept of property (E.A) was initially introduced in 2002 by Aamari and Moutawakil [1]. This concept was employed by a number of authors to show that common fixed points exist; see [5, 6, 34, 35, 36].

The sets of all natural numbers, $\mathbb{N}$, and $\mathbb{R}^+ = [0,\infty)$ are denoted.

Definition 1.1. [21] Let § be a set that is not empty. If the following criteria are met, a function $\eth : \S \times \S \to \mathbb{R}^+$ is referred to be a b-metric: for any $\xi, \zeta, \eta \in \S$

(b_1) $0 \leq \eth(\xi,\zeta)$ and $\eth(\xi,\zeta) = 0$ if and only if $\xi = \zeta$,

(b_2) $\eth(\xi,\zeta) = \eth(\zeta,\xi)$,

(b_3) there exists $s \geq 1$ such that $\eth(\xi,\eta) \leq s[\eth(\xi,\zeta) + \eth(\zeta,\eta)]$.

A b-metric space with coefficient s is defined in this instance for the pair $(\S,\eth)$.

All metric spaces have $s = 1$ and are b-metric spaces. Generally speaking, not all b-metric spaces are metric spaces.

Definition 1.2. [18] Assume that the b-metric space is $(\S,\eth)$.

(i) If there exists $\xi \in \S$ such that $\eth(\xi_n,\xi) \to 0$ as $n \to \infty$, then a sequence $\{\xi_n\}$ in § is termed b-convergent. Here, we write $\lim_{n\to\infty} \xi_n = \xi$.

(ii) In §, a sequence $\{\xi_n\}$ is considered b-Cauchy if $\eth(\xi_n, \xi_m) \to 0$ as $n, m \to \infty$.

It is not always the case that a b-metric is continuous.

Example 1.3. [28] For $\S = \mathbb{N} \cup \{\infty\}$, let them be. The following is how we define a mapping $\eth : \S \times \S \to \mathbb{R}^+$:

$$\eth(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then $(\S, \eth)$ is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [29] When $\lim_{n\to\infty} \eth(\Xi\Lambda\xi_n, \Lambda\Xi\xi_n) = 0$, then a pair (Λ, Ξ) of selfmaps on a metric space $(\S, \eth)$ is considered compatible. Whenever $\{\xi_n\}$ is a sequence in §, such that, for some $\eta \in \S$, $\lim_{n\to\infty} \Lambda\xi_n = \lim_{n\to\infty} \Xi\xi_n = \eta$.

Definition 1.5. [1] Given a sequence $\{\xi_n\}$ in § such that, for each $\eta \in \xi$, $\lim_{n\to\infty} \Lambda\xi_n = \lim_{n\to\infty} \Xi\xi_n = \eta$, a pair (Λ, Ξ) of selfmaps on a metric space $(\S, \eth)$ is said to satisfy (E.A)-property.

Definition 1.6. [34] b-(E.A)-property is satisfied by a pair (Λ, Ξ) of selfmaps on a b-metric space $(\S, \eth)$ if there exists a sequence $\{\xi_n\}$ in § such that $\lim_{n\to\infty} \Lambda\xi_n = \lim_{n\to\infty} \Xi\xi_n = \eta$ for some $\eta \in \S$.

Definition 1.7. [30] Weakly compatible selfmaps are pairs (Λ, Ξ) on a set § if and only if $\Lambda\Xi\xi = \Xi\Lambda\xi$ whenever $\Lambda\xi = \Xi\xi$ for any $\xi \in \S$.

Geraghty [26] in 1973 introduced the family of functions as follows:
$\mathfrak{t} = \{\beta : [0,\infty) \to [0,1) / \lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0\}$.

Theorem 1.8. [26] Assume that the metric space $(\S, \eth)$ is complete. Consider a selfmap $\Upsilon : \S \to \S$ that satisfies the following:For any $\xi, \zeta \in \S$, there exists $\beta \in \mathfrak{t}$ such that

$$\eth(\Upsilon\xi, \Upsilon\zeta) \leq \beta(\eth(\xi,\zeta))\eth(\xi,\zeta)$$

. Then there is a unique fixed point for Υ.

We denote $\mathfrak{g} = \{\alpha : [0,\infty) \to [0, \frac{1}{s}) / \lim_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}$.

Theorem 1.8 was extended to the case of b-metric spaces by Dukic et al. [24] in 2011.

Theorem 1.9. [24] With coefficient $s \geq 1$, let $(\S, \eth)$ be a complete b-metric space. Let $\Upsilon : \S \to \S$ be a selfmap of $\S$. Assume that for every $\xi, \zeta \in \S$, there exists $\alpha \in \mathfrak{g}$ such that

$$\eth(\Upsilon\xi, \Upsilon\zeta) \leq \alpha(\eth(\xi,\zeta))\eth(\xi,\zeta).$$

In $\S$, there is a unique fixed point for Υ.

The lemmas that follow are helpful in demonstrating our primary findings.

Lemma 1.10. [27] Let b-metric space $(\S, \eth)$ have coefficient $s \geq 1$. Assume that for every $n \in \mathbb{N}$, where $k \in [0,1)$ is a constant, there exists a sequence $\{\xi_n\}$ in $\S$ such that $\eth(\xi_n, \xi_{n+1}) \leq k\eth(\xi_{n-1}, \xi_n)$. It follows that $\{\xi_n\}$ in $\S$ is a b-Cauchy sequence.

Lemma 1.11. [3] Consider a b-metric space with coefficient $s \geq 1$, denoted as $(\S, \eth)$. Assuming that $\{\xi_n\}$ and $\{\zeta_n\}$ are b-convergent to ξ and ζ, respectively, we obtain the following:

$$\frac{1}{s^2}\eth(\xi,\zeta) \leq \liminf_{n\to\infty} \eth(\xi_n,\zeta_n) \leq \limsup_{n\to\infty} \eth(\xi_n,\zeta_n) \leq s^2\eth(\xi,\zeta).$$

Specifically, $\lim_{n\to\infty} \eth(\xi_n,\zeta_n) = 0$ if $\xi = \zeta$. Additionally, we have

$$\frac{1}{s}\eth(\xi,\eta) \leq \liminf_{n\to\infty} \eth(\xi_n,\eta) \leq \limsup_{n\to\infty} \eth(\xi_n,\eta) \leq s\eth(\xi,\eta).$$

This is true for each $\eta \in \S$.

According to Latif et al. [32], it was established in b-metric spaces that fixed points of a single selfmap meeting the Suzuki type contraction requirement exist and are unique.

Theorem 1.12. [32] Assume that $(\S, \eth)$ is the complete b-metric space and let $f : \S \to \S, \alpha : \S \times \S \to [0,\infty)$ satisfying

(a) $\alpha(\xi,\zeta) \geq 1 \implies \alpha(f\xi, f\zeta) \geq 1$,

(b) $\alpha(\xi,\eta) \geq 1, \alpha(\eta,\zeta) \geq 1 \implies \alpha(\xi,\zeta) \geq 1, \xi,\zeta,\eta \in \S$. Suppose that $\beta \in \mathfrak{g}$ such that $\frac{1}{2s}\eth(\xi, f\xi) \leq \eth(\xi,\zeta) \implies s\alpha(\xi,\zeta)\eth(f\xi, f\zeta) \leq \beta(M(\xi,\zeta))M(\xi,\zeta)$ for all $\xi,\zeta \in \S$, where $M(\xi,\zeta) = \max\{\eth(\xi,\zeta), \frac{\eth(\xi,f\xi)\eth(\xi,f\zeta)+\eth(\zeta,f\zeta)\eth(\zeta,f\xi)}{1+s[\eth(\xi,\zeta)+\eth(f\xi,f\zeta)]}, \frac{\eth(\xi,f\xi)\eth(\xi,f\zeta)+\eth(\zeta,f\zeta)\eth(\zeta,f\xi)}{1+\eth(\xi,f\zeta)+\eth(\zeta,f\xi)}\}$.

Also, suppose that the following assertions hold:

(i) there exists $\xi_0 \in \S$ such that $\alpha(\xi_0, f\xi_0) \geq 1$;

(ii) for any sequence $\{\xi_n\}$ in $\S$ with $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $\xi_n \to \xi$ as $n \to \infty$, we have $\alpha(\xi_n, \xi) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

$O_f(\xi_0)$ denotes the set $\{\xi_0, f\xi_0, f^2\xi_0, f^3\xi_0, \ldots\}$, which is known as an orbit of f at the point ξ_0 [14].

Definition 1.13. [20] In a b-metric space $\S$, every Cauchy sequence in $O_f(\xi_0)$ that converges in $\S$, where f is a selfmapping on $\S$ and $\xi_0 \in \S$, is called f-orbitally complete.

Definition 1.14. [33] Assume that $\alpha : \S \times \S \to \mathbb{R}$ and that $\S$ is any nonempty set. For each $\xi, \zeta \in \S$ with $\xi \neq \zeta$, there exists $\S \in \xi$ such that $\alpha(\xi, \eta) \geq 1, \alpha(\zeta, \eta) \geq 1$ and $\alpha(\eta, f\eta) \geq 1$. This is known as a property (H) of a selfmap $f : \S \to \S$.

Definition 1.15. [33] Suppose we have a b-metric space $(\S, \eth)$ and $\alpha : \S \times \S \to \mathbb{R}$. For any $\xi, \zeta \in \S$, there exists a $\beta \in \mathfrak{g}$ such that $f : \S \to \S$ is called a generalized α-Suzuki-Geraghty contraction, $\frac{1}{2s}\eth(\xi, f\xi) \leq s\eth(\xi, \zeta) \implies \eth(f\xi, f\zeta) \leq \beta(M(\xi, \zeta))M(\xi, \zeta)$,
where

$$M(\xi, \zeta) = \max\Big\{\eth(\xi, \zeta), \eth(\xi, f\xi), \eth(\zeta, f\zeta), \eth(f^2\xi, f\xi), \eth(f^2\xi, \zeta), \frac{\eth(f^2\xi, f\zeta)}{s}, \frac{\eth(f^2\xi, \xi)}{2s},$$
$$\frac{\eth(\xi, f\zeta) + \eth(\zeta, f\xi)}{2s}, \frac{\eth(\xi, f\xi)\eth(\xi, f\zeta) + \eth(\zeta, f\zeta)\eth(\zeta, f\xi)}{1 + s[\eth(\xi, \zeta) + \eth(f\xi, f\zeta)]}, \frac{\eth(\xi, f\xi)\eth(\xi, f\zeta) + \eth(\zeta, f\zeta)\eth(\zeta, f\xi)}{1 + \eth(\xi, f\zeta) + \eth(\zeta, f\xi)}\Big\}.$$

Theorem 1.16. [33] With parameter $s \geq 1, \alpha : \S \times \S \to \mathbb{R}$ and $f : \S \to \S$, let $(\S, \eth)$ be the complete b-metric space. Assume the following requirements are met and $\S$ is f-orbitally complete:

(i) there exists $\xi_0 \in \S$ such that $\alpha(\xi_0, f\xi_0) \geq 1$;

(ii) f is a generalized α-Suzuki-Geraghty contraction and a triangular α-orbital admissible;

(iii) either f is continuous or for any sequence $\{\xi_n\}$ in $\S$ with $\alpha(\xi_n, \xi_{n+1}) \geq 1$ such that $\xi_n \to \xi$ as $n \to \infty$ we have $\alpha(\xi_n, \xi) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, $\{f^n\xi_0\}$ converges to η, and f has a fixed point η in $\S$. Furthermore, if the property (H) is substituted for condition (i), then f has a unique fixed point.

In this work, we indicate

$\mathfrak{F} = \{\beta : [0,\infty) \to [0,\frac{1}{s})/\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}$.

The following theorem was proved in 2019 by Faraji et al. [25].

Theorem 1.17. [25] Consider a complete b-metric space with parameter $s \geq 1$, denoted as $(\S, \eth)$. Let $\Upsilon, \Sigma : \S \to \S$ be selfmaps on $\S$ that satisfy the following: there exists $\beta \in \mathfrak{F}$ such that

$$s\eth(\Upsilon\xi, \Sigma\zeta) \leq \beta(M(\xi,\zeta))M(\xi,\zeta)$$

for all $\xi, \zeta \in \S$, where $M(\xi,\zeta) = \max\{\eth(\xi,\zeta), \eth(\xi,\Upsilon\xi), \eth(\zeta,\Sigma\zeta)\}$.
There exists a unique common fixed point for Υ and Σ if any of them is continuous.

The Geraghty-Suzuki type contraction was introduced in b-metric spaces in 2020 by Babu and Babu [9].

Definition 1.18. [9] Let $(\S, \eth)$ be a b-metric space, and let $\Lambda, \Xi, \Sigma$and Υ be selfmaps of $\S$. If there exists $\beta \in \mathfrak{F}$ such that

$$\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta)\} \leq \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Lambda\xi,\Xi\zeta)\} \implies s^4\eth(\Lambda\xi,\Xi\zeta) \leq \beta(M(\xi,\zeta))M(\xi,\zeta)$$

where

$$M(\xi,\zeta) = \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \frac{\eth(\Sigma\xi,\Xi\zeta)}{2s}, \frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s}, \frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)}, \frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}, \text{ for all } \xi,\zeta \in \S$$

then the pairs (Λ, Σ) and (Ξ, Υ) are Geraghty-Suzuki type contraction maps.

The following theorems are due to Babu and Babu [9].

Theorem 1.19. [9] Let Λ, Ξ, Σ, and Υ be selfmaps on the complete b-metric space $(\S, \eth)$ that satisfy Geraghty-Suzuki type contraction maps and $A(\S) \subseteq \Upsilon(\S)$ and $\Xi(\S) \subseteq \Sigma(\S)$. Should either

(i) (Ξ, Υ) is weakly compatible, Λ (or) Σ is b-continuous, and the pair (Λ, Σ) is compatible

or

(ii) Ξ (or) Υ is b-continuous, the pair (Ξ, Υ) is compatible, and the pair (Λ, Σ) is weakly compatible

therefore there is a single common fixed point in $\S$ for Λ, Ξ, Σ, and Υ.

Theorem 1.20. [9] Let $s \geq 1$ be a coefficient in a b-metric space $(\S, \eth)$. Allow $\Lambda, \Xi, \Sigma, \Upsilon : \S \to \S$ to be selfmaps of $\S$ that fulfill Geraghty-Suzuki type contraction maps and $\Lambda(\S) \subseteq \Upsilon(\S)$ and $\Xi(\S) \subseteq \Sigma(\S)$. Assume that one of the subspaces $\Lambda(\S), \Xi(\S), \Sigma(\S)$, and $\Upsilon(\S)$ is b-closed in $\S$ and that one of the pairs (Λ, Σ) and (Ξ, Υ) fulfills the b-(E.A)-property. Then, there is a point of coincidence in $\S$ between the pairs (Λ, Σ) and (Ξ, Υ). Furthermore, Λ, Ξ, Σ and Υ have a single common fixed point in $\S$ if the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible.

In the second part of this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps that satisfy a contraction condition of the Geraghty-Berinde-Suzuki type: one pair is compatible and b-continous, while the other pair is weakly compatible in complete b-metric spaces. Our work is motivated by the works of Babu and Babu [9]. Additionally, we demonstrate the same using various hypotheses on two pairs of selfmaps that meet the b-(E.A)-property. We provide some instances and corollaries for our findings in Section 3. The significance of L in our contracting situation is also covered. Some of the results in the literature are extended to two pairs of self maps by our theorems. From our findings, we derive a few corrollaries and offer evidence to back up our findings.

2. Main Results

Given a b-metric space $(\S, \eth)$, let Λ, Ξ, Σ, and Υ be mappings from it into itself that fulfill

$$\Lambda(\S) \subseteq \Upsilon(\S) \text{ and } \Xi(\S) \subseteq \Sigma(\S). \tag{2.1}$$

According to (2.1), there exists $\xi_1 \in \S$ such that $\zeta_0 = \Lambda\xi_0 = \Upsilon\xi_1$ for every $\xi_0 \in \S$. Similarly, we can select a point $\xi_2 \in \S$ for this ξ_1 such that $\zeta_1 = \Xi\xi_1 = \Sigma\xi_2$ and so forth. Generally speaking, we define

$$\zeta_{2n} = \Lambda\xi_{2n} = \Upsilon\xi_{2n+1} \text{ and } \zeta_{2n+1} = \Xi\xi_{2n+1} = \Sigma\xi_{2n+2} \text{ for } n = 0, 1, 2, \ldots. \tag{2.2}$$

Proposition 2.1. *Assume that the b-metric space $(\S, \eth)$ has a coefficient $s \geq 1$. Assume that $\S$ has selfmaps Λ, Ξ, Σ, and Υ that meet the following criteria: $\beta \in \mathfrak{F}$ exists such that*

$$\begin{aligned} &\tfrac{1}{2s}\min\{\eth(\Sigma\xi, \Lambda\xi), \eth(\Upsilon\zeta, \Xi\zeta)\} \leq \max\{\eth(\Sigma\xi, \Upsilon\zeta), \eth(\Lambda\xi, \Xi\zeta)\} \\ &\qquad\qquad \Longrightarrow s^4\eth(\Lambda\xi, \Xi\zeta) \leq \beta(M(\xi, \zeta))M(\xi, \zeta) + LN(\xi, \zeta) \end{aligned} \tag{2.3}$$

where

$$M(\xi,\zeta)=\max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\frac{\eth(\Sigma\xi,\Xi\zeta)}{2s},\frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s},\frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)},\frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\},$$

$$N(\xi,\zeta)=\min\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\eth(\Sigma\xi,\Xi\zeta),\eth(\Upsilon\zeta,\Lambda\xi)\}.$$

for all $\xi,\zeta\in\S$. Then we have the following:

- (i) *If $\Lambda(\S)\subseteq\Upsilon(\S)$ and the pair (Ξ,Υ) is weakly compatible and if η is a common fixed point of Λ and Σ then η is a common fixed point of Λ,Ξ,Σ and Υ and it is unique.*
- (ii) *If $\Xi(\S)\subseteq\Sigma(\S)$ and the pair (Λ,Σ) is weakly compatible and if η is a common fixed point of Ξ and Υ then η is a common fixed point of Λ,Ξ,Σ and Υ and it is unique.*

Proof. First, we assume that (i) holds. Let η be a common fixed point of Λ and Σ.

Then $\Lambda\eta=\Sigma\eta=\eta$. Since $\Lambda(\S)\subseteq\Upsilon(\S)$, there exists $u\in\S$ such that $\Upsilon u=\eta$.

Therefore $\Lambda\eta=\Sigma\eta=\Upsilon u=\eta$.

We now prove that $\Lambda\eta=Bu$. Suppose that $A\eta\neq Bu$.

Since $\frac{1}{2s}\min\{\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon u,\Xi u)\}\leq\max\{\eth(\Sigma\eta,\Upsilon u),\eth(\Lambda\eta,\Xi u)\}$.

From the inequality (2.3), we have

$$s^4\eth(\Lambda\eta,\Xi u)\leq\beta(M(\eta,u))M(\eta,u)+LN(\eta,u)\tag{2.4}$$

where

$$\begin{aligned}M(\eta,u)&=\max\{\eth(\Sigma\eta,\Upsilon u),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon u,\Xi u),\frac{\eth(\Sigma\eta,\Xi u)}{2s},\frac{\eth(\Upsilon u,\Lambda\eta)}{2s},\frac{\eth(\Sigma\eta,\Lambda\eta)\eth(\Upsilon u,\Xi u)}{1+\eth(\Sigma\eta,\Upsilon u)+\eth(\Lambda\eta,\Xi u)},\frac{\eth(\Sigma\eta,\Xi u)\eth(\Upsilon u,\Lambda\eta)}{1+s^4[\eth(\Sigma\eta,\Upsilon u)+\eth(\Lambda\eta,\Xi u)]}\}\\&=\max\{0,0,\eth(\Lambda\eta,\Xi u),\frac{\eth(\Lambda\eta,\Xi u)}{2s},0,0,0\}=\eth(\Lambda\eta,\Xi u),\end{aligned}$$

$$N(\eta,u)=\min\{\eth(\Sigma\eta,\Upsilon u),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon u,\Xi u),\eth(\Sigma\eta,\Xi u),\eth(\Upsilon u,\Lambda\eta)\}=0.$$

From the inequality (2.4), we have

$s^4\eth(\Lambda\eta,\Xi u)\leq\beta(\eth(\eta,u))\eth(\eta,u)\leq\frac{\eth(\Lambda\eta,\Xi u)}{s}$ so that $(s^5-1)\eth(\Lambda\eta,\Xi u)\leq 0$.

Since $(s^5-1)\geq 0$, it follows that $\eth(\Lambda\eta,\Xi u)=0$.

Hence $\Lambda\eta=\Xi u$. Therefore $\Lambda\eta=\Xi u=\Sigma\eta=\Upsilon u=\eta$.

Since the pair (Ξ,Υ) is weakly compatible and $\Xi u=\Upsilon u$, we have $\Xi\Upsilon u=\Upsilon\Xi u$. i.e., $\Xi\eta=\Upsilon\eta$.

Now we show that $\Xi\eta=\eta$.

If $\Xi\eta\neq\eta$, then we have

$\frac{1}{2s}\min\{\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\eta,\Xi\eta)\}\leq\max\{\eth(\Sigma\eta,\Upsilon\eta),\eth(\Lambda\eta,\Xi\eta)\}$

From the inequality (2.3), we have

$$s^4\eth(\eta,\Xi\eta)=s^4\eth(\Lambda\eta,\Xi\eta)\leq\beta(M(\eta,\eta))M(\eta,\eta)+LN(\eta,\eta) \tag{2.5}$$

where

$$\begin{aligned}M(\eta,\eta)=\max\{&\eth(\Sigma\eta,\Upsilon\eta),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\eta,\Xi\eta),\tfrac{\eth(\Sigma\eta,\Xi\eta)}{2s},\tfrac{\eth(\Upsilon\eta,\Lambda\eta)}{2s},\\ &\tfrac{\eth(\Sigma\eta,\Lambda\eta)\eth(\Upsilon\eta,\Xi\eta)}{1+\eth(\Sigma\eta,\Upsilon\eta)+\eth(\Lambda\eta,\Xi\eta)},\tfrac{\eth(\Sigma\eta,\Xi\eta)\eth(\Upsilon\eta,\Lambda\eta)}{1+s^4[\eth(\Sigma\eta,\Upsilon\eta)+\eth(\Lambda\eta,\Xi\eta)]}\}\\ =\max\{&\eth(\eta,\Xi\eta),0,0,\tfrac{\eth(\eta,\Xi\eta)}{2s},\tfrac{\eth(\eta,\Xi\eta)}{2s},0,\tfrac{[\eth(\eta,\Xi\eta)]^2}{1+2s^4[\eth(\eta,\Xi\eta)]}\}=\eth(\eta,\Xi\eta),\end{aligned}$$

$N(\eta,\eta)=\min\{\eth(\Sigma\eta,\Upsilon\eta),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\eta,\Xi\eta),\eth(\Sigma\eta,\Xi\eta),\eth(\Upsilon\eta,\Lambda\eta)\}=0$

From the inequality (2.5), we have

$s^4\eth(\eta,\Xi\eta)\leq\beta(M(\eta,\eta))M(\eta,\eta)=\beta(\eth(\eta,\Xi\eta))\eth(\eta,\Xi\eta)\leq\frac{\eth(\eta,\Xi\eta)}{s}$ so that

$(s^5-1)\eth(\eta,\Xi\eta)\leq 0$.

Since $(s^5-1)\geq 0$, it follows that $\eth(\eta,\Xi\eta)=0$.

Hence $\Xi\eta=\eta$. Therefore $\Lambda\eta=\Xi\eta=\Sigma\eta=\Upsilon\eta=\eta$.

Since Λ,Ξ,Σ and Υ all have common fixed points, η is one of them. Analogously, the proposition's conclusion flows from (ii) assumption.

The inequality (2.3) implies uniqueness. □

Remark 2.2. Geraghty-Berinde-Suzuki type contraction maps on § are defined as selfmaps Λ,Ξ,Σ and Υ of a b-metric space § that fulfill (2.3).

Proposition 2.3. *Let Λ,Ξ,Σ and Υ be selfmaps that fulfill* (2.1) *and Geraghty-Berinde-Suzuki type contraction maps of a b-metric space* $(\S,\eth)$. *Then, the sequence $\{\zeta_n\}$ described by* (2.2) *is b-Cauchy in § for every $\xi_0\in\S$.*

Proof. Given that $\{\zeta_n\}$ is defined by (2.2), let $\xi_0\in\S$. Let us assume that for a given n, $\zeta_n=\zeta_{n+1}$.

Case(i): n even. Given some $m\in\mathbb{N}$, we write $n=2m$. $\eth(\zeta_{n+1},\zeta_{n+2})>0$ is assumed. Since

$$\frac{1}{2s}\min\{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})\}\leq\max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})\}$$

The inequality (2.3) gives us

$$
\begin{aligned}
s^4\eth(\zeta_{n+1},\zeta_{n+2}) &= s^4\eth(\zeta_{2m+1},\zeta_{2m+2})\\
&= s^4\eth(\zeta_{2m+2},\zeta_{2m+1})\\
&= s^4\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})\\
&\le \beta(M(\xi_{2m+2},\xi_{2m+1}))M(\xi_{2m+2},\xi_{2m+1})+LN(\xi_{2m+2},\xi_{2m+1})
\end{aligned} \tag{2.6}
$$

where

$$
\begin{aligned}
M(\xi_{2m+2},\xi_{2m+1}) &= \max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\\
&\quad \eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\tfrac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1})}{2s},\tfrac{\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})}{2s},\\
&\quad \tfrac{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})}{1+\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})},\\
&\quad \tfrac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1})\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})}{1+s^4[\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})]}\}\\
&= \max\{0,\eth(\zeta_{n+1},\zeta_{n+2}),0,0,\tfrac{\eth(\zeta_n,\zeta_{n+2})}{2s},0,0\} = \eth(\zeta_{n+1},\zeta_{n+2}),\\
N(\xi_{2m+2},\xi_{2m+1}) &= \min\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\\
&\quad \eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1}),\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})\}\\
&= \min\{0,\eth(\zeta_{n+1},\zeta_{n+2}),0,0,\eth(\zeta_n,\zeta_{n+2})\} = 0.
\end{aligned}
$$

We obtain the inequality (2.6) from it

$$
\begin{aligned}
s^4\eth(\zeta_{n+1},\zeta_{n+2}) &\le \beta(M(\xi_{2m+2},\xi_{2m+1}))M(\xi_{2m+1},\xi_{2m+1})\\
&\le \beta(\eth(\zeta_{n+1},\zeta_{n+2}))\eth(\zeta_{n+1},\zeta_{n+2}) \le \tfrac{\eth(\zeta_{n+1},\zeta_{n+2})}{s}
\end{aligned}
$$

which implies that $(s^5-1)\eth(\zeta_{n+1},\zeta_{n+2}) \le 0$.

Since $(s^5-1)\ge 0$, we have $\eth(\zeta_{n+1},\zeta_{n+2}) \le 0$.

$\zeta_{n+2}=\zeta_{n+1}=\zeta_n$, as a result. Generally speaking, for $k=0,1,2,\ldots$, we obtain $\zeta_{n+k}=\zeta_n$.

Case (ii): n odd.

$n=2m+1$ is written for a given $m\in\mathbb{N}$.

$\frac{1}{2s}\min\{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\eth(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})\} \le \max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3}),\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})\}$,

the inequality (2.3) gives us

$$
\begin{aligned}
s^4\eth(\zeta_{n+1},\zeta_{n+2}) &= s^4\eth(\zeta_{2m+2},\zeta_{2m+3})\\
&= \eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})\\
&\le \beta(M(\xi_{2m+2},\xi_{2m+3}))M(\xi_{2m+2},\xi_{2m+3})LN(\xi_{2m+2},\xi_{2m+3})
\end{aligned} \tag{2.7}
$$

where

$$
M(\xi_{2m+2},\xi_{2m+3}) = \max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),
$$

$$\eth(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3}),\frac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+3})}{2s},\frac{\eth(\Upsilon\xi_{2m+3},\Lambda\xi_{2m+2})}{2s},$$
$$\frac{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})\eth(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3})}{1+\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})},$$
$$\frac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+3})\eth(\Upsilon\xi_{2m+3},\Lambda\xi_{2m+2})}{1+s^4[\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+3})]}\}$$
$$=\max\{0,0,\eth(\zeta_{n+1},\zeta_{n+2}),\frac{\eth(\zeta_n,\zeta_{n+2})}{2s},0,0,0\}=\eth(\zeta_{n+1},\zeta_{n+2}),$$

$$\begin{aligned}N(\xi_{2m+2},\xi_{2m+3})&=\min\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+3}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\\&\qquad\eth(\Upsilon\xi_{2m+3},\Xi\xi_{2m+3}),\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+3}),\eth(\Upsilon\xi_{2m+3},\Lambda\xi_{2m+2})\}\\&=\min\{0,0,\eth(\zeta_{n+1},\zeta_{n+2}),\eth(\zeta_n,\zeta_{n+2}),0\}=0.\end{aligned}$$

From the inequality (2.7), we have

$$\begin{aligned}s^4\eth(\zeta_{n+1},\zeta_{n+2})&\le\beta(M(\xi_{2m+2},\xi_{2m+3}))M(\xi_{2m+2},\xi_{2m+3})\\&\le\beta(\eth(\zeta_{n+1},\zeta_{n+2}))\eth(\zeta_{n+1},\zeta_{n+2})\le\frac{\eth(\zeta_{n+1},\zeta_{n+2})}{s}\end{aligned}$$

which implies that $(s^5-1)\eth(\zeta_{n+1},\zeta_{n+2})\le 0$.

Since $(s^5-1)\ge 0$, we have $\eth(\zeta_{n+1},\zeta_{n+2})\le 0$.

As a result, $\zeta_{n+2}=\zeta_{n+1}=\zeta_n$.

For $k=0,1,2,\ldots$, we generally obtain $\zeta_{n+k}=\zeta_n$.

$\zeta_{n+k}=\zeta_n$ for all $k=0,1,2,\ldots$ follows from Cases (i) and (ii). Due to the fact that $\{\zeta_{n+k}\}$ is a constant sequence, $\{\zeta_n\}$ is Cauchy.

For every $n\in\mathbb{N}$, we now suppose that $\zeta_{n-1}\ne\zeta_n$. For all $m\in\mathbb{N}$, $n=2m+1$ if n is odd.

Since

$$\frac{1}{2s}\min\{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})\}\le\max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})\}.$$

From the inequality (2.3), we have

$$\begin{aligned}s^4\eth(\zeta_n,\zeta_{n+1})&=s^4\eth(\zeta_{2m+1},\zeta_{2m+2})\\&=s^4\eth(\zeta_{2m+2},\zeta_{2m+1})\\&=s^4\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})\\&\le\beta(M(\xi_{2m+2},\xi_{2m+1}))M(\xi_{2m+2},\xi_{2m+1})+LN(\xi_{2m+2},\xi_{2m+1})\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}M(\xi_{2m+2},\xi_{2m+1})&=\max\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),\\&\qquad\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\frac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1})}{2s},\frac{\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})}{2s},\\&\qquad\frac{\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2})\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})}{1+\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})},\\&\qquad\frac{\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1})\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})}{1+s^4[\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1})+\eth(\Lambda\xi_{2m+2},\Xi\xi_{2m+1})]}\}\end{aligned}$$

$$\leq \max\{\eth(\zeta_{n-1},\zeta_n),\eth(\zeta_n,\zeta_{n+1}),\eth(\zeta_{n-1},\zeta_n),0,\frac{\eth(\zeta_n,\zeta_n)+\eth(\zeta_n,\zeta_{n+1})}{2},$$
$$\frac{\eth(\zeta_n,\zeta_{n+1})\eth(\zeta_{n-1},\zeta_n)}{1+\eth(\zeta_{n-1},\zeta_n)+\eth(\zeta_n,\zeta_{n+1})},0\}$$
$$\leq \max\{\eth(\zeta_{n-1},\zeta_n),\eth(\zeta_n,\zeta_{n+1})\},$$

$$N(\xi_{2m+2},\xi_{2m+1}) = \min\{\eth(\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Lambda\xi_{2m+2}),$$
$$\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\eth(\Sigma\xi_{2m+2},\Xi\xi_{2m+1}),\eth(\Upsilon\xi_{2m+1},\Lambda\xi_{2m+2})\}$$
$$= \min\{\eth(\zeta_{n-1},\zeta_n),\eth(\zeta_n,\zeta_{n+1}),\eth(\zeta_{n-1},\zeta_n),0,\eth(\zeta_{n-1},\zeta_{n+1})\} = 0.$$

Suppose $M(\xi_{2m+2},\xi_{2m+1}) = \eth(\zeta_n,\zeta_{n+1})$.

Then from the inequality (2.8), we have

$$s^4\eth(\zeta_n,\zeta_{n+1}) \leq \beta(M(\xi_{2m+2},\xi_{2m+1}))M(\xi_{2m+2},\xi_{2m+1})$$
$$\leq \beta(\eth(\zeta_n,\zeta_{n+1}))\eth(\zeta_n,\zeta_{n+1}) \leq \frac{\eth(\zeta_n,\zeta_{n+1})}{s}$$

which implies that $(s^5-1)\eth(\zeta_n,\zeta_{n+1}) \leq 0$.

Since $(s^5-1) \geq 0$, we have $\eth(\zeta_n,\zeta_{n+1}) \leq 0$.

Consequently, $\eth(\zeta_{n-1},\zeta_n) = M(\xi_{2m+2},\xi_{2m+1})$.

We obtain the inequality (2.8) from it

$$\begin{aligned} s^4\eth(\zeta_n,\zeta_{n+1}) &\leq \beta(M(\xi_{2m+2},\xi_{2m+1}))M(\xi_{2m+2},\xi_{2m+1}) \\ &\leq \beta(\eth(\zeta_{n-1},\zeta_n))\eth(\zeta_{n-1},\zeta_n) \leq \frac{\eth(\zeta_{n-1},\zeta_n)}{s}. \end{aligned} \tag{2.9}$$

Furthermore, it is evident that (2.9) holds when n is even. Thus, for every $n \in \mathbb{N}$, we obtain $\eth(\zeta_n,\zeta_{n+1}) \leq \frac{1}{s^5}\eth(\zeta_{n-1},\zeta_n)$.

The sequence $\{\zeta_n\}$ is a b-Cauchy sequence in §, according to Lemma 1.10. □

The principal outcome of this paper is as follows.

Theorem 2.4. *Let* (2.1) *and Geraghty-Berinde-Suzuki type contractive maps be satisfied by* Λ,Ξ,S *and* Υ *selfmaps on a complete b-metric space* $(§,\eth)$. *Should either*

(i) *the pair* (Ξ,Υ) *is weakly compatible,* Λ *(or)* Σ *is b-continuous, and the pair* (Λ,Σ) *is compatible*

or

(ii) Ξ *(or)* Υ *is b-continuous, the pair* (Ξ,Υ) *is compatible, and the pair* (Λ,Σ) *is weakly compatible*

then Λ,Ξ,Σ *and* Υ *have a unique common fixed point in* §.

Proof. The sequence $\{\zeta_n\}$ is b-Cauchy in § according to Proposition 2.3. Given that § is b-complete, $\lim_{n\to\infty}\zeta_n = \eta$ for every $\eta \in$ §. Thus

$$\begin{cases} \lim_{n\to\infty}\zeta_{2n} = \lim_{n\to\infty}\Lambda\xi_{2n} = \lim_{n\to\infty}\Upsilon\xi_{2n+1} = \eta \text{ and} \\ \lim_{n\to\infty}\zeta_{2n+1} = \lim_{n\to\infty}\Xi\xi_{2n+1} = \lim_{n\to\infty}\Sigma\xi_{2n+2} = \eta. \end{cases} \tag{2.10}$$

Let us assume that (i).

It implies that Σ is b-continuous. So, $\lim_{n\to\infty}\Sigma\Sigma\xi_{2n+2} = \Sigma\eta$, $\lim_{n\to\infty}\Sigma\Lambda\xi_{2n} = \Sigma\eta$.

By the b-triangle inequality, we have $\eth(\Lambda\Sigma\xi_{2n},\Sigma\eta) \leq s[\eth(\Lambda\Sigma\xi_{2n},\Sigma\Lambda\xi_{2n}) + \eth(\Sigma\Lambda\xi_{2n},\Sigma\eta)]$.

Since the pair (Λ,Σ) is compatible, $\lim_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n},\Sigma\Lambda\xi_{2n}) = 0$.

Limit superior taken as $n\to\infty$ gives us

$\limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n},\Sigma\eta) \leq s[\limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n},\Sigma\Lambda\xi_{2n}) + \limsup_{n\to\infty}\eth(\Sigma\Lambda\xi_{2n},\Sigma\eta)] = 0.$

Since $\lim_{n\to\infty}\Lambda\Sigma\xi_{2n} = \Sigma\eta$, this follows.

The proof of $\Sigma\eta = \eta$ is now given.

Let $\Sigma\eta \neq \eta$ be the case. Since

$$\frac{1}{2s}\min\{\eth(\Sigma\Sigma\xi_{2m+2},\Lambda\Sigma\xi_{2m+2}),\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})\} \leq \max\{\eth(\Sigma\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}), \eth(\Lambda\Sigma\xi_{2m+2},\Xi\xi_{2m+1})\}$$

The inequality (2.3) gives us

$$s^4\eth(\Lambda\Sigma\xi_{2n+2},\Xi\xi_{2n+1}) \leq \beta(M(\Sigma\xi_{2n+2},\xi_{2n+1}))M(\Sigma\xi_{2n+2},\xi_{2n+1}) + LN(\Sigma\xi_{2n+2},\xi_{2n+1}) \tag{2.11}$$

where

$$\begin{aligned} M(\Sigma\xi_{2n+2},\xi_{2n+1}) = \max\{&\eth(\Sigma\Sigma\xi_{2n+2},\Upsilon\xi_{2n+1}),\eth(\Sigma\Sigma\xi_{2n+2},\Lambda\Sigma\xi_{2n+s}),\\ &\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1}),\frac{\eth(\Sigma\Sigma\xi_{2n+2},\Xi\xi_{2n+1})}{2s},\frac{\eth(\Upsilon\xi_{2n+1},\Lambda\Sigma\xi_{2n+2})}{2s},\\ &\frac{\eth(\Sigma\Sigma\xi_{2n+2},\Lambda\Sigma\xi_{2n+2})\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1})}{1+\eth(\Sigma\Sigma\xi_{2n+2},\Upsilon\xi_{2n+1})+\eth(\Lambda\Sigma\xi_{2n+2},\Xi\xi_{2n+1})},\\ &\frac{\eth(\Sigma\Sigma\xi_{2n+2},\Xi\xi_{2n+1})\eth(\Upsilon\xi_{2n+1},\Lambda\Sigma\xi_{2n+2})}{1+s^4[\eth(\Sigma\Sigma\xi_{2n+2},\Upsilon\xi_{2n+1})+\eth(\Lambda\Sigma\xi_{2n+2},\Xi\xi_{2n+1})]}\},\\ N(\Sigma\xi_{2m+2},\xi_{2m+1}) = \min\{&\eth(\Sigma\Sigma\xi_{2m+2},\Upsilon\xi_{2m+1}),\eth(\Sigma\Sigma\xi_{2m+2},\Lambda\Sigma\xi_{2m+2}),\\ &\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\eth(\Sigma\Sigma\xi_{2m+2},\Xi\xi_{2m+1}),\eth(\Upsilon\xi_{2m+1},\Lambda\Sigma\xi_{2m+2})\}. \end{aligned}$$

Using Lemma 1.11 and assuming limit superior as $n\to\infty$ on $M(\Sigma\xi_{2n+2},\xi_{2n+1}), N(\Sigma\xi_{2m+2},\xi_{2m+1})$, we obtain

$$\limsup_{n\to\infty}M(\Sigma\xi_{2n+2},\xi_{2n+1}) \leq \max\{s^2\eth(\Sigma\eta,\eta),0,0,\frac{s^2\eth(\Sigma\eta,\eta)}{2s},\frac{s^2\eth(\Sigma\eta,\eta)}{2s},0,\frac{s^4[\eth(\Sigma\eta,\eta)]^2}{1+2s^4\eth(\Sigma\eta,\eta)}\} = s^2\eth(\Sigma\eta,\eta),$$

$$\limsup_{n\to\infty}N(\Sigma\xi_{2n+2},\xi_{2n+1}) \leq \min\{s^2\eth(\Sigma\eta,\eta),0,0,s^2\eth(\Sigma\eta,\eta),s^2\eth(\Sigma\eta,\eta)\} = 0.$$

Therefore

(2.12)

$$\frac{1}{s^2}\eth(\Sigma\eta,\eta) \le \liminf_{n\to\infty} M(\Sigma\xi_{2n+2},\xi_{2n+1}) \le \limsup_{n\to\infty} M(\Sigma\xi_{2n+2},\xi_{2n+1}) \le s^2\eth(\Sigma\eta,\eta) \text{ and}$$

$$\limsup_{n\to\infty} N(\Sigma\xi_{2n+2},\xi_{2n+1}) = 0.$$

Using Lemma 1.11 and the inequality (2.11), where the limit superior is $n\to\infty$, we obtain

$$\begin{aligned}
s^4\frac{1}{s^2}\eth(\Sigma\eta,\eta) &\le s^4 \limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n+2},\Xi\xi_{2n+1})\\
&= \limsup_{n\to\infty} s^4\eth(\Lambda\Sigma\xi_{2n+2},\Xi\xi_{2n+1})\\
&\le \limsup_{n\to\infty}[\beta(M(\Sigma\xi_{2n+2},\xi_{2n+1}))M(\Sigma\xi_{2n+2},\xi_{2n+1}) + LN(\Sigma\xi_{2n+2},\xi_{2n+1})]\\
&= \limsup_{n\to\infty}\beta(M(\Sigma\xi_{2n+2},\xi_{2n+1}))\limsup_{n\to\infty} M(\Sigma\xi_{2n+2},\xi_{2n+1})\\
&\qquad + L\limsup_{n\to\infty} N(\Sigma\xi_{2n+2},\xi_{2n+1})\\
&\le \limsup_{n\to\infty}\beta(M(\Sigma\xi_{2n+2},\xi_{2n+1}))s^2\eth(\Sigma\eta,\eta).
\end{aligned}$$

Therefore

$\frac{1}{s} \le 1 \le \limsup_{n\to\infty}\beta(M(\Sigma\xi_{2n+2},\xi_{2n+1})) \le \frac{1}{s}$ which implies that

$\limsup_{n\to\infty}\beta(M(\Sigma\xi_{2n+2},\xi_{2n+1})) = \frac{1}{s}$.

It follows that $\lim_{n\to\infty} M(\Sigma\xi_{2n+2},\xi_{2n+1}) = 0$ since $\beta\in\mathfrak{F}$.

Thus, using the inequality (2.12), we may obtain

$\frac{1}{s^2}\eth(\Sigma\eta,\eta) \le \lim_{n\to\infty} M(\Sigma\xi_{2n+2},\xi_{2n+1}) = 0$ which implies that $\eth(\Sigma\eta,\eta)\le 0$.

Consequently, $\Sigma\eta = \eta$. Now, we demonstrate that $\Lambda\eta = \eta$. Assume $\Lambda\eta \ne \eta$.

Since

$\frac{1}{2s}\min\{\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1})\} \le \max\{\eth(\Sigma\eta,\Upsilon\xi_{2m+1}),\eth(\Lambda\eta,\Xi\xi_{2m+1})\}$

From the inequality (2.3), we have

$$s^4\eth(\Lambda\eta,\Xi\xi_{2n+1}) \le \beta(M(\eta,\xi_{2n+1}))M(\eta,\xi_{2n+1}) + LN(\eta,\xi_{2n+1}) \tag{2.13}$$

where

$$\begin{aligned}
M(\eta,\xi_{2n+1}) = \max\{&\eth(\Sigma\eta,\Upsilon\xi_{2n+1}),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1}),\\
&\tfrac{\eth(\Sigma\eta,\Xi\xi_{2n+1})}{2s},\tfrac{\eth(\Upsilon\xi_{2n+1},\Lambda\eta)}{2s},\tfrac{\eth(\Sigma\eta,\Lambda\eta)\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1})}{1+\eth(\Sigma\eta,\Upsilon\xi_{2n+1})+\eth(\Lambda\eta,\Xi\xi_{2n+1})},\\
&\tfrac{\eth(\Sigma\eta,\Xi\xi_{2n+1})\eth(\Upsilon\xi_{2n+1},\Lambda\eta)}{1+s^4[\eth(\Sigma\eta,\Upsilon\xi_{2n+1})+\eth(\Lambda\eta,\Xi\xi_{2n+1})]}\},
\end{aligned}$$

$N(\eta,\xi_{2m+1}) = \min\{\eth(\Sigma\eta,\Upsilon\xi_{2m+1}),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon\xi_{2m+1},\Xi\xi_{2m+1}),\eth(\Sigma\eta,\Xi\xi_{2m+1}),\eth(\Upsilon\xi_{2m+1},\Lambda\eta)\}$.

With Lemma 1.11 and limit superior taken as $n\to\infty$ on $M(\eta,\xi_{2n+1}),N(\eta,\xi_{2n+1})$, we obtain

$$\limsup_{n\to\infty} M(\eta,\xi_{2n+1}) \leq \max\{s^2\eth(\Lambda\eta,\eta),0,0,\tfrac{s^2\eth(\Lambda\eta,\eta)}{2s},\tfrac{s^2\eth(\Lambda\eta,\eta)}{2s},0,\tfrac{s^4[\eth(\Lambda\eta,\eta)]^2}{1+2s^4\eth(\Lambda\eta,\eta)}\}$$
$$= s^2\eth(\Lambda\eta,\eta),$$

$$\limsup_{n\to\infty} M(\eta,\xi_{2n+1}) \leq \min\{s^2\eth(\Lambda\eta,\eta),0,0,0,s^2\eth(\Lambda\eta,\eta)\} = 0.$$

Therefore

$$\begin{aligned}&\tfrac{1}{s^2}\eth(\Lambda\eta,\eta) \leq \liminf_{n\to\infty} M(\eta,\xi_{2n+1}) \leq \limsup_{n\to\infty} M(\eta,\xi_{2n+1}) \leq s^2\eth(\Lambda\eta,\eta) \text{ and}\\ &\limsup_{n\to\infty} N(\eta,\xi_{2n+1}) = 0.\end{aligned} \tag{2.14}$$

Lemma 1.11, (2.14), and the inequality (2.13) are used to obtain the limit superior, which is $n\to\infty$

$$\begin{aligned}
s^4\tfrac{1}{s^2}\eth(\Lambda\eta,\eta) &\leq s^4\limsup_{n\to\infty}\eth(\Lambda\eta,\Xi\xi_{2n+1})\\
&= \limsup_{n\to\infty} s^4\eth(\Lambda\eta,\Xi\xi_{2n+1})\\
&\leq \limsup_{n\to\infty}[\beta(M(\eta,\xi_{2n+1}))M(\eta,\xi_{2n+1}) + LN(\eta,\xi_{2n+1})]\\
&= \limsup_{n\to\infty}\beta(M(\eta,\xi_{2n+1}))\limsup_{n\to\infty} M(\eta,\xi_{2n+1}) + L\limsup_{n\to\infty} N(\eta,\xi_{2n+1})\\
&\leq \limsup_{n\to\infty}\beta(M(\eta,\xi_{2n+1}))s^2\eth(A\eta,\eta).
\end{aligned}$$

Hence

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty}\beta(M(\eta,\xi_{2n+1})) \leq \frac{1}{s}$ which implies that

$\limsup_{n\to\infty}\beta(M(\eta,\xi_{2n+1})) = \frac{1}{s}$.

It follows that $\lim_{n\to\infty} M(\eta,\xi_{2n+1}) = 0$ since $\beta \in \mathfrak{F}$. Thus, based on the inequality (2.14), we obtain $\frac{1}{s^2}\eth(\Lambda\eta,\eta) \leq \lim_{n\to\infty} M(\eta,\xi_{2n+1}) = 0$ which implies that $\eth(\Lambda\eta,\eta) \leq 0$.

$\Lambda\eta = \Sigma\eta = \eta$ as a result. We now know that η is a unique common fixed point of Λ,Ξ,Σ and Υ according to Proposition 2.1. Assuming b-continuous Λ, it follows that

$\lim_{n\to\infty}\Lambda\Lambda\xi_{2n} = \Lambda\eta,\ \lim_{n\to\infty}\Lambda\Sigma\xi_{2n+2} = \Lambda\eta.$

The b-triangle inequality gives us

$\eth(\Sigma\Lambda\xi_{2n},\Lambda\eta) \leq s[\eth(\Sigma\Lambda\xi_{2n},\Lambda\Sigma\xi_{2n}) + \eth(\Lambda\Sigma\xi_{2n},\Lambda\eta)].$

Since the pair (Λ,Σ) is compatible, $\lim_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n},\Sigma\Lambda\xi_{2n}) = 0$.

Limit superior taken as $n\to\infty$ gives us

$\limsup_{n\to\infty}\eth(\Sigma\Lambda\xi_{2n},\Lambda\eta) \leq s[\limsup_{n\to\infty}\eth(\Sigma\Lambda\xi_{2n},\Lambda\Sigma\xi_{2n}) + \limsup_{n\to\infty}\eth(\Lambda\Sigma\xi_{2n},\Lambda\eta)] = 0.$

Therefore $\lim_{n\to\infty}\Sigma\Lambda\xi_{2n} = \Lambda\eta$.

We now establish $\Lambda\eta = \eta$. Assume that $\Lambda\eta \neq \eta$.

Since

$\frac{1}{2s}\min\{\eth(\Sigma\Lambda\xi_{2n},\Lambda\Lambda\xi_{2n}),\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1})\}\leq\max\{\eth(\Sigma\Lambda\xi_{2n},\Upsilon\xi_{2n+1}),\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})\}$

From the inequality (2.3), we have

$$s^4\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})\leq\beta(M(\Lambda\xi_{2n},\xi_{2n+1}))M(\Lambda\xi_{2n},\xi_{2n+1})+LN(\Lambda\xi_{2n},\xi_{2n+1}) \tag{2.15}$$

where

$$\begin{aligned}M(\Lambda\xi_{2n},\xi_{2n+1})=\max\{&\eth(\Sigma\Lambda\xi_{2n},\Upsilon\xi_{2n+1}),\eth(\Sigma\Lambda\xi_{2n},\Lambda\Lambda\xi_{2n}),\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1}),\\&\tfrac{\eth(\Sigma\Lambda\xi_{2n},\Xi\xi_{2n+1})}{2s},\tfrac{\eth(\Upsilon\xi_{2n+1},\Lambda\Lambda\xi_{2n})}{2s},\tfrac{\eth(\Sigma\Lambda\xi_{2n},\Lambda\Lambda\xi_{2n})\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1})}{1+\eth(\Sigma\Lambda\xi_{2n},\Upsilon\xi_{2n+1})+\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})},\\&\tfrac{\eth(\Sigma\Lambda\xi_{2n},\Xi\xi_{2n+1})\eth(\Upsilon\xi_{2n+1},\Lambda\Lambda\xi_{2n})}{1+s^4[\eth(\Sigma\Lambda\xi_{2n},\Upsilon\xi_{2n+1})+\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})]}\},\\N(\Lambda\xi_{2n},\xi_{2n+1})=\min\{&\eth(\Sigma\Lambda\xi_{2n},\Upsilon\xi_{2n+1}),\eth(\Sigma\Lambda\xi_{2n},\Lambda\Lambda\xi_{2n}),\\&\eth(\Upsilon\xi_{2n+1},\Xi\xi_{2n+1}),\eth(\Sigma\Lambda\xi_{2n},\Xi\xi_{2n+1}),\eth(\Upsilon\xi_{2n+1},\Lambda\Lambda\xi_{2n})\}.\end{aligned}$$

Taking the limit superior as $n\to\infty$ on $M(\Lambda\xi_{2n},\xi_{2n+1}),N(\Lambda\xi_{2n},\xi_{2n+1})$ and applying Lemma 1.11, we obtain

$$\begin{aligned}\limsup_{n\to\infty}M(\Lambda\xi_{2n},\xi_{2n+1})&\leq\max\{s^2\eth(\Lambda\eta,\eta),0,0,\tfrac{s^2\eth(\Lambda\eta,\eta)}{2s},\tfrac{s^2\eth(\Lambda\eta,\eta)}{2s},0,\tfrac{s^4[\eth(\Lambda\eta,\eta)]^2}{1+2s^2\eth(\Lambda\eta,\eta)}\}\\&=s^2\eth(\Lambda\eta,\eta),\end{aligned}$$

$$\limsup_{n\to\infty}N(\Lambda\xi_{2n},\xi_{2n+1})\leq\min\{s^2\eth(\Lambda\eta,\eta),0,0,s^2\eth(\Lambda\eta,\eta),s^2\eth(\eta,\Lambda\eta)\}=0.$$

Therefore

$$\begin{aligned}&\tfrac{1}{s^2}\eth(\Lambda\eta,\eta)\leq\liminf_{n\to\infty}M(\Lambda\xi_{2n},\xi_{2n+1})\leq\limsup_{n\to\infty}M(\Lambda\xi_{2n},\xi_{2n+1})\leq s^2\eth(\Lambda\eta,\eta)\text{ and}\\&\limsup_{n\to\infty}N(\Lambda\xi_{2n},\xi_{2n+1})=0.\end{aligned} \tag{2.16}$$

Using (2.16), Lemma 1.11, and the inequality (2.15), where the limit superior is $n\to\infty$, we obtain

$$\begin{aligned}s^4\tfrac{1}{s^2}\eth(\Lambda\eta,\eta)&\leq s^4\limsup_{n\to\infty}\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})\\&=\limsup_{n\to\infty}s^4\eth(\Lambda\Lambda\xi_{2n},\Xi\xi_{2n+1})\\&\leq\limsup_{n\to\infty}[\beta(M(\Lambda\xi_{2n},\xi_{2n+1}))M(\Lambda\xi_{2n},\xi_{2n+1})+LN(\Lambda\xi_{2n},\xi_{2n+1})]\\&=\limsup_{n\to\infty}\beta(M(\Lambda\xi_{2n},\xi_{2n+1}))\limsup_{n\to\infty}M(\Lambda\xi_{2n},\xi_{2n+1})\\&\quad+L\limsup_{n\to\infty}N(\Lambda\xi_{2n},\xi_{2n+1})\\&\leq\limsup_{n\to\infty}\beta\Lambda(M(\Lambda\xi_{2n},\xi_{2n+1}))s^2\eth(\Lambda\eta,\eta).\end{aligned}$$

Thus

$\frac{1}{s}\leq 1\leq\limsup_{n\to\infty}\beta(M(\Lambda\xi_{2n},\xi_{2n+1}))\leq\frac{1}{s}$ which implies that

$\limsup_{n\to\infty}\beta(M(\Lambda\xi_{2n},\xi_{2n+1}))=\frac{1}{s}$.

Given that $\beta \in \mathfrak{F}$, $\lim_{n\to\infty} M(\Lambda\xi_{2n}, \xi_{2n+1}) = 0$ follows. Consequently, based on the inequality (2.16), we have

$\frac{1}{s^2}\eth(\Lambda\eta, \eta) \leq \lim_{n\to\infty} M(\Lambda\xi_{2n}, \xi_{2n+1}) = 0$ which implies that $\eth(\Lambda\eta, \eta) \leq 0$.

Therefore $\Lambda\eta = \eta$.

Since $\Lambda(\xi) \subseteq \Upsilon(\S)$, there exists $u \in \S$ such that $\eta = \Upsilon u$.

We now show that $\Xi u = \eta$. Suppose that $\Xi u \neq \eta$.

Since

$\frac{1}{2s}\min\{\eth(\Sigma\xi_{2n}, \Lambda\xi_{2n}), \eth(\Upsilon u, \Xi u)\} \leq \max\{\eth(\Sigma\xi_{2n}, \Upsilon u), \eth(\Lambda\xi_{2n}, \Xi u)\}$

From the inequality (2.3), we have

$$s^4\eth(\Lambda\xi_{2n}, \Xi u) \leq \beta(M(\xi_{2n}, u))M(\xi_{2n}, u) + LN(\xi_{2n}, u) \tag{2.17}$$

where

$$M(\xi_{2n}, u) = \max\{\eth(\Sigma\xi_{2n}, \Upsilon u), \eth(\Sigma\xi_{2n}, \Lambda\xi_{2n}), \eth(\Upsilon u, \Xi u), \tfrac{\eth(\Sigma\xi_{2n}, \Xi u)}{2s}, \tfrac{\eth(\Upsilon u, \Lambda\xi_{2n})}{2s},$$
$$\tfrac{\eth(\Sigma\xi_{2n}, \Lambda\xi_{2n})\eth(\Upsilon u, \Xi u)}{1+\eth(\Sigma\xi_{2n}, \Upsilon u)+\eth(\Lambda\xi_{2n}, \Xi u)}, \tfrac{\eth(\Sigma\xi_{2n}, \Xi u)\eth(\Upsilon u, \Lambda\xi_{2n})}{1+s^4[\eth(\Sigma\xi_{2n}, \Upsilon u)+\eth(\Lambda\xi_{2n}, \Xi u)]}\},$$

$$N(\xi_{2n}, u) = \min\{\eth(\Sigma\xi_{2n}, \Upsilon u), \eth(\Sigma\xi_{2n}, \Lambda\xi_{2n}), \eth(\Upsilon u, \Xi u), \eth(\Sigma\xi_{2n}, \Xi u), \eth(\Upsilon u, \Lambda\xi_{2n})\}.$$

Using Lemma 1.11 and interpreting limit superior as $n \to \infty$ on $M(\xi_{2n}, u), N(\xi_{2n}, u)$, we arrive at

$$\begin{aligned}\limsup_{n\to\infty} M(\xi_{2n}, u) &\leq \max\{s^2\eth(\eta, \Xi u), 0, 0, \tfrac{s^2\eth(\eta, \Xi u)}{2s}, \tfrac{s^2\eth(\eta, \Xi u)}{2s}, 0, \tfrac{s^4[\eth(\eta, \Xi u)]^2}{1+2s^2\eth(\eta, \Xi u)}\}\\ &= s^2\eth(\Lambda\eta, \eta),\end{aligned}$$

$$\limsup_{n\to\infty} N(\xi_{2n}, u) \leq \min\{s^2\eth(\eta, \Xi u), 0, 0, s^2\eth(\eta, \Xi u), s^2\eth(\eta, \Xi u)\}.$$

Therefore

$$\begin{aligned}&\tfrac{1}{s^2}\eth(\eta, \Xi u) \leq \liminf_{n\to\infty} M(\xi_{2n}, u) \leq \limsup_{n\to\infty} M(\xi_{2n}, u) \leq s^2\eth(\eta, \Xi u) \text{ and}\\ &\leq \limsup_{n\to\infty} N(\xi_{2n}, u) = 0\end{aligned} \tag{2.18}$$

Using (2.18) and Lemma 1.11, we obtain the limit superior, which we take to be $n \to \infty$ in the inequality (2.17)

$$\begin{aligned}s^4\tfrac{1}{s^2}\eth(\eta, \Xi u) &\leq s^4 \limsup_{n\to\infty} \eth(\Lambda\xi_{2n}, \Xi u)\\ &= \limsup_{n\to\infty} s^4\eth(\Lambda\xi_{2n}, \Xi u)\\ &\leq \limsup_{n\to\infty}[\beta(M(\xi_{2n}, u))M(\xi_{2n}, u) + LN(\xi_{2n}, u)]\\ &= \limsup_{n\to\infty} \beta(M(\xi_{2n}, u)) \limsup_{n\to\infty} M(\xi_{2n}, u) + L \limsup_{n\to\infty} N(\xi_{2n}, u)\end{aligned}$$

$$\leq \limsup_{n\to\infty} \beta(M(\xi_{2n},u))s^2\eth(\eta,\Xi u).$$

Therefore

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(\xi_{2n},u)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(\xi_{2n},u)) = \frac{1}{s}$.

It follows that $\lim_{n\to\infty} M(\xi_{2n},u) = 0$ since $\beta \in \mathfrak{F}$. Thus, based on the inequality (2.18), we can obtain

$\frac{1}{s^2}\eth(\eta,\Xi u) \leq \lim_{n\to\infty} M(\xi_{2n},u) = 0.$ implies that $\eth(\eta,\Xi u) \leq 0$.

Therefore $\Xi u = \Upsilon u = \eta$. Since the pair (Ξ,Υ) is weakly compatible and $\Xi u = \Upsilon u$, we have

$\Xi\Upsilon u = \Upsilon\Xi u$. i.e., $\Xi\eta = \Upsilon\eta$.

Now we demonstrate that $\Xi\eta = \eta$. Let us assume $\Xi\eta \neq \eta$.

As $\frac{1}{2s}\min\{\eth(\Sigma\xi_{2n},\Lambda\xi_{2n}),\eth(\Upsilon\eta,\Xi\eta)\} \leq \max\{\eth(\Sigma\xi_{2n},\Upsilon\eta),\eth(\Lambda\xi_{2n},\Xi\eta)\}$.

From the inequality (2.3), we have

$$s^4\eth(\Lambda\xi_{2n},\Xi\eta) \leq \beta(M(\xi_{2n},\eta))M(\xi_{2n},\eta) + LN(\xi_{2n},\eta) \tag{2.19}$$

where

$$M(\xi_{2n},\eta) = \max\{\eth(\Sigma\xi_{2n},\Upsilon\eta),\eth(\Sigma\xi_{2n},\Lambda\xi_{2n}),\eth(\Upsilon\eta,\Xi\eta),\frac{\eth(\Sigma\xi_{2n},\Xi\eta)}{2s},\frac{\eth(\Upsilon\eta,\Lambda\xi_{2n})}{2s},$$
$$\frac{\eth(\Sigma\xi_{2n},\Lambda\xi_{2n})\eth(\Upsilon\eta,\Xi\eta)}{1+\eth(\Sigma\xi_{2n},\Upsilon\eta)+\eth(\Lambda\xi_{2n},\Xi\eta)},\frac{\eth(\Sigma\xi_{2n},\Xi\eta)\eth(\Upsilon\eta,\Lambda\xi_{2n})}{1+s^4\eth(\Sigma\xi_{2n},\Upsilon\eta)+\eth(\Lambda\xi_{2n},\Xi\eta)}\},$$

$$N(\xi_{2n},\eta) = \min\{\eth(\Sigma\xi_{2n},\Upsilon\eta),\eth(\Sigma\xi_{2n},\Lambda\xi_{2n}),\eth(\Upsilon\eta,\Xi\eta),\eth(\Sigma\xi_{2n},\Xi\eta),\eth(\Upsilon\eta,\Lambda\xi_{2n})\}$$

Utilizing Lemma 1.11 and defining limit superior as $n\to\infty$ on $M(\xi_{2n},\eta),N(\xi_{2n},\eta)$, we arrive at

$$\limsup_{n\to\infty} M(\xi_{2n},\eta) \leq \max\{s^2\eth(\eta,\Xi\eta),0,0,\frac{s^2\eth(\eta,\Xi\eta)}{2s},\frac{s^2\eth(\eta,\Xi\eta)}{2s},0,\frac{s^6[\eth(\eta,\Xi\eta)]^2}{1+2s^2\eth(\eta,\Xi\eta)}\} = s^2\eth(\Lambda\eta,\eta),$$
$$\limsup_{n\to\infty} N(\xi_{2n},\eta) \leq \min\{s^2\eth(\eta,\Xi\eta),0,0,s^2\eth(\eta,\Xi\eta),s^2\eth(\eta,\Xi\eta)\} = 0.$$

Therefore

$$\frac{1}{s^2}\eth(\eta,\Xi\eta) \leq \liminf_{n\to\infty} M(\xi_{2n},\eta) \leq \limsup_{n\to\infty} M(\xi_{2n},\eta) \leq s^2\eth(\eta,\Xi\eta) \text{ and } \limsup_{n\to\infty} N(\xi_{2n},\eta) = 0. \tag{2.20}$$

Lemma 1.11 and the inequality (2.19) allow us to obtain the limit superior as $n\to\infty$.

$$\begin{aligned}
s^4\frac{1}{s^2}\eth(\eta,\Xi\eta) &\leq s^4\limsup_{n\to\infty}\eth(\Lambda\xi_{2n},\Xi\eta)\\
&= \limsup_{n\to\infty} s^4\eth(\Lambda\xi_{2n},\Xi\eta)\\
&\leq \limsup_{n\to\infty}[\beta(M(\xi_{2n},\eta))M(\xi_{2n},\eta) + LN(\xi_{2n},\eta)]\\
&= \limsup_{n\to\infty}\beta(M(\xi_{2n},\eta))\limsup_{n\to\infty} M(\xi_{2n},\eta) + L\limsup_{n\to\infty} N(\xi_{2n},\eta)
\end{aligned}$$

$\leq \limsup_{n\to\infty} \beta(M(\xi_{2n},\eta))s^2\eth(\eta,\Xi\eta).$

Therefore

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(\xi_{2n},\eta)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(\xi_{2n},\eta)) = \frac{1}{s}$.

It follows that $\lim_{n\to\infty} M(\xi_{2n},\eta) = 0$ since $\beta \in \mathfrak{F}$. Thus, based on the inequality (2.20), we can obtain

$\frac{1}{s^2}\eth(\eta,\Xi\eta) \leq \lim_{n\to\infty} M(\xi_{2n},\eta) = 0.$ implies that $\eth(\eta,\Xi\eta) \leq 0$.

Hence $\Xi\eta = \eta$.

Therefore $\Xi\eta = \Upsilon\eta = \eta$.

A common fixed point of Λ and Σ is hence η.

We now see that, in accordance with Proposition 2.1, η is a unique common fixed point of Λ, Ξ, Σ and Υ.

Similarly, the theorem's conclusion holds with the assumption (ii). □

Theorem 2.5. *Assume that the b-metric space* $(\S,\eth)$ *has a coefficient* $s \geq 1$. *Assume that the selfmaps* $\Lambda,\Xi,\Sigma,\Upsilon : \S \to \S$ *satisfy both Geraghty-Berinde-Suzuki type contractive maps and* (2.1) *for* $\S$. *Assume that one of the subspaces* $\Lambda(\S),\Xi(\S),\Sigma(\S)$ *and* $\Upsilon(\S)$ *is b-closed in* $\S$ *and that one of the pairs* (Λ,Σ) *and* (Ξ,Υ) *fulfills the b-(E.A)-property. Then, there is a point of coincidence in* $\S$ *between the pairs* (Λ,Σ) *and* (Ξ,Υ). *Furthermore,* Λ,Ξ,Σ *and* Υ *have a unique common fixed point in* $\S$ *if the pairs* (Λ,Σ) *and* (Ξ,Υ) *are weakly compatible.*

Proof. First, we suppose that the b-(E.A)-property is satisfied by the pair (Λ,Σ). Thus, in $\S$, there is a series $\{\xi_n\}$ such that

$$\lim_{n\to\infty} \Lambda\xi_n = \lim_{n\to\infty} \Sigma\xi_n = q \tag{2.21}$$

for some $q \in \xi$.

There is a sequence $\{\zeta_n\}$ in $\S$ such that $\Lambda\xi_n = T\zeta_n$ since $\Lambda(\S) \subseteq T(\S)$. Consequently,

$$\lim_{n\to\infty} T\zeta_n = q. \tag{2.22}$$

We now demonstrate that $\lim_{n\to\infty} \Xi\zeta_n = q$.

Since $\frac{1}{2s}\min\{\eth(\Sigma\xi_n,\Lambda\xi_n),\eth(\Upsilon\zeta_n,\Xi\zeta_n)\} \leq \max\{\eth(\Sigma\xi_n,\Upsilon\zeta_n),\eth(\Lambda\xi_n,\Xi\zeta_n)\}$.

The inequality (2.3) gives us

$$s^4 \eth(\Lambda\xi_n, \Xi\zeta_n) \leq \beta(M(\xi_n,\zeta_n))M(\xi_n,\zeta_n) + LN(\xi_n,\zeta_n) \tag{2.23}$$

where

$$M(\xi_n,\zeta_n) = \max\{\eth(\Sigma\xi_n,\Upsilon\zeta_n), \eth(\Sigma\xi_n,\Lambda\xi_n), \eth(\Upsilon\zeta_n,\Xi\zeta_n), \tfrac{\eth(\Sigma\xi_n,\Xi\zeta_n)}{2s}, \tfrac{\eth(\Upsilon\zeta_n,\Lambda\xi_n)}{2s}, \tfrac{\eth(\Sigma\xi_n,\Lambda\xi_n)\eth(\Upsilon\zeta_n,\Xi\zeta_n)}{1+\eth(\Sigma\xi_n,\Upsilon\zeta_n)+\eth(\Lambda\xi_n,\Xi\zeta_n)}, \tfrac{\eth(\Sigma\xi_n,\Xi\zeta_n)\eth(\Upsilon\zeta_n,\Lambda\xi_n)}{1+s^4[\eth(\Sigma\xi_n,\Upsilon\zeta_n)+\eth(\Lambda\xi_n,\Xi\zeta_n)]}\},$$

$$N(\xi_n,\zeta_n) = \min\{\eth(\Sigma\xi_n,\Upsilon\zeta_n), \eth(\Sigma\xi_n,\Lambda\xi_n), \eth(\Upsilon\zeta_n,\Xi\zeta_n), \eth(\Sigma\xi_n,\Xi\zeta_n), \eth(\Upsilon\zeta_n,\Lambda\xi_n)\}.$$

Limit superior on $M(\xi_n,\zeta_n), N(\xi_n,\zeta_n)$ is taken as $n\to\infty$. Using (2.21) and (2.22), we obtain

$$\begin{cases} \limsup_{n\to\infty} M(\xi_n,\zeta_n) = \max\{0,0,\limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n), \frac{\limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n)}{2s},0,0,0\} \\ \qquad = \limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n) \text{ and} \\ \limsup_{n\to\infty} N(\xi_n,\zeta_n) = 0 \end{cases} \tag{2.24}$$

Using (2.24) and considering limit superior as $n\to\infty$ in (2.23), we obtain

$$\begin{aligned} s^4 \limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n) &= \limsup_{n\to\infty}[\beta(M(\xi_n,\zeta_n))M(\xi_n,\zeta_n) + LN(\xi_n,\zeta_n)] \\ &= \limsup_{n\to\infty}\beta(M(\xi_n,\zeta_n))\limsup_{n\to\infty}M(\xi_n,\zeta_n) \\ &\quad + L\limsup_{n\to\infty}N(\xi_n,\zeta_n) \\ &= \limsup_{n\to\infty}\beta(M(\xi_n,\zeta_n))\limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n). \end{aligned}$$

Therefore

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty}\beta(M(\xi_n,\zeta_n)) \leq \frac{1}{s^5} \leq \frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(\xi_n,\zeta_n)) = \frac{1}{s}$.
Given that $\beta\in\mathfrak{F}$, we have $\lim_{n\to\infty}M(\xi_n,\zeta_n) = 0$. i.e., $\limsup_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n) = 0$.
Therefore

$$\lim_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n) = 0. \tag{2.25}$$

We have

$$\eth(q,\Xi\zeta_n) \leq s[\eth(q,\Lambda\xi_n) + \eth(\Lambda\xi_n,\Xi\zeta_n)]. \tag{2.26}$$

Using (2.21) and (2.25) and assuming limits as $n\to\infty$ in (2.26), we obtain

$\lim_{n\to\infty}\eth(q,\Xi\zeta_n) \leq s[\lim_{n\to\infty}\eth(q,\Lambda\xi_n) + \lim_{n\to\infty}\eth(\Lambda\xi_n,\Xi\zeta_n)] = 0.$
Therefore $\lim_{n\to\infty}\eth(q,\Xi\zeta_n) = 0$.

Case (i): Suppose that § has a b-closed subset, which is $\Upsilon(X)$. We can select $r \in$ § in this instance, $q \in \Upsilon(\S)$, so that $\Upsilon r = q$. Now, we establish that $\Xi r = q$. Assume $\eth(\Xi r, q) > 0$.

Since $\frac{1}{2s}\min\{\eth(\Sigma\xi_n, \Lambda\xi_n), \eth(\Upsilon r, \Xi r)\} \leq \max\{\eth(\Sigma\xi_n, \Upsilon r), \eth(\Lambda\xi_n, \Xi r)\}$

From the inequality (2.3), we have

$$s^4\eth(\Lambda\xi_n, \Xi r) \leq \beta(M(\xi_n, r))M(\xi_n, r) + LN(\xi_n, r) \tag{2.27}$$

where

$$M(\xi_n, r) = \max\{\eth(\Sigma\xi_n, \Upsilon r), \eth(\Sigma\xi_n, \Lambda\xi_n), \eth(\Upsilon r, \Xi r), \frac{\eth(\Sigma\xi_n, \Xi r)}{2s}, \frac{\eth(\Upsilon r, \Lambda\xi_n)}{2s}, \frac{\eth(\Sigma\xi_n, \Lambda\xi_n)\eth(\Upsilon r, \Xi r)}{1+\eth(\Sigma\xi_n, \Upsilon r)+\eth(\Lambda\xi_n, \Xi r)}, \frac{\eth(\Sigma\xi_n, \Xi r)\eth(\Upsilon r, \Lambda\xi_n)}{1+s^4[\eth(\Sigma\xi_n, \Upsilon r)+\eth(\Lambda\xi_n, \Xi r)]}\},$$

$$N(\xi_n, r) = \min\{\eth(\Sigma\xi_n, \Upsilon r), \eth(\Sigma\xi_n, \Lambda\xi_n), \eth(\Upsilon r, \Xi r), \eth(\Sigma\xi_n, \Xi r), \eth(\Upsilon r, \Lambda\xi_n)\}.$$

Using $M(\xi_n, r), N(\xi_n, r)$ and $n \to \infty$ as the limit superior, along with (2.21), (2.22), and Lemma 1.11, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} M(\xi_n, r) \leq \max\{0, 0, \eth(q, \Xi r), \frac{\eth(q, \Xi r)}{2}, 0, 0, 0\} = \eth(q, \Xi r) \text{ and} \\ &\limsup_{n\to\infty} N(\xi_n, r) = 0. \end{aligned} \tag{2.28}$$

We have

$$\begin{aligned} \eth(\Xi r, q) &\leq s[\eth(\Xi r, \Sigma\xi_n) + \eth(\Sigma\xi_n, q)] \\ &= 2s^2[\frac{\eth(\Xi r, \Sigma\xi_n)}{2s}] + s\eth(\Sigma\xi_n, q) \leq 2s^2 M(\xi_n, r) + s\eth(\Sigma\xi_n, q). \end{aligned}$$

When we consider limit inferior as $n \to \infty$, we obtain

Therefore $\frac{1}{2s^2}\eth(\Xi r, q) \leq \liminf_{n\to\infty} M(\xi_n, r)$.

By utilizing (2.28) and Lemma 1.11 and taking limit superior as $n \to \infty$ in (2.27), we have

$$\begin{aligned} s^4(\tfrac{1}{s}\eth(q, \Xi r)) &\leq s^4 \limsup_{n\to\infty} \eth(\Lambda\xi_n, \Xi r) \\ &= \limsup_{n\to\infty} [\beta(M(\xi_n, r))M(\xi_n, r) + LN(\xi_n, r)] \\ &= \limsup_{n\to\infty} \beta(M(\xi_n, r)) \limsup_{n\to\infty} M(\xi_n, r) + L \limsup_{n\to\infty} N(\xi_n, r) \\ &\leq \limsup_{n\to\infty} \beta(M(\xi_n, r))\eth(q, \Xi r). \end{aligned}$$

Therefore

$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(\xi_n, r)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(\xi_n, r)) = \frac{1}{s}$.

$\lim_{n\to\infty} M(\xi_n, r) = 0$ is the result of $\beta \in \mathfrak{F}$. Consequently, $\frac{1}{2s^2}\eth(\Xi r, q) \leq \lim_{n\to\infty} M(\xi_n, r) = 0$.

Consequently, $\Xi r = q$.

As a result, $\Xi r = \Upsilon r = q$, meaning that q is the Ξ and Υ coincidence point.

There exists $\eta \in \S$ such that $\Sigma\eta = q = \Xi r$ since $\Xi(\S) \subseteq \Sigma(\S)$, and we have $q \in \Sigma(\S)$.

We now demonstrate that $\Lambda\eta = q$.

Assume that $\Lambda\eta \neq q$.

Since

$\frac{1}{2s}\min\{\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon r,\Xi r)\} \leq \max\{\eth(\Sigma\eta,\Upsilon r),\eth(\Lambda\eta,\Xi r)\}$

As a result of the inequality (2.3), we get

$$s^4\eth(\Lambda\eta,q) = s^4\eth(\Lambda\eta,\Xi r) \leq \beta(M(\eta,r))M(\eta,r) + LN(\eta,r) \tag{2.29}$$

where

$$\begin{aligned} M(\eta,r) &= \max\{\eth(\Sigma\eta,\Upsilon r),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon r,\Xi r),\tfrac{\eth(\Sigma\eta,\Xi r)}{2s},\tfrac{\eth(\Upsilon r,\Lambda\eta)}{2s}, \\ &\quad \tfrac{\eth(\Sigma\eta,\Lambda\eta)\eth(\Upsilon r,\Xi r)}{1+\eth(\Sigma\eta,\Upsilon r)+\eth(\Lambda\eta,\Xi r)},\tfrac{\eth(\Sigma\eta,\Xi r)\eth(\Upsilon r,\Lambda\eta)}{1+s^4[\eth(\Sigma\eta,\Upsilon r)+\eth(\Lambda\eta,\Xi r)]}\} \\ &= \max\{0,\eth(q,\Lambda\eta),0,0,\tfrac{\eth(q,\Lambda\eta)}{2s},0,0\} = \eth(q,\Lambda\eta), \end{aligned}$$

$N(\eta,r) = \min\{\eth(\Sigma\eta,\Upsilon r),\eth(\Sigma\eta,\Lambda\eta),\eth(\Upsilon r,\Xi r),\eth(\Sigma\eta,\Xi r),\eth(\Upsilon r,\Lambda\eta)\} = 0.$

From the inequality (2.29), we have

$s^4\eth(\Lambda\eta,q) \leq \beta(\eth(\Lambda\eta,q)\eth(\Lambda\eta,q)) < \eth(\Lambda\eta,q),$

a contradiction.

So, η is a coincidence point of Λ and Σ since $\Lambda\eta = \Sigma\eta = q$.

We have $\Lambda q = \Sigma q$ and $\Xi q = \Upsilon q$ because the pairs (Λ,Σ) and (Ξ,Υ) are weakly compatible.

As a result, q also serves as a coincidence point for the (Ξ,Υ) and (Λ,Σ) pairs.

Specifically, we demonstrate that q is a common fixed point of Λ,Ξ,Σ and Υ.

Suppose $\Lambda q \neq q$.

Since $\frac{1}{2s}\min\{\eth(\Sigma q,\Lambda q),\eth(\Upsilon r,\Xi r)\} \leq \max\{\eth(\Sigma q,\Upsilon r),\eth(\Lambda q,\Xi r)\}$,

according to the inequality (2.3), we have

$$s^4\eth(\Lambda q,q) = s^4\eth(\Lambda q,\Xi r) \leq \beta(M(q,r))M(q,r) + LN(q,r) \tag{2.30}$$

where

$$\begin{aligned} M(q,r) &= \max\{\eth(\Sigma q,\Upsilon r),\eth(\Sigma q,\Lambda q),\eth(\Upsilon r,\Xi r),\tfrac{\eth(\Sigma q,\Xi r)}{2s},\tfrac{\eth(\Upsilon r,\Lambda q)}{2s}, \\ &\quad \tfrac{\eth(\Sigma q,\Lambda q)\eth(\Upsilon r,\Xi r)}{1+\eth(\Sigma q,\Upsilon r)+\eth(\Lambda q,\Xi r)},\tfrac{\eth(\Sigma q,\Xi r)\eth(\Upsilon r,\Lambda q)}{1+s^4[\eth(\Sigma q,\Upsilon r)+\eth(\Lambda q,\Xi r)]}\} \\ &= \max\{\eth(\Lambda q,q),0,0,\tfrac{\eth(\Lambda q,q)}{2s},\tfrac{\eth(\Lambda q,q)}{2s},0,0\} \\ &= \eth(\Lambda q,q), \end{aligned}$$

$N(q,r) = \min\{\eth(\Sigma q, \Upsilon r), \eth(\Sigma q, \Lambda q), \eth(\Upsilon r, \Xi r), \eth(\Sigma q, \Xi r), \eth(\Upsilon r, \Lambda q)\} = 0.$

We now have a contradiction:

$s^4\eth(\Lambda q, q) \leq \beta(\eth(\Lambda q, q)\eth(\Lambda q, q)) < \eth(\Lambda q, q)$, derived from the inequality (2.30).

To ensure that q is a common fixed point of Λ and Σ, $\Lambda q = \Sigma q = q$.

q is a unique common fixed point of Λ, Ξ, Σ and Υ, according to Proposition 2.1.

Case (ii): Assume that $\Lambda(\S)$ is b-closed.Since $\Lambda(\S) \subseteq \Upsilon(\S)$ in this instance, we select $r \in \S$ such that $q = \Upsilon r$. The evidence is presented as it was in Case (i).

Case (iii): Assume that $\sum(\S)$ is b-closed. We draw a conclusion by following the reasoning in a manner similar to Case (i).

Case (iv): It is assumed that $\Xi(\S)$ is b-closed. We receive the conclusion, just like in Case (ii).

The argument for the situation when (Ξ, Υ) satisfies the b-(E.A)-property is the same as the one used when (Λ, Σ) satisfies the b-(E.A)-property. □

3. Corollaries and Examples

We derive a few corollaries from our primary findings in this part, along with supporting data.

If we select $\Lambda = \Xi = f$ and $\Sigma = \Upsilon = g$, we derive Corollary 3.1 and Corollary 3.2, respectively, from Theorem 2.4 and Theorem 2.5.

Corollary 3.1. *Let f and g be selfmaps of $\S$, and let $(\S, \eth)$ be a b-metric space. Let us assume that $\beta \in \mathfrak{F}$ exists and that*

$$\frac{1}{2s}\min\{\eth(f\xi, g\xi), \eth(f\zeta, g\zeta)\} \leq \max\{\eth(g\xi, g\zeta), \eth(f\xi, f\zeta)\} \implies s^4\eth(f\xi, f\zeta) \leq \beta(M(\xi,\zeta))M(\xi,\zeta) + LN(\xi,\zeta) \tag{3.1}$$

where

$M(\xi,\zeta) = \max\{\eth(g\xi, g\zeta), \eth(g\xi, f\xi), \eth(g\zeta, f\zeta), \frac{\eth(g\xi, f\zeta)}{2s}, \frac{\eth(g\zeta, f\xi)}{2s}, \frac{\eth(g\xi, f\xi)\eth(g\zeta, f\zeta)}{1+\eth(g\xi, g\zeta)+\eth(f\xi, f\zeta)}, \frac{\eth(g\xi, f\zeta)\eth(g\zeta, f\xi)}{1+s^4[\eth(g\xi, g\zeta)+\eth(f\xi, f\zeta)]}\},$

$N(\xi,\zeta) = \min\{\eth(g\xi, g\zeta), \eth(g\xi, f\xi), \eth(g\zeta, f\zeta), \eth(g\xi, f\zeta), \eth(g\zeta, f\xi)\}$

for all $\xi, \zeta \in \S$. When $f(\S) \subseteq g(\S)$, f or g is b-continuous, and f and g has a single common fixed point in $\S$, the pair (f, g) is compatible.

Corollary 3.2. *Assume that the b-metric space $(\S,\eth)$ has a coefficient $s \geq 1$. Let $f,g:\S\to\S$ be selfmaps of $\S$ that meet the inequality* (3.1) *and $f(\S)\subseteq g(\S)$. Assume that one of the subspaces $f(\S)$ and $g(\S)$ is b-closed in $\S$, and that the pair (f,g) fulfills the b-(E.A)-property. Then, in $\S$, there is a point of coincidence for the pairings (f,g). Furthermore, in $\S$, f and g have a single shared fixed point if the pair (f,g) is weakly compatible.*

The following is an example in support of Theorem 2.4.

Example 3.3. Let $\S=[0,1]$ and let $\eth:\S\times\S\to\mathbb{R}^+$ defined by

$$\eth(\xi,\zeta)=\begin{cases} 0 & \text{if } \xi=\zeta, \\ (\xi+\zeta)^2 & \text{if } \xi\neq\zeta. \end{cases}$$

Then clearly $(\S,\eth)$ is a complete b-metric space with coefficient $s=2$.

We define $\Lambda,\Xi,\Sigma,\Upsilon:\S\to\S$ by

$$\Lambda(\xi)=\begin{cases} \frac{1}{2} & \text{if } \xi\in[0,\frac{1}{2}] \\ 1 & \text{if } \xi\in(\frac{1}{2},1], \end{cases},\ \Xi(\xi)=\begin{cases} \frac{1}{2} & \text{if } \xi\in[0,\frac{1}{2}] \\ \frac{9}{10} & \text{if } \xi\in(\frac{1}{2},1], \end{cases}$$

$$\Sigma(\xi)=\begin{cases} \frac{1}{4}+\frac{\xi}{2} & \text{if } \xi\in[0,\frac{1}{2}] \\ \xi & \text{if } \xi\in(\frac{1}{2},1], \end{cases} \text{ and } \Upsilon(\xi)=\begin{cases} 1-\xi & \text{if } \xi\in[0,\frac{1}{2}] \\ \frac{\xi}{2} & \text{if } \S\in(\frac{1}{2},1]. \end{cases}$$

Clearly $\Lambda(\S)\subseteq\Upsilon(\S)$ and $\Xi(\S)\subseteq\Sigma(\S)$.

Here Λ is b-continuous.

We choose a sequence $\{\xi_n\}$ with $\{\xi_n\}=\frac{1}{2}-\frac{1}{3n}, n\geq 1$, we have

$\Lambda\Sigma\xi_n=\Lambda(\frac{1}{4}+\frac{\frac{1}{2}-\frac{1}{3n}}{2})=\frac{1}{2}$ and $\Sigma\Lambda\xi_n=\Sigma\frac{1}{2}=\frac{1}{2}$.

Consequently, $\lim_{n\to\infty}\eth(\Lambda\Sigma\xi_n,\Sigma\Lambda\xi_n)=0$, indicating the compatibility of the pair (Λ,Σ), whereas the pair (Ξ,Υ) is evidently weakly compatible.

$\beta(t)=\frac{1}{2}e^{-t}$ is how we define $\beta:[0,\infty)\to[0,\frac{1}{s})$. Thus, $\beta\in\mathfrak{F}$ is what we have.

Case (i): $\xi,\zeta\in[0,\frac{1}{2}]$.

$\eth(\Lambda\xi,\Xi\zeta)=1, \eth(\Sigma\xi,\Upsilon\zeta)=(\frac{5}{4}+\frac{\xi}{2}-\zeta)^2, \eth(\Upsilon\zeta,\Xi\zeta)=(\frac{3}{2}-\zeta)^2,$

$\eth(\Sigma\xi,\Lambda\xi)=(\frac{3}{2}+\frac{\xi}{2})^2, \eth(\Lambda\xi,\Upsilon\zeta)=(\frac{3}{2}-\zeta)^2, \eth(\Sigma\xi,\Xi\zeta)=(\frac{3}{2}+\frac{\xi}{2})^2,$

$$\begin{aligned} M(\xi,\zeta)&=\max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\frac{\eth(\Sigma\xi,\Xi\zeta)}{2s},\frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s},\\ &\qquad \frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)},\frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}\\ &=\max\{(\frac{5}{4}+\frac{\xi}{2}-\zeta)^2,(\frac{3}{2}+\frac{\xi}{2})^2,(\frac{3}{2}-\zeta)^2),\frac{(\frac{3}{2}+\frac{\xi}{2})^2}{4},\frac{(\frac{3}{2}-\zeta)^2}{4},\\ &\qquad \frac{(\frac{3}{2}+\frac{\xi}{2})^2(\frac{3}{2}-\zeta)^2}{2+(\frac{5}{4}+\frac{\xi}{2}-\zeta)^2},\frac{(\frac{3}{2}+\frac{\xi}{2})^2(\frac{3}{2}-\zeta)^2}{1+16[1+(\frac{5}{4}+\frac{\xi}{2}-\zeta)^2]}\} \end{aligned}$$

$$= (\tfrac{3}{2}+\tfrac{\xi}{2})^2 \text{and}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\eth(\Sigma\xi,\Xi\zeta),\eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(\tfrac{5}{4}+\tfrac{\xi}{2}-\zeta)^2,(\tfrac{3}{2}+\tfrac{\xi}{2})^2,(\tfrac{3}{2}-\zeta)^2,(\tfrac{3}{2}+\tfrac{\xi}{2})^2,(\tfrac{3}{2}-\zeta)^2\} = (\tfrac{3}{2}-\zeta)^2.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{\eth(\Sigma\xi,A\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} &= \tfrac{1}{4}\min\{(\tfrac{3}{2}+\tfrac{\xi}{2})^2,(\tfrac{3}{2}-\zeta)^2\}\\ &= (\tfrac{1}{4})(\tfrac{3}{2}-\zeta)^2\\ &\leq \max\{(\tfrac{5}{4}+\tfrac{\xi}{2}-\zeta)^2,1\}\\ &= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}.\end{aligned}$$

Now we consider

$$\begin{aligned}s^4\eth(\Lambda\xi,\Xi\zeta) = 16 &\leq \tfrac{1}{2}e^{-(\frac{3}{2}+\frac{\xi}{2})^2}(\tfrac{3}{2}+\tfrac{\xi}{2})^2+36(\tfrac{3}{2}-\zeta)^2\\ &= \beta(M(\xi,\zeta))M(\xi,\zeta)+LN(\xi,\zeta).\end{aligned}$$

Case (ii): $\xi\in[0,\frac{1}{2}],\zeta\in(\frac{1}{2},1]$.

$\eth(\Lambda\xi,\Xi\zeta) = (\frac{14}{10})^2,\eth(\Sigma\xi,\Upsilon\zeta) = (\frac{1}{4}+\frac{\xi}{2}+\frac{\zeta}{2})^2,\eth(\Upsilon\zeta,\Xi\zeta) = (\frac{9}{10}+\frac{\zeta}{2})^2,$

$\eth(\Sigma\xi,\Lambda\xi) = (\frac{3}{4}+\frac{\xi}{2})^2,\eth(\Lambda\xi,\Upsilon\zeta) = (\frac{1}{2}+\frac{\zeta}{2})^2,\eth(\Sigma\xi,\Xi\zeta) = (\frac{19}{40}+\frac{\xi}{2})^2,$

$$\begin{aligned}M(\xi,\zeta) &= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\tfrac{\eth(\Sigma\xi,\Xi\zeta)}{2s},\tfrac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s},\\ &\qquad \tfrac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)},\tfrac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}\\ &= \max\{(\tfrac{1}{4}+\tfrac{\xi}{2}+\tfrac{\zeta}{2})^2,(\tfrac{3}{4}+\tfrac{\xi}{2})^2,(\tfrac{9}{10}+\tfrac{\zeta}{2})^2,(\tfrac{1}{4})((\tfrac{19}{40}+\tfrac{\xi}{2})^2),(\tfrac{1}{4})(\tfrac{1}{2}+\tfrac{\zeta}{2})^2,\\ &\qquad \tfrac{(\frac{3}{4}+\frac{\xi}{2})^2(\frac{9}{10}+\frac{\zeta}{2})^2}{1+(\frac{1}{4}+\frac{\xi}{2}+\frac{\zeta}{2})^2+(\frac{14}{10})^2},\tfrac{(\frac{19}{40}+\frac{\xi}{2})^2(\frac{1}{2}+\frac{\zeta}{2})^2}{1+16(\frac{1}{4}+\frac{\xi}{2}+\frac{\zeta}{2})^2+(\frac{14}{10})^2}\}\\ &= (\tfrac{9}{10}+\tfrac{\zeta}{2})^2 \text{ and}\end{aligned}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\eth(\Sigma\xi,\Xi\zeta),\eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(\tfrac{1}{4}+\tfrac{\xi}{2}+\tfrac{\zeta}{2})^2,(\tfrac{3}{4}+\tfrac{\xi}{2})^2,(\tfrac{9}{10}+\tfrac{\zeta}{2})^2,(\tfrac{19}{40}+\tfrac{\xi}{2})^2,(\tfrac{1}{2}+\tfrac{\zeta}{2})^2\} = (\tfrac{19}{40}+\tfrac{\xi}{2})^2.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} &= \tfrac{1}{4}\min\{(\tfrac{3}{4}+\tfrac{\xi}{2})^2,(\tfrac{9}{10}+\tfrac{\zeta}{2})^2\}\\ &= (\tfrac{1}{4})(\tfrac{3}{4}+\tfrac{\xi}{2})^2\\ &\leq \max\{(\tfrac{1}{4}+\tfrac{\xi}{2}+\tfrac{\zeta}{2})^2,(\tfrac{14}{10})^2\}\\ &= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}.\end{aligned}$$

Now, we consider

$$\begin{aligned}s^4\eth(\Lambda\xi,\Xi\zeta) = (16)(\tfrac{14}{10})^2 &\leq \tfrac{1}{2}e^{-(\frac{9}{10}+\frac{\zeta}{2})^2}(\tfrac{9}{10}+\tfrac{\zeta}{2})^2+36(\tfrac{19}{40}+\tfrac{\xi}{2})^2\\ &= \beta(M(\xi,\zeta))M(\xi,\zeta)+LN(\xi,\zeta).\end{aligned}$$

Case (iii): $\zeta \in [0, \frac{1}{2}], \xi \in (\frac{1}{2}, 1]$.

$\eth(\Lambda\xi, \Xi\zeta) = (\frac{3}{2})^2, \eth(\Sigma\xi, \Upsilon\zeta) = (1+\xi-\zeta)^2, \eth(\Upsilon\zeta, \Xi\zeta) = (\frac{3}{2}-\zeta)^2,$

$\eth(\Sigma\xi, \Lambda\xi) = (1+\xi)^2, \eth(\Lambda\xi, \Upsilon\zeta) = (2-\zeta)^2, \eth(\Sigma\xi, \Xi\zeta) = (\frac{1}{2}+\xi)^2,$

$$M(\xi,\zeta) = \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \frac{\eth(\Sigma\xi,\Xi\zeta)}{2s}, \frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s}, \frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)}, \frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}$$

$$= \max\{(1+\xi-\zeta)^2, (1+\xi)^2, (\frac{3}{2}-\zeta)^2, \frac{(\frac{1}{2}+\xi)^2}{4}, \frac{(2-\zeta)^2}{4}, \frac{(1+\xi)^2(\frac{3}{2}-\zeta)^2}{1+(1+\xi-\zeta)^2+(\frac{3}{2})^2}, \frac{(\frac{1}{2}+\xi)^2(2-\zeta)^2}{1+16[(1+\xi-\zeta)^2+(\frac{3}{2})^2]}\} = (1+\xi)^2 \text{ and}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \eth(\Sigma\xi,\Xi\zeta), \eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(1+\xi-\zeta)^2, (1+\xi)^2, (\frac{3}{2}-\zeta)^2, (\frac{1}{2}+\xi)^2, (2-\zeta)^2\} = (\frac{1}{2}+\xi)^2.$$

Since

$$\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta)\} = \frac{1}{4}\min\{(1+\xi)^2, (\frac{3}{2}-\zeta)^2\}$$
$$= (\frac{1}{4})(\frac{3}{2}-\zeta)^2$$
$$\leq \max\{(1+\xi-\zeta)^2, (\frac{3}{2})^2\}$$
$$= \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Lambda\xi,\Xi\zeta)\}.$$

Now, we consider

$$s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\frac{3}{2})^2 \leq \frac{1}{2}e^{-(1+\xi)^2}(1+\xi)^2 + 36(\frac{1}{2}+\xi)^2$$
$$= \beta(M(\xi,\zeta))M(\xi,\zeta) + LN(\xi,\zeta).$$

Case (iv): $\xi, \zeta \in (\frac{1}{2}, 1]$.

$\eth(\Lambda\xi, \Xi\zeta) = (\frac{19}{10})^2, \eth(\Sigma\xi, \Upsilon\zeta) = (\xi+\frac{\zeta}{2})^2, \eth(\Upsilon\zeta, \Xi\zeta) = (\frac{9}{10}+\frac{\zeta}{2})^2,$

$\eth(\Sigma\xi, \Lambda\xi) = (1+\xi)^2, \eth(\Lambda\xi, \Upsilon\zeta) = (1+\frac{\zeta}{2})^2, \eth(\Sigma\xi, \Xi\zeta) = (\frac{9}{10}+\xi)^2,$

$$M(\xi,\zeta) = \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \frac{\eth(\Sigma\xi,\Xi\zeta)}{2s}, \frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s}, \frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)}, \frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}$$

$$= \max\{(\xi+\frac{\zeta}{2})^2, (1+\xi)^2, (\frac{9}{10}+\frac{\zeta}{2})^2, \frac{(\frac{9}{10}+\xi)^2}{4}, \frac{(1+\frac{\zeta}{2})^2}{4}, \frac{(1+\xi)^2(\frac{9}{10}+\frac{\zeta}{2})^2}{1+(\xi+\frac{\zeta}{2})^2+(\frac{19}{10})^2}, \frac{(\frac{9}{10}+\xi)^2(1+\frac{\zeta}{2})^2}{1+16[(\xi+\frac{\zeta}{2})^2+(\frac{19}{10})^2]}\}$$
$$= (1+\xi)^2 \text{ and}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \eth(\Sigma\xi,\Xi\zeta), \eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(\xi+\frac{\zeta}{2})^2, (1+\xi)^2, (\frac{9}{10}+\frac{\zeta}{2})^2, (\frac{9}{10}+\xi)^2, (1+\frac{\zeta}{2})^2\}$$
$$= (\frac{9}{10}+\frac{\zeta}{2})^2.$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} &= \frac{1}{4}\min\{(1+\xi)^2,(\frac{9}{10}+\frac{\zeta}{2})^2\}\\ &= (\frac{1}{4})(\frac{9}{10}+\frac{\zeta}{2})^2\\ &\le \max\{(\xi+\frac{\zeta}{2})^2,(\frac{19}{10})^2\}\\ &= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}.\end{aligned}$$

Now, we consider

$$\begin{aligned}s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\frac{19}{10})^2 &\le \frac{1}{2}e^{-(1+\xi)^2}(1+\xi)^2+36(\frac{9}{10}+\frac{\zeta}{2})^2\\ &= \beta(M(\xi,\zeta))M(\xi,\zeta)+LN(\xi,\zeta).\end{aligned}$$

Λ,Ξ,Σ and Υ are contraction maps of the Geraghty-Berinde-Suzuki type from each of the four situations mentioned above. Consequently, all of the assumptions of Theorem 2.4 are satisfied by Λ,Ξ,Σ and Υ, and $\frac{1}{2}$ is their unique common fixed point.

Here, we note that the inequality (2.3) is not true if $L = 0$.

For, using $\zeta = 1$ and $\xi = 0$, we have

$$\begin{aligned}\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} &= \frac{1}{4}\min\{(\frac{3}{4})^2,(\frac{14}{10})^2\}\\ &= (\frac{1}{4})(\frac{3}{4})^2\\ &\le \max\{(\frac{3}{4})^2,(\frac{14}{10})^2\}\\ &= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}\end{aligned}$$

implies that

$$s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\frac{14}{10})^2 \not\le \beta((\frac{14}{10})^2)(\frac{14}{10})^2 = \beta(M(\xi,\zeta))M(\xi,\zeta) \text{ for any } \beta\in\mathfrak{F}.$$

An example supporting Theorem 2.5 is as follows.

Example 3.4. Let $\S = [0,1]$ and let $\eth : \S\times\S\to\mathbb{R}^+$ defined by

$$\eth(\xi,\zeta) = \begin{cases} 0 & \text{if } \xi = \zeta,\\ (\xi+\zeta)^2 & \text{if } \xi\ne\zeta.\end{cases}$$

We define $\Lambda,\Xi,\Sigma,\Upsilon : \S\to\S$ by

$$\Lambda(\S) = \frac{2}{3} \text{ if } \xi\in[0,1],\ \Xi(\xi) = \begin{cases}\frac{1}{2} & \text{if } \xi\in[0,\frac{2}{3})\\ 1-\frac{\xi}{2} & \text{if } \xi\in[\frac{2}{3},1],\end{cases}$$

$$\Sigma(\S) = \begin{cases}\xi & \text{if } \xi\in[0,\frac{2}{3})\\ \frac{4}{3}-\xi & \text{if } \xi\in[\frac{2}{3},1],\end{cases} \text{ and } \Upsilon(\xi) = \begin{cases}\frac{1}{4} & \text{if } \xi\in[0,\frac{2}{3})\\ \frac{4}{3}-\xi & \text{if } \xi\in[\frac{2}{3},1].\end{cases}$$

Clearly $\Lambda(\S)\subseteq\Upsilon(\S)$ and $\Xi(\S)\subseteq\Sigma(\S)$. $\Lambda(\S) = \{\frac{2}{3}\}$ is b-closed.

We choose a sequence $\{\xi_n\}$ with $\{\xi_n\} = \frac{2}{3}+\frac{1}{2n}, n\ge 2$ with

$\lim_{n\to\infty} \Lambda\xi_n = \lim_{n\to\infty} \Sigma\xi_n = \frac{2}{3}$, hence the pair (Λ, Σ) satisfies the b-(E.A)-property.

Clearly the pairs (Λ, Σ) and (Ξ, Υ) are weakly compatible.

$\beta(t) = \frac{1}{2}e^{-t}$ is how we define $\beta : [0, \infty) \to [0, \frac{1}{s})$. Thus, $\beta \in \mathfrak{F}$ is what we have.

Case (i): $\xi, \zeta \in [0, \frac{2}{3})$.

$\eth(\Lambda\xi, \Xi\zeta) = (\frac{7}{6})^2, \eth(\Sigma\xi, \Upsilon\zeta) = (\xi + \frac{1}{4})^2, \eth(\Upsilon\zeta, \Xi\zeta) = (\frac{3}{4})^2,$

$\eth(\Sigma\xi, \Lambda\xi) = (\xi + \frac{2}{3})^2, \eth(\Lambda\xi, \Upsilon\zeta) = (\frac{11}{12})^2, \eth(\Sigma\xi, \Xi\zeta) = (\xi + \frac{1}{2})^2,$

$$\begin{aligned} M(\xi,\zeta) = \max\{&\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \frac{\eth(\Sigma\xi,\Xi\zeta)}{2s}, \frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s}, \\ &\frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)}, \frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\} \\ = \max\{&(\xi+\frac{1}{4})^2, (\xi+\frac{2}{3})^2, (\frac{3}{4})^2, \frac{(\xi+\frac{1}{2})^2}{4}, \frac{(\frac{11}{12})^2}{4}, \frac{(\xi+\frac{2}{3})^2(\frac{3}{4})^2}{1+(\xi+\frac{1}{4})^2+(\frac{7}{6})^2}, \\ &\frac{(\xi+\frac{1}{2})^2(\frac{11}{12})^2}{1+16[(\xi+\frac{1}{4})^2+(\frac{7}{6})^2]}\} \\ = (\xi&+\frac{2}{3})^2, \end{aligned}$$

$$\begin{aligned} N(\xi,\zeta) &= \min\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \eth(\Sigma\xi,\Xi\zeta), \eth(\Upsilon\zeta,\Lambda\xi)\} \\ &= \min\{(\xi+\frac{1}{4})^2, (\xi+\frac{2}{3})^2, (\frac{3}{4})^2, (\xi+\frac{1}{2})^2, (\frac{11}{12})^2\} \\ &= (\xi+\frac{1}{4})^2, \text{ if } \xi < \frac{1}{2} \text{ and} \end{aligned}$$

$N(\xi, \zeta) = (\frac{3}{4})^2 \ \xi \geq \frac{1}{2}$.

Since

$$\begin{aligned} \frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta)\} &= \frac{1}{4}\min\{(\xi+\frac{2}{3})^2, (\frac{3}{4})^2\} \\ &= (\frac{1}{4})(\frac{3}{4})^2 \\ &\leq \max\{(\xi+\frac{\zeta}{2})^2, (\frac{19}{10})^2\} \\ &= \max\{\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Lambda\xi,\Xi\zeta)\}. \end{aligned}$$

Now, we consider

$$\begin{aligned} s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\frac{7}{6})^2 &\leq \frac{1}{2}e^{-(\xi+\frac{2}{3})^2}(\xi+\frac{2}{3})^2 + 350(\xi+\frac{1}{4})^2 \\ &= \beta(M(\xi,\zeta))M(\xi,\zeta) + LN(\xi,\zeta). \end{aligned}$$

Case (ii): $\xi, \zeta \in [\frac{2}{3}, 1]$.

$\eth(\Lambda\xi, \Xi\zeta) = (\frac{5}{3} - \frac{\zeta}{2})^2, \eth(\Sigma\xi, \Upsilon\zeta) = (1 - \xi - \zeta)^2, \eth(\Upsilon\zeta, \Xi\zeta) = (\frac{7}{3} - \frac{3\zeta}{2})^2,$

$\eth(\Sigma\xi, \Lambda\xi) = (2 - \xi)^2, \eth(\Lambda\xi, \Upsilon\zeta) = (2 - \zeta)^2, \eth(\Sigma\xi, \Xi\zeta) = (\frac{7}{3} - \xi - \frac{\zeta}{2})^2,$

$$\begin{aligned} M(\xi,\zeta) = \max\{&\eth(\Sigma\xi,\Upsilon\zeta), \eth(\Sigma\xi,\Lambda\xi), \eth(\Upsilon\zeta,\Xi\zeta), \frac{\eth(\Sigma\xi,\Xi\zeta)}{2s}, \frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s}, \\ &\frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)}, \frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\} \\ = \max\{&(1-\xi-\zeta)^2, (2-\xi)^2, (\frac{7}{3}-\frac{3\zeta}{2})^2, \frac{(\frac{7}{3}-\xi-\frac{\zeta}{2})^2}{4}, \frac{(2-\zeta)^2}{4}, \end{aligned}$$

$$\frac{(2-\xi)^2(\frac{7}{3}-\frac{3\zeta}{2})^2}{1+(1-\xi-\zeta)^2+(\frac{5}{3}-\frac{\zeta}{2})^2}, \frac{(\frac{7}{3}-\xi-\frac{\zeta}{2})^2(2-\zeta)^2}{1+16[(1-\xi-\zeta)^2+(\frac{5}{3}-\frac{\zeta}{2})^2]}\}$$
$$= (\tfrac{7}{3}-\tfrac{3\zeta}{2})^2 \text{ and}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\eth(\Sigma\xi,\Xi\zeta),\eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(1-\xi-\zeta)^2,(2-\xi)^2,(\tfrac{7}{3}-\tfrac{3\zeta}{2})^2,(\tfrac{7}{3}-\xi-\tfrac{\zeta}{2})^2,(2-\zeta)^2\}$$
$$= (1-\xi-\zeta)^2.$$

Since

$$\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} = \frac{1}{4}\min\{(2-\xi)^2,(\tfrac{7}{3}-\tfrac{3\zeta}{2})^2\}$$
$$= (\tfrac{1}{4})(2-\xi)^2$$
$$\le \max\{(1-\xi-\zeta)^2,(\tfrac{5}{3}-\tfrac{\zeta}{2})^2\}$$
$$= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}.$$

Now, we consider

$$s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\tfrac{5}{3}-\tfrac{\zeta}{2})^2 \le \tfrac{1}{2}e^{-(\frac{7}{3}-\frac{3\zeta}{2})^2}(\tfrac{7}{3}-\tfrac{3\zeta}{2})^2+350(1-\xi-\zeta)^2$$
$$= \beta(M(\xi,\zeta))M(\xi,\zeta)+LN(\xi,\zeta).$$

Case (iii): $\zeta\in[0,\frac{2}{3}),\xi\in[\frac{2}{3},1]$.

$\eth(\Lambda\xi,\Xi\zeta) = (\frac{7}{6})^2, \eth(\Sigma\xi,\Upsilon\zeta) = (\frac{19}{12}-\xi)^2, \eth(\Upsilon\zeta,\Xi\zeta) = (\frac{3}{4})^2,$

$\eth(\Sigma\xi,\Lambda\xi) = (2-\xi)^2, \eth(\Lambda\xi,\Upsilon\zeta) = (\frac{11}{12})^2, \eth(\Sigma\xi,\Xi\zeta) = (\frac{11}{6}-\xi)^2,$

$$M(\xi,\zeta) = \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\frac{\eth(\Sigma\xi,\Xi\zeta)}{2s},\frac{\eth(\Upsilon\zeta,\Lambda\xi)}{2s},$$
$$\frac{\eth(\Sigma\xi,\Lambda\xi)\eth(\Upsilon\zeta,\Xi\zeta)}{1+\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)},\frac{\eth(\Sigma\xi,\Xi\zeta)\eth(\Upsilon\zeta,\Lambda\xi)}{1+s^4[\eth(\Sigma\xi,\Upsilon\zeta)+\eth(\Lambda\xi,\Xi\zeta)]}\}$$
$$= \max\{(\tfrac{19}{12}-\xi)^2,(2-\xi)^2,(\tfrac{3}{4})^2,\frac{(\frac{11}{6}-\xi)^2}{4},\frac{(\frac{11}{12})^2}{4},$$
$$\frac{(2-\xi)^2(\frac{3}{4})^2}{1+(\frac{19}{12}-\xi)^2+(\frac{7}{6})^2},\frac{(\frac{11}{6}-\xi)^2(\frac{11}{12})^2}{1+16[(\frac{19}{12}-\xi)^2+(\frac{7}{6})^2]}\}$$
$$= (2-\xi)^2 \text{ and}$$

$$N(\xi,\zeta) = \min\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta),\eth(\Sigma\xi,\Xi\zeta),\eth(\Upsilon\zeta,\Lambda\xi)\}$$
$$= \min\{(\tfrac{19}{12}-\xi)^2,(2-\xi)^2,(\tfrac{3}{4})^2,(\tfrac{11}{6}-\xi)^2,(\tfrac{11}{12})^2\} = (\tfrac{3}{4})^2.$$

Since

$$\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} = \frac{1}{4}\min\{(2-\xi)^2,(\tfrac{3}{4})^2\}$$
$$= (\tfrac{1}{4})(\tfrac{3}{4})^2$$
$$\le \max\{(\tfrac{19}{12}-\xi)^2,(\tfrac{7}{6})^2\}$$
$$= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\}.$$

Now, we consider

$$s^4\eth(\Lambda\xi,\Xi\zeta) = 16(\tfrac{7}{6})^2 \leq \tfrac{1}{2}e^{-(2-\xi)^2}(2-\xi)^2 + 350(\tfrac{3}{4})^2$$
$$= \beta(M(\xi,\zeta))M(\xi,\zeta) + LN(\xi,\zeta).$$

Λ, Ξ, Σ and Υ are contraction maps of the Geraghty-Berinde-Suzuki type from each of the four cases mentioned above. Consequently, all of the assumptions of Theorem 2.5 are satisfied by Λ, Ξ, Σ and Υ, and $\frac{2}{3}$ is their unique common fixed point.

Here, we note that the inequality (2.3) is not true if $L = 0$.

For, selecting $\zeta = 0$ and $\xi = 1$, we have

$$\frac{1}{2s}\min\{\eth(\Sigma\xi,\Lambda\xi),\eth(\Upsilon\zeta,\Xi\zeta)\} = \frac{1}{4}\min\{1,(\tfrac{3}{4})^2\}$$
$$= (\tfrac{1}{4})(\tfrac{3}{4})^2$$
$$\leq \max\{(\tfrac{7}{12})^2,(\tfrac{7}{6})^2\}$$
$$= \max\{\eth(\Sigma\xi,\Upsilon\zeta),\eth(\Lambda\xi,\Xi\zeta)\} \text{ implies that}$$
$$s^4\eth(\Lambda\xi,\Xi\zeta) = (16)(\tfrac{7}{6})^2 \nleq \beta(1)1 = \beta(M(\xi,\zeta))M(\xi,\zeta) \text{ for any } \beta \in \mathfrak{F}.$$

Remark 3.5. Theorem 1.19 and Theorem 1.20 follows as corollaries to Theorem 2.4 and Theorem 2.5 respectively by choosing $L = 0$. Hence Example 3.3 and Example 3.4 suggests that Theorem 2.4 and Theorem 2.5 are generalizations of Theorem 1.19 and Theorem 1.20 respectively.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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