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SOME FIXED POINT THEOREMS FOR Ψ-CHATTERJEA CONTRACTIONS IN C^*-ALGEBRA-VALUED GENERALIZED METRIC SPACES

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Abstract. Based on the concept and properties of C^*-algebras, this article introduces a concept of C^*-algebra-valued generalized metric space type of Jleli–Samet and give some Chatterjea fixed point theorems for linear positive mapping. Examples are given to illustrate our results.

Keywords: complete C^*-algebras spaces; fixed point theorems; Chaterjea fixed point theorem.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Since 1922, the renowned Banach contraction principle, often referred as Banach's fixed point theorem [1], has made fixed point theory an appealing area of study for numerous researchers. From then on, extending Banach's fixed-point theorem has been the subject of extensive research. In 1968, Kannan [2] introduced a new class of contractive mappings yielding a unique fixed point theorem in a complete metric space. Kannan's theorem holds significance as it provides a characterization of metric completeness. Specifically, a metric space is complete if and only if every Kannan mapping has a fixed point. A mapping T on a metric space $(X;d)$

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is called Chatterjea if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$. In 1972, Chatterjea [3] proved that if (X, d) is complete then T has a unique fixed point theorem and the orbite of T converges to the fixed point.

In 2015 Jleli and Samet introduced a new concept of generalized metric spaces (also known as JS-metric spaces) recovering various topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces (see [4, 5]). In this space, Jleli and Samet gave some generalized versions of metric fixed-point theorems. Recently, in 2019 K.Chaira et al [6] extend some common fixed point theorems of Banach, Chatterjea, and Kannan contractions in generalized metric space of Samet-Jleli endowed with a graph.

On the other hand, Ma et al. [7], using the positive elements of a C^*-algebra, introduced the concept of C^*-algebra-valued metric space and yielding some fixed-point results for contractive and expansive mappings. In 2021, the authors in [8] demonstrated the utility of C^*-algebra-valued metric space across various application domains, illustrating several applications derived from the obtained results. Given the significance of exploring fixed point results within the framework of C^*-algebra-valued metric space, researchers have introduced a plethora of new generalized spaces beyond metric spaces, for more details see [9, 10, 11, 12, 13, 14, 15, 16].

Motivated by the above ideas, we introduce a new concept of generalized metric spaceswe C^* algebra-valued generalized metric space. This novel concept of generalized metric spaces encompasses various topological spaces, including standard C^*-algebra-valued metric spaces, C^*-algebra-valued b-metric spaces, C^*-algebra-valued dislocated metric spaces, and C^*-algebra-valued modular spaces. Furthermore, a new class of Chatterjea-type mappings was introduced, and several related fixed point theorems were presented.

The paper is organized as follows: we begin by revisit several definitions, lemmas, and theorems relevant to C^* algebra-valued generalized metric space and explore their associated properties. We then study some fixed point theorems of Chatterjea-type mappings from which we deduce several known results as corollaries. Throughout the paper, we provide illustrative examples.

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2. PRELIMINARIES

Throughout this paper, $\mathbb{A}$ is a unital by an unital (i.e ,unity element $1_{\mathbb{A}}$) C^*-algebra with linear involution $*$, such that for all $u, v \in \mathbb{A}$,

$$(uv)^* = v^*u^* \text{ and } u^{**} = u.$$

We call an element $a \in \mathbb{A}$ a positive element, denote it by $0_{\mathbb{A}} \preceq a$, if $a \in \mathscr{A}_h = \{u \in \mathbb{A} : u = u^*\}$ and $\sigma(a) \subseteq \mathbb{R}_+$, where $\sigma(a)$ is the spectrum of a. Using positive element, we can define a partial ordering "$\preceq$" on $\mathscr{A}_h$ as follows:

$$u \preceq v \text{ if and only if } 0_{\mathbb{A}} \preceq v - u,$$

where $0_{\mathbb{A}}$ means the zero element of $\mathbb{A}$.
We denote the set $\{a \in \mathbb{A} : 0_{\mathbb{A}} \preceq a\}$ by $\mathbb{A}_+$ and $|a| = (a^*a)^{\frac{1}{2}}$. Set $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba \text{ for all } b \in \mathbb{A}\}$.

Remark 2.1. [17] *When $\mathbb{A}$ is an unital C^*-algebra, then for any $a \in \mathbb{A}_+$ we have*

$$(a \preceq 1_{\mathbb{A}} \Leftrightarrow \|a\| \leqslant 1).$$

Lemma 2.1. [18, 17] *Let $\mathbb{A}$ is an unital C^*-algebra with unit $1_{\mathbb{A}}$:*

(1) If $a, b \in \mathbb{A}$ such that $0_{\mathbb{A}} \preceq a \preceq b$, then $\|a\| \leqslant \|b\|$.
(2) Suppose that $a, b \in \mathbb{A}$ with $0_{\mathbb{A}} \preceq a, b$ and $ab = ba$, then $0 \preceq ab$.
(3) If $0 \preceq a \preceq b$ and $c \in \mathbb{A}$, then $0 \preceq c^*ac \preceq c^*bc$.
(4) If a is an element of a C^*-algebra $\mathbb{A}$, then $\|a\| = \|a^*\| = \|aa^*\|^{\frac{1}{2}}$.

Proposition 2.1. [19] *$\mathbb{A}_+$ is closed in a C^*-algebra $\mathbb{A}$.*

Definition 2.1. *Let $\mathscr{X}$ be a nonempty set and $\mathscr{D} : \mathscr{X} \times \mathscr{X} \to \mathbb{A}$ be a given mapping. Let $x \in \mathscr{X}$ and $(x_n)_{n\geqslant 0}$ a sequence in $\mathscr{X}$. We say $(x_n)_{n\geqslant 0}$ is $\mathscr{D}$-convergent to x (with respect to $\mathbb{A}$) and we write $\lim\limits_{n\to+\infty} \|\mathscr{D}(x_n,x)\| = 0_{\mathbb{A}}$, if and only if for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all intger $n \in \mathbb{N}$,*

$$(n \geqslant n_0 \Rightarrow \|\mathscr{D}(x,x_n)\| < \varepsilon).$$

For every $x \in \mathscr{X}$, let us define the set

$$C(\mathscr{D},\mathscr{X},x) = \{(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{X} : \lim_{n\to+\infty} \|\mathscr{D}(x_n,x)\| = 0\}.$$

We now introduce the notion of C^*-algebra-valued generalized metric, extending the classical metric type spaces.

Definition 2.2. *Let $\mathscr{X}$ be a nonempty set. Suppose that the mapping $\mathscr{D} : \mathscr{X} \times \mathscr{X} \to \mathbb{A}$ is defined, with the following properties:*

- (D_1) $0_{\mathbb{A}} \preceq \mathscr{D}(x,y)$, *for all x and y in $\mathscr{X}$;*
- (D_2) $\mathscr{D}(x,y) = 0_{\mathbb{A}}$ *implies that $x = y$;*
- (D_3) $\mathscr{D}(x,y) = \mathscr{D}(y,x)$, *for all x and y in $\mathscr{X}$;*
- (D_4) *there exists $c \succ 0_{\mathbb{A}}$, such that, for all $(x,y) \in \mathscr{X} \times \mathscr{X}$ and $(x_n)_{n\in\mathbb{N}} \in \mathscr{C}(\mathscr{D},\mathscr{X},x)$, we have*

$$\begin{cases} \limsup\limits_{n\to+\infty} \|\mathscr{D}(x_n,y)\|) < \infty \text{ and} \\ \mathscr{D}(x,y) \preceq (\limsup\limits_{n\to+\infty} \|\mathscr{D}(x_n,y)\|) \cdot c \end{cases}$$

In this case, $\mathscr{D}$ is said to be a C^*-algebra-valued generalized metric on $\mathscr{X}$, and $(\mathscr{X},\mathbb{A},\mathscr{D})$ is said to be a C^*-algebra-valued generalized metric space.

Remark 2.2. *Let $(\mathscr{X},\mathbb{A},\mathscr{D})$ be a C^*-algebra-valued generalized metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{X}$. If $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-convergent to x, then this limit is unique.*

Proof. Suppose the sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to x and y with respect to $\mathbb{A}$. By the property (D_4), we have.

$$\mathscr{D}(x,y) \preceq \big(\limsup_{n\to+\infty} \|\mathscr{D}(x_n,y)\|\big).c = 0_{\mathbb{A}}.$$

So $\mathscr{D}(x,y) = 0_{\mathbb{A}}$. Which shows that $x = y$.

□

Remark 2.3. *Note that the set $C(\mathscr{D},\mathscr{X},x)$ can empty for all $x \in \mathscr{X}$, in this case we consider by convention that $(\mathscr{X},\mathbb{A},D)$ is a C^*-algebra-valued generalized metric space if and only if the three axioms (D_1),(D_2) and (D_3) are verified.*

Example 2.1. *Let $\mathscr{X} = [0,1]$ and $\mathbb{A} = \mathscr{M}_2(\mathbb{R})$ with $\|M\|_2 = \left(\sum_{i=1}^{4} |\alpha_i|^2\right)^{\frac{1}{2}}$, where α_i are the coefficients of the matrix $M \in \mathbb{A}$. Then $\mathbb{A}$ is a C^*-algebra.*

$$M \in \mathscr{A}_+ \Leftrightarrow ({}^tM = M \text{ and } spec(M) \subseteq \mathbb{R}_+).$$

Consider the mapping $\mathscr{D}$ defined by

$$\mathscr{D}(x,y) = \begin{cases} \begin{pmatrix} 2(x+y) & 0 \\ 0 & 2(x+y) \end{pmatrix} & \text{if } x = 0 \text{ or } y = 0 \\ \begin{pmatrix} \frac{x+y}{2} & 0 \\ 0 & \frac{x+y}{2} \end{pmatrix} & \text{otherwise} \end{cases}$$

We verify the axioms of the C^*-algebra-valued generalized metric space.

- *(D_1) Obvious.*
- *(D_2) $\mathscr{D}(x,y) = 0_{\mathbb{A}}$, then $x+y = 0$ and $x = y = 0$.*
- *(D_3) Symmetry is clearly verified $\mathscr{D}(x,y) = \mathscr{D}(y,x)$, for all x and y in $\mathscr{X}$.*
- *(D_4) We distinguish two cases*

Case 1: *$x \neq 0$, then the set $C(\mathscr{D},\mathscr{X},x)$ is empty. Otherwise, there exists a sequence $(x_n)_{n\geqslant 0}$ of $\mathscr{X}$ such that $\lim_{n\to\infty} \|\mathscr{D}(x_n,x)\|_2 = 0$.*

• If the sequence admits an infinite of zero, there exists a subsequence $(x_{\phi(n)})_{n\geqslant 0}$ of $(x_n)_{n\geqslant 0}$ such that $x_{\phi(n)} = 0$, for all $n \in \mathbb{N}$, so $\lim_{n\to+\infty} \|\mathscr{D}(x_{\phi(n)},x)\|_2 = 2\sqrt{2}x$ is different from 0, which is impossible, since $x \neq 0$.

• If the sequence $(x_n)_{n\geqslant 0}$ does not admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$, for any integer $n \geqslant n_0$. We have $\mathscr{D}(x_n,x) = \begin{pmatrix} \frac{x_n+x}{2} & 0 \\ 0 & \frac{x_n+x}{2} \end{pmatrix} \succ 0_{\mathbb{A}}$. As a

result

$$\lim_{n\to+\infty}\|\mathscr{D}(x_n,x)\|_2 = \lim_{n\to+\infty}\sqrt{2}(\frac{x_n+x}{2}) > \frac{\sqrt{2}x}{2},$$

which is absurd, since $\frac{\sqrt{2}x}{2} > 0$.

Case 2: $x=0$. *In this case* $\mathscr{C}(\mathscr{D},\mathscr{X},x)\neq\emptyset$. *Let* $(u_n)_{n\geqslant 0}\in\mathscr{C}(\mathscr{D},\mathscr{X},0)$ *and* $y\in\mathbb{R}_+\setminus\{0\}$.

• *If the sequence* $(u_n)_{n\geqslant 0}$ *admits an infinity of zero, there exists a subsequence* $(u_{\psi(n)})_{n\geqslant 0}$ *of* $(u_n)_{n\geqslant 0}$ *such that* $u_{\psi(n)}=0$ *for all* $n\in\mathbb{N}$. *We have*

$$\mathscr{D}(0,y)=\begin{pmatrix}2y & 0\\ 0 & 2y\end{pmatrix}=\mathscr{D}(u_{\psi(n)},y)$$

for all $n\in\mathbb{N}$. *So,*

$$\begin{cases}\mathscr{D}(0,y)\preceq(\lim_{n\to+\infty}\|\mathscr{D}(u_{\psi(n)},y)\|_2).1_{\mathbb{A}}\leqslant(\limsup_{n\to+\infty}\|\mathscr{D}(u_n,y)\|_2).1_{\mathbb{A}}\\ \text{and}\ \limsup_{n\to+\infty}\|\mathscr{D}(u_n,y)\|_2\leqslant 2\sqrt{2}<\infty,\ \text{since}\ u_n,y\in[0,1]\end{cases}.$$

• *If the sequence* $(u_n)_{n\geqslant 0}$ *does not admit an infinity of zero, then there exists* $n_0\in\mathbb{N}$ *such that* $u_n\neq 0$, *for any integer* $n\geqslant n_0$. *We have*

$$\begin{aligned}\mathscr{D}(0,y)&\preceq\mathscr{D}(0,u_n)+4\mathscr{D}(u_n,y)\\ &\preceq(\|\mathscr{D}(0,u_n)\|_2+4\|\mathscr{D}(u_n,y)\|_2).1_{\mathbb{A}}\end{aligned}$$

As a result

$$\mathscr{D}(0,y)\preceq(\limsup_{n\to+\infty}\|\mathscr{D}(u_n,y)\|_2).(4.1_{\mathbb{A}}).$$

For $y=0$, $\mathscr{D}(x,0)=0_{\mathbb{A}}\prec\limsup_{n\to+\infty}\|\mathscr{D}(u_n,0)\|_2.1_{\mathbb{A}}$.

Hence, for all $y\in X$,

$$\mathscr{D}(x,y)\preceq\left(\limsup_{n\to\infty}\|\mathscr{D}(u_n,y)\|\right).c,\ \text{where}\ c=4.1_{\mathbb{A}}\ \text{and}\ \limsup_{n\to+\infty}\|\mathscr{D}(u_n,y)\|_2<\infty.$$

Therefore $(\mathscr{X},\mathbb{A},\mathscr{D})$ *is a* C^**-algebra-valued generalized metric space.*

Remark 2.4. C^**-algebra-valued metric spaces and* C^**-algebra-valued b-metric spaces are* C^**-algebra-valued generalized metric spaces. Indeed, let b be a real number such that* $b\geqslant 1$ *and* $(\mathscr{X},\mathbb{A},\mathscr{D})$ *a* C^**-algebra-valued b-metric space. Then* $\mathscr{D}$ *satisfies the conditions* (D_1), (D_2) *and*

(D_3). For the condition (D_4), let $(x,y)\in\mathscr{X}\times\mathscr{X}$ and $(x_n)_{n\in\mathbb{N}}\in\mathscr{C}(\mathscr{D},\mathscr{X},x)$. We have, for all $n\in\mathbb{N}$,

$$\|\mathscr{D}(x_n,y)\|\leqslant\|b\|(\|\mathscr{D}(x_n,x)\|+\|\mathscr{D}(x,y)\|),$$

$$\text{so }\limsup_{n\to+\infty}\|\mathscr{D}(x_n,y)\|\leqslant\|b\|(\limsup_{n\to+\infty}\|\mathscr{D}(x_n,x)\|+\|\mathscr{D}(x,y)\|)=\|\mathscr{D}(x,y)\|<+\infty.$$

We also have

$$\begin{aligned}\mathscr{D}(x,y)&\preceq b(\mathscr{D}(x,x_n)+\mathscr{D}(x_n,y))\\&\preceq(\|\mathscr{D}(x,x_n)+\mathscr{D}(x_n,y)\|).(\|b\|.1_{\mathbb{A}}),\end{aligned}$$

$$\text{so }\|\mathscr{D}(x,y)\|\leqslant(\|\mathscr{D}(x,x_n)\|+\|\mathscr{D}(x_n,y)\|)\|b\|,$$

which implies that $\|\mathscr{D}(x,y)\|\leqslant(\lim_{n\to+\infty}\|\mathscr{D}(x,x_n)\|+\limsup_{n\to+\infty}\|\mathscr{D}(x_n,y)\|)\|b\|$. Hence,

$$\mathscr{D}(x,y)\preceq\|\mathscr{D}(x,y)\|.1_{\mathbb{A}}\preceq(\limsup_{n\to+\infty}\|\mathscr{D}(x_n,y)\|).(\|b\|.1_{\mathbb{A}})$$

Definition 2.3. *Let $(\mathscr{X},\mathbb{A},\mathscr{D})$ is a C^*-algebra-valued metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{X}$.*

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be a $\mathscr{D}$-Cauchy sequence (with respect to $\mathbb{A}$) whenever, for every $\varepsilon>0$, there is a natural number $N\in\mathbb{N}$, such that

$$\|D(x_n,x_m)\|<\varepsilon,\text{ for all }n,m\geqslant N,$$

(ii) The space $(\mathscr{X},\mathbb{A},\mathscr{D})$ is said to be $\mathscr{D}$-complete if every $\mathscr{D}$-Cauchy sequence is $\mathscr{D}$-convergent to some element in $\mathscr{X}$ (with respect to $\mathbb{A}$).

Example 2.2. *We return to Example 1.7. We show that $(\mathscr{X},\mathbb{A},\mathscr{D})$ is $\mathscr{D}$-Complete. Let $(u_n)_{n\in\mathbb{N}}$ be a $\mathscr{D}$-Cauchy sequence of $\mathscr{X}$ (with respect to $\mathbb{A}$).*
Let $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\left(n,m\geqslant n_0\Rightarrow\|\mathscr{D}(u_n,u_m)\|_2<\frac{\varepsilon}{4}\right)$, then

$$\left(n,m\geqslant n_0\Rightarrow\|\mathscr{D}(u_n,0)\|_2=2\sqrt{2}u_n\leqslant4\|\mathscr{D}(u_n,u_m)\|_2<\varepsilon\right).$$

Which shows that $\lim_{n\to+\infty}\|\mathscr{D}(u_n,0)\|_2=0$. Hence, the sequence $(u_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to 0 in $\mathscr{X}$ (with respect to $\mathbb{A}$), finally $(\mathscr{X},\mathbb{A},D)$ is $\mathscr{D}$-complete (with respect to $\mathbb{A}$).

Definition 2.4. *Let $\mathbb{A}$ be a C^*-algebra. A linear mapping $\psi:\mathbb{A}\to\mathbb{A}$ is said to be positive if $\psi(\mathbb{A}^+)\subseteq\mathbb{A}^+$.*

Proposition 2.2. *Let $\mathbb{A}$ be a C^*-algebra with unit $1_{\mathbb{A}}$, then the linear positive function $\psi : \mathbb{A} \to \mathbb{A}$ is bounded, continuous, contractive and $\psi_{\mathbb{A}}(1_{\mathbb{A}}) = \|\psi_{\mathbb{A}}\|.1_{\mathbb{A}}$.*

Definition 2.5. *Let $(\mathscr{X}, \mathbb{A}, \mathscr{D})$ is a C^*-algebra-valued generalized metric space and A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is ψ-Chatterjea contraction if for every $(x,y) \in \mathscr{X}^2$,*

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) \preceq \psi\Big(\mathscr{D}(\mathscr{T}x, y) + \mathscr{D}(x, \mathscr{T}y)\Big), \tag{2.1}$$

where ψ the linear positive mapping on $\mathbb{A}$, with $0 < \|\psi\| < \frac{1}{2}$.

3. Main Results

The following notation will be used in the sequel

$$\delta(\mathscr{D}, \mathscr{T}, x_0) := \sup\Big\{\|\mathscr{D}(\mathscr{T}^i x_0, x_0)\| : i \in \mathbb{N} \setminus \{0\}\Big\}, \text{ where } x_0 \in \mathscr{X}.$$

The forthcoming lemma will stand as the cornerstone of what ensues. Let $(x_n)_{n \geqslant 0}$ be the Picard-sequence defined by $x_{n+1} = \mathscr{T}x_n$, for all $n \in \mathbb{N}$.

Lemma 3.1. *Let $(\mathscr{X}, \mathbb{A}, \mathscr{D})$ be a complete C^*-algebra-valued generalized metric space and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a ψ-Chatterjea contraction. Then, for any $(m,n) \in (\mathbb{N} \setminus \{0\})^2$,*

$$\mathscr{D}(x_n, x_m) \preceq \delta_0.\left(\sum_{m}^{n+m-1} \binom{j-1}{m-1}.\psi^j(1_{\mathbb{A}}) + \sum_{n}^{n+m-1} \binom{j-1}{n-1}\psi^j(1_{\mathbb{A}})\right), \tag{3.1}$$

where $\delta_0 = \delta(\mathscr{D}, \mathscr{T}, x_0)$.

Proof. (By induction). Let p be an integer greater than or equal to 2 and $(n,m) \in \mathbb{N}^2$ such that $p = n + m$. Since

$$\begin{aligned}
\mathscr{D}(x_1, x_1) &\preceq \psi(\mathscr{D}(x_1, x_0) + \mathscr{D}(x_0, x_1)) \\
&= 2\psi(\mathscr{D}(x_1, x_0)) \\
&\preceq 2\psi(\|\mathscr{D}(\mathscr{T}x_0, x_0)\|.1_{\mathbb{A}}) \\
&\preceq 2\psi(\delta_0.1_{\mathbb{A}}) \\
&\preceq 2\delta_0.\psi(1_{\mathbb{A}}).
\end{aligned}$$

It is clear that the inequality (3.1) holds for $p=2$ with $(m,n)=(1,1)$.
Assume next that inequality (3.1) holds for any $(m',n')\in(\mathbb{N}\setminus\{0\})^2$ be chosen in such a way that $n'+m'=p$; let $(m,n)\in(\mathbb{N}\setminus\{0\})^2$ with $n+m=p+1$. By hypothesis,

$$\mathscr{D}(x_n,x_m)\preceq\psi\big(\mathscr{D}(x_n,x_{m-1})+\mathscr{D}(x_{n-1},x_m)\big).$$

Since $n+(m-1)=p$ and $(n-1)+m=p$, the inductive hypothesis yields

$$\begin{aligned}\mathscr{D}(x_n,x_m)\preceq\delta_0.\Bigg(&\sum_{j=m}^{n+m-2}\binom{j-1}{m-1}.\psi^j(1_{\mathbb{A}})+\sum_{j=n-1}^{n+m-2}\binom{j-1}{n-2}.\psi^j(1_{\mathbb{A}})\\&+\sum_{j=m-1}^{n+m-2}\binom{j-1}{m-2}.\psi^j(1_{\mathbb{A}})+\sum_{j=n}^{n+m-2}\binom{j-1}{n-1}.\psi^j(1_{\mathbb{A}})\Bigg)\end{aligned}$$

$$\begin{aligned}\mathscr{D}(x_n,x_m)\preceq\delta_0\Bigg(&\sum_{j=m}^{n+m-2}\left(\binom{j-1}{m-1}.\psi^j(1_{\mathbb{A}})+\binom{j-1}{m-2}.\psi^j(1_{\mathbb{A}})\right)+\psi^{m-1}(1_{\mathbb{A}})\\&+\sum_{j=n}^{n+m-2}\left(\binom{j-1}{n-1}.\psi^j(1_{\mathbb{A}})+\binom{j-1}{n-2}.\psi^j(1_{\mathbb{A}})\right)+\psi^{n-1}(1_{\mathbb{A}})\Bigg)\end{aligned}$$

$$\begin{aligned}&\preceq\delta_0.\Bigg(\sum_{j=m}^{n+m-2}\binom{j}{m-1}.\psi^j(1_{\mathbb{A}})+\sum_{j=n}^{n+m-2}\binom{j}{n-1}.\psi^j(1_{\mathbb{A}})\\&+\psi^{n-1}(1_{\mathbb{A}})+\psi^{m-1}(1_{\mathbb{A}})\Bigg)\end{aligned}$$

$$\preceq\delta_0.\left(\sum_{j=m-1}^{n+m-2}\binom{j}{m-1}.\psi^j(1_{\mathbb{A}})+\sum_{j=n-1}^{n+m-2}\binom{j}{n-1}.\psi^j(1_{\mathbb{A}})\right)$$

$$\preceq\delta_0.\left(\sum_{j=m}^{n+m-1}\binom{j-1}{m-1}.\psi^j(1_{\mathbb{A}})+\sum_{j=n}^{n+m-1}\binom{j-1}{n-1}.\psi^j(1_{\mathbb{A}})\right),$$

Finally the inequality (3.1) holds for $(n,m)\in(\mathbb{N}^*)^2$ such that $n+m=p+1$. □

Lemma 3.2. *Let $(\mathscr{X},\mathbb{A},\mathscr{D})$ be a complete C^*-algebra-valued generalized metric space and $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ is a ψ-Chatterjea contraction. Then, for every $(m,n)\in\mathbb{N}\setminus\{0\}^2$ such that $m\leqslant n$, we have*

$$\mathscr{D}(x_n,x_m)\preceq 2^m\delta_0\|\psi\|^m(1-2\|\psi\|)^{-1}.1_A, \tag{3.2}$$

where $\delta_0 = \delta(\mathscr{D}, \mathscr{T}, x_0)$.

Proof. Let $n, m \in \mathbb{N} \setminus \{0\}$; assume that $m \leqslant n$.

Since $\binom{j-1}{m-1} \leqslant 2^{j-1}$ for any $j \in [\![m, n+m-1]\!]$ and $\binom{j-1}{n-1} \leqslant 2^{j-1}$ for any $j \in [\![n, n+m-1]\!]$, it follows that

$$\begin{aligned}\sum_{j=m}^{n+m-1} \binom{j-1}{m-1} \psi^j(1_{\mathbb{A}}) &\preceq \frac{1}{2} \sum_{j=m}^{n+m-1} 2^j \psi^j(1_{\mathbb{A}}) \\ &\preceq \frac{1}{2} \sum_{j=m}^{n+m-1} \|2^j \psi^j(1_{\mathbb{A}})\|.1_A \\ &\preceq \frac{1}{2} \sum_{j=m}^{n+m-1} 2^j \|\psi\|^j.1_A \\ &\preceq \frac{1}{2} \frac{(2^m \|\psi\|^m)}{(1-2\|\psi\|)}.1_A,\end{aligned}$$

and that

$$\sum_{j=n}^{n+m-1} \binom{j-1}{n-1} \psi^j(1_{\mathbb{A}}) \preceq \frac{1}{2} \frac{2^n \|\psi\|^n}{(1-2\|\psi\|)}.1_A \preceq \frac{1}{2} \frac{2^m \|\psi\|^m}{(1-2\|\psi\|)}.1_A.$$

It follows from inequality (3.1) of Lemma 3.1 that

$$\mathscr{D}(x_n, x_m) \preceq 2^m \delta_0 \|\psi\|^m (1-2\|\psi\|)^{-1}.1_A$$

□

Theorem 3.1. *Let* $(\mathscr{X}, \mathbb{A}, \mathscr{D})$ *be a complete* C^**-algebra-valued generalized metric space and* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be a* ψ*-Chatterjea contraction. Suppose that there exists* $x_0 \in \mathscr{X}$ *such that* $\delta(\mathscr{D}, \mathscr{T}, x_0) < \infty$. *Then,* $\mathscr{T}$ *has unique fixed point* ω *of* $\mathscr{X}$ *and the sequence* $(\mathscr{T}^n x_0)_{n\in\mathbb{N}}$ *converges to* ω.

Proof. Select $(m,n) \in (\mathbb{N} \setminus \{0\})^2$ such that $m \leqslant n$.

According to the Lemma 3.2,

$$\mathscr{D}(x_n, x_m) = \mathscr{D}(\mathscr{T}^n x_0, \mathscr{T}^m x_0) \preceq 2^m \delta_0 \|\psi\|^m (1-2\|\psi\|)^{-1}.1_A$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}$-Cauchy sequence. Since $(\mathscr{X}, \mathscr{D})$ is complete, the sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to some $\omega \in \mathscr{X}$.

Step 1: We show by induction that for all $n \in \mathbb{N} \setminus \{0\}$

$$\|\mathscr{D}(\mathscr{T}\omega, x_n)\| \leqslant \|\psi\|^n \|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=1}^{n} \|\psi\|^k \|\mathscr{D}(x_{n-k+1}, \omega)\|, \tag{3.3}$$

For $n = 1$, we have

$$\begin{aligned} \|\mathscr{D}(\mathscr{T}\omega, x_1)\| &\leqslant \|\psi(\mathscr{D}(\mathscr{T}\omega, x_0) + \mathscr{D}(\omega, \mathscr{T}x_0))\| \\ &\leqslant \|\psi\|\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \|\psi\|\|\mathscr{D}(\omega, \mathscr{T}x_0)\|. \end{aligned}$$

For $n = 2$, we have

$$\begin{aligned} \|\mathscr{D}(\mathscr{T}\omega, x_2)\| &\leqslant \|\psi(\mathscr{D}(\mathscr{T}\omega, x_1) + \mathscr{D}(\omega, \mathscr{T}x_1))\| \\ &\leqslant \|\psi\|^2\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \|\psi\|^2\|\mathscr{D}(\omega, x_1)\| + \|\psi\|\|\mathscr{D}(\omega, x_2)\|. \end{aligned}$$

Suppose that inequality (3.3) holds for some $n \geqslant 2$. We show that

$$\|\mathscr{D}(\mathscr{T}\omega, x_{n+1})\| \leqslant \|\psi\|^{n+1}\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=1}^{n+1} \|\psi\|^k \|\mathscr{D}(x_{n-k+2}, \omega)\|.$$

We have

$$\begin{aligned} \|\mathscr{D}(\mathscr{T}\omega, x_{n+1})\| &\leqslant \|\psi\|(\|\mathscr{D}(\mathscr{T}\omega, x_n)\| + \|\mathscr{D}(\omega, x_{n+1}))\| \\ &\leqslant \|\psi\|^{n+1}\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \|\psi\| \sum_{k=1}^{n} \|\psi\|^k \|\mathscr{D}(x_{n-k+1}, \omega)\| + \|\psi\|\|\mathscr{D}(\omega, x_{n+1}))\| \\ &\leqslant \|\psi\|^{n+1}\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=1}^{n} \|\psi\|^{k+1} \|\mathscr{D}(x_{n-k+1}, \omega)\| + \|\psi\|\|\mathscr{D}(\omega, x_{n+1}))\| \\ &\leqslant \|\psi\|^{n+1}\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=0}^{n} \|\psi\|^{k+1} \|\mathscr{D}(x_{n-k+1}, \omega)\| \\ &\leqslant \|\psi\|^{n+1}\|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=1}^{n+1} \|\psi\|^{k} \|\mathscr{D}(x_{n-k+2}, \omega)\| \end{aligned}$$

Hence, for any positive integer $n \geqslant 2$,

$$\|\mathscr{D}(\mathscr{T}\omega, x_n)\| \leqslant \|\psi\|^n \|\mathscr{D}(\mathscr{T}\omega, x_0)\| + \sum_{k=1}^{n} \|\psi\|^k \|\mathscr{D}(x_{n-k+1}, \omega)\|.$$

Step 2: Let $\varepsilon > 0$. Since $(\|\mathscr{D}(\omega, x_n)\|)_{n\geqslant 0}$ and $(\|\psi\|^n)_{n\in\mathbb{N}}$ converges to 0, there exists $N \in \mathbb{N}$ such that

$$\left(n \geqslant N+1 \implies \|\mathscr{D}(\omega, x_n)\| \leqslant \varepsilon' \text{ and } \|\psi\|^n \leqslant \varepsilon'\right),$$

where $\varepsilon = \dfrac{\varepsilon^{'}}{1-\|\psi\|}\left(1+\max\limits_{n-N+1\leqslant k\leqslant n}\|\mathscr{D}(x_{n-k+1},\omega)\|\ \|\psi\|^{1-N}\right)$. We have

$$\begin{aligned}
\sum_{k=1}^{n}\|\psi\|^{k}\|\mathscr{D}(x_{n-k+1},\omega)\| &= \sum_{k=1}^{n-N}\|\psi\|^{k}\|\mathscr{D}(x_{n-k+1},\omega)\| + \sum_{k=n-N+1}^{n}\|\psi\|^{k}\|\mathscr{D}(x_{n-k+1},\omega)\| \\
&\leqslant \varepsilon^{'}\left(\sum_{k=1}^{n-N}\|\psi\|^{k}\right) + \sum_{k=n-N+1}^{n}\|\psi\|^{k}\|\mathscr{D}(x_{n-k+1},\omega)\| \\
&\leqslant \varepsilon^{'}\left(\sum_{k=1}^{n-N}\|\psi\|^{k}\right) + \max_{n-N+1\leqslant k\leqslant n}\|\mathscr{D}(x_{n-k+1},\omega)\|\left(\sum_{k=n-N+1}^{n}\|\psi\|^{k}\right) \\
&\leqslant \left(\frac{\varepsilon^{'}}{1-\|\psi\|} + \max_{n-N+1\leqslant k\leqslant n}\|\mathscr{D}(x_{n-k+1},\omega)\|\left(\sum_{k=0}^{N-1}\|\psi\|^{k+n-N+1}\right)\right) \\
&\leqslant \frac{\varepsilon^{'}}{1-\|\psi\|} + \max_{n-N+1\leqslant k\leqslant n}\|\mathscr{D}(x_{n-k+1},\omega)\|\|\psi\|^{n-N+1}\left(\sum_{k=0}^{+\infty}\|\psi\|^{k}\right) \\
&\leqslant \frac{\varepsilon^{'}}{1-\|\psi\|}\left(1+\max_{n-N+1\leqslant k\leqslant n}\|\mathscr{D}(x_{n-k+1},\omega)\|\|\psi\|^{1-N}\right) \\
&\leqslant \varepsilon.
\end{aligned}$$

Which shows that $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\|\psi\|^{k}\|\mathscr{D}(x_{n-k+1},\omega)\| = 0$. Hence, $\lim\limits_{n\to+\infty}\mathscr{D}(\mathscr{T}\omega,x_n) = 0$. According to the condition (D_4), we have $\mathscr{D}(\mathscr{T}\omega,\omega) \preceq \limsup\limits_{n\to+\infty}\|\mathscr{D}(\omega,x_n)\|.c = 0_{\mathbb{A}}$. Thus, $\mathscr{T}\omega = \omega$.

Uniqueness: Suppose that $u,v\in\mathscr{X}$ are two fixed points of $\mathscr{T}$. Since $\mathscr{T}$ is a ψ-Chatterjea contraction, we have

$$\mathscr{D}(u,v) = \mathscr{D}(\mathscr{T}u,\mathscr{T}v) \preceq \psi\big(\mathscr{D}(\mathscr{T}u,v)+\mathscr{D}(u,\mathscr{T}v)\big),$$

which implies that

$$\mathscr{D}(u,v) \preceq 2\psi(\mathscr{D}(u,v)).$$

$$\|\mathscr{D}(u,v)\| \leqslant 2\|\psi\|\ \|\mathscr{D}(u,v)\|.$$

Hence

$$(1-2\|\psi\|)\|\mathscr{D}(u,v)\| \leqslant 0.$$

Therefore $\mathscr{D}(u,v)=0$, i.e., $u=v$. □

The following example illustrates the above Theorem.

Example 3.1. *Let $\mathscr{X} = \mathbb{R}_+$ and $\mathbb{A} = \mathscr{M}_2(\mathbb{R})$.*

$\mathbb{A}$ is a C^*-algebra endowed with the Euclidean norm $\|.\|_2$. Let $\mathscr{D} : \mathscr{X} \times \mathscr{X} \longrightarrow \mathbb{A}$ be given by

$$\mathscr{D}(x,y) = \begin{pmatrix} 2(\max\{x,y\})^p & 0 \\ 0 & 3(\min\{x,y\})^p. \end{pmatrix}$$

where $x,y \in \mathscr{X}$ and $p \in \mathbb{N} \setminus \{0\}$.

• *It is easy to verify the axioms (D_1), (D_2) and (D_3) of the C^*-algebra-valued generalized metric space.*

Let's show the condition (D_4). Let $(x_n)_{n\geqslant 0} \in C(\mathscr{D},X,x)$, then

$$\lim_{n\to+\infty} \|\mathscr{D}(x_n,x)\|_2 = \lim_{n\to+\infty} \sqrt{4(\max\{x_n,x\})^{2p} + 9(\min\{x_n,x\})^{2p}} = 0.$$

As

$$0 \leqslant x^{2p} \leqslant 2(\max\{x_n,x\})^{2p} \leqslant \|\mathscr{D}(x_n,x)\|_2 \text{ and } \lim_{n\to+\infty} \mathscr{D}(x_n,x)\|_2 = 0,$$

we obtain $x = 0$ and $C(\mathscr{D},\mathscr{X},x) \neq \emptyset$.

Let $y \in X$, we have

$$\mathscr{D}(0,y) = \begin{pmatrix} 2y^p & 0 \\ 0 & 0 \end{pmatrix} \preceq$$

$$\begin{pmatrix} 2(\max\{x_n,0\})^p & 0 \\ 0 & 3(\min\{x_n,0\})^p \end{pmatrix} + \begin{pmatrix} 2(\max\{x_n,y\})^p & 0 \\ 0 & 3(\min\{x_n,y\})^p \end{pmatrix},$$

so $\mathscr{D}(0,y) \preceq \mathscr{D}(x_n,0) + \mathscr{D}(x_n,y)$. We have also $\mathscr{D}(x_n,y) \preceq \mathscr{D}(x_n,0) + \mathscr{D}(y,y)$. Hence,

$$\mathscr{D}(0,y) \preceq \limsup_{n\to+\infty}(\|\mathscr{D}(x_n,y)\|_2).1_{\mathbb{A}} \text{ and } \limsup_{n\to+\infty} \|\mathscr{D}(x_n,y)\|_2 \leqslant \|\mathscr{D}(y,y)\|_2 < +\infty.$$

• *Now we prove that $(\mathscr{X},\mathbb{A},\mathscr{D})$ is $\mathscr{D}$-complete (with respect to $\mathbb{A}$). Let $(x_n)_{n\in\mathbb{N}}$ be a $\mathscr{D}$-Cauchy sequence of $\mathscr{X}$ (with respect to $\mathbb{A}$).*

Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n,m \in \mathbb{N}$,

$$(n,m \geqslant n_0 \implies \|\mathscr{D}(x_n,x_m)\|_2 < \varepsilon),$$

so, for all integers $n,m \geqslant n_0$,

$$0 \leqslant 2x_n^p \leqslant 2(\max\{x_n,x_m\})^p \leqslant \sqrt{4(\max\{x_n,x_m\})^{2p} + 9(\min\{x_n,x_m\})^{2p}} < \varepsilon,$$

which implies $0 \leqslant x_n < \sqrt[p]{\frac{\varepsilon}{2}}$, *for all integer* $n \geqslant n_0$. *Hence,*

$$\left(n \geqslant n_0 \implies \|\mathscr{D}(x_n,0)\|_2 = \left\| \begin{pmatrix} 2x_n^p & 0 \\ 0 & 0 \end{pmatrix} \right\|_2 = 2x_n^p < \varepsilon \right).$$

Which shows that $(x_n)_{n\in\mathbb{N}}$ *is* $\mathscr{D}$*-converge to* 0 *in* $\mathscr{X}$ *(with respect to* $\mathbb{A}$*).*

- *Let* $\mathscr{T}$ *(resp.* ψ*) be the function defined on* $\mathscr{X}$ *(resp.* $\mathbb{A}$*) by* $\mathscr{T}(x) = \frac{x}{2}$ *(resp.* $\psi(M) = \frac{k}{1-k}.1_{\mathbb{A}}$, *with* $k \in \left]\frac{1}{4}, \frac{1}{3}\right[$ *fixed). We have* $\frac{1}{3} < \frac{k}{1-k} < \frac{1}{2}$ *and* $\|\psi\| = \frac{k}{1-k} < \frac{1}{2}$.

Let $x, y \in \mathbb{R}_+$.

- *If* $x \geqslant \frac{x}{2} \geqslant y \geqslant \frac{y}{2}$, *then*

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) = \mathscr{D}\left(\frac{x}{2}, \frac{y}{2}\right) = \begin{pmatrix} 2\left(\frac{x}{2}\right)^p & 0 \\ 0 & 3\left(\frac{y}{2}\right)^p. \end{pmatrix}$$

and $$\psi(\mathscr{D}(\mathscr{T}x,y) + \mathscr{D}(x,\mathscr{T}y)) = \frac{k}{1-k} \begin{pmatrix} 2\left(\left(\frac{x}{2}\right)^p + x^p\right) & 0 \\ 0 & 3\left(y^p + \left(\frac{y}{2}\right)^p\right) \end{pmatrix}.$$

Since,

$$\begin{aligned} \frac{k}{1-k} \times 2\left(\left(\frac{x}{2}\right)^p + x^p\right) &= \frac{k}{1-k}(1+2^p) \times 2\left(\frac{x}{2}\right)^p \\ &\geqslant \frac{1+2^p}{3} \times 2\left(\frac{x}{2}\right)^p \\ &\geqslant 2\left(\frac{x}{2}\right)^p. \end{aligned}$$

then

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) \preceq \psi\Big(\mathscr{D}(\mathscr{T}x,y) + \mathscr{D}(x,\mathscr{T}y)\Big), \tag{3.4}$$

- *If* $x \geqslant y \geqslant \frac{x}{2} \geqslant \frac{y}{2}$, *then*

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) = \mathscr{D}\left(\frac{x}{2}, \frac{y}{2}\right) = \begin{pmatrix} 2\left(\frac{x}{2}\right)^p & 0 \\ 0 & 3\left(\frac{y}{2}\right)^p. \end{pmatrix}$$

and $\psi(\mathscr{D}(\mathscr{T}x,y)+\mathscr{D}(x,\mathscr{T}y)) = \dfrac{k}{1-k}\begin{pmatrix} 2(y^p+x^p) & 0 \\ 0 & 3\Big((\frac{x}{2})^p+(\frac{y}{2})^p\Big). \end{pmatrix}$

We have also

$$\begin{aligned}\frac{k}{1-k}\times 2(y^p+x^p) &\geqslant \frac{k}{1-k}\times 2\Big((\frac{x}{2})^p+x^p\Big)\\ &\geqslant \frac{k}{1-k}(2^p+1)\times 2\Big(\frac{x}{2}\Big)^p\\ &\geqslant \frac{1}{3}(2^p+1)\times 2\Big(\frac{x}{2}\Big)^p \geqslant 2\Big(\frac{x}{2}\Big)^p.\end{aligned}$$

We obtain,

$$\mathscr{D}(\mathscr{T}x,\mathscr{T}y) \preceq \psi\Big(\mathscr{D}(\mathscr{T}x,y)+\mathscr{D}(x,\mathscr{T}y)\Big), \tag{3.5}$$

- *Similary for cases* $y \geqslant \frac{y}{2} \geqslant x \geqslant \frac{x}{2}$ *and* $y \geqslant x \geqslant \frac{y}{2} \geqslant \frac{x}{2}$.

Hence, $\mathscr{T}$ *be a* ψ*-Chatterjea contraction,* $\delta(\mathscr{D},\mathscr{T},0)<\infty$ *and* $\mathscr{T}$ *has unique fixed point* 0.

Corollary 3.1. *Let* $(\mathscr{X},\mathbb{A},\mathscr{D})$ *be a* C^**-algebra-valued generalized metric space and* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *a* $\mathscr{D}$*-Chatterjea contraction i.e. for every* $(x,y)\in\mathscr{X}^2$,

$$\mathscr{D}(\mathscr{T}x,\mathscr{T}y) \preceq a\Big(\mathscr{D}(\mathscr{T}x,y)+\mathscr{D}(x,\mathscr{T}y)\Big). \tag{3.6}$$

where $a\in\mathbb{A}'_+$*, with* $0<\|a\|<\frac{1}{2}$.
Moreover, if there exists $x_0\in\mathscr{X}$ *such that* $\delta(\mathscr{D},\mathscr{T},x_0)<\infty$*, then* $\mathscr{T}$ *has unique fixed point* ω *and the sequence* $(\mathscr{T}^n x_0)_{n\in\mathbb{N}}$ *converges to* ω.

Proof. Lemma 2.1-2 justifies that the linear map $\psi: u\mapsto au$ defined on $\mathbb{A}$ is positive. Moreover $\mathscr{T}$ is ψ-Chatterjea contraction and $\|\psi\|=\|a\|<\frac{1}{2}$. According to Theorem 3.1, $\mathscr{T}$ has unique fixed point of $\mathscr{X}$. □

Corollary 3.2. *Let* $(\mathscr{X},\mathbb{A},\mathscr{D})$ *be a* C^**-algebra-valued generalized metric space and a map* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *satisfies for every* $(x,y)\in\mathscr{X}^2$,

$$\mathscr{D}(\mathscr{T}x,\mathscr{T}y) \preceq a\Big(\mathscr{D}(\mathscr{T}x,y)+\mathscr{D}(x,\mathscr{T}y)\Big)a^*. \tag{3.7}$$

where $a \in \mathbb{A}$, with $0 < \|a\| < \frac{\sqrt{2}}{2}$.
Suppose that there exists $x_0 \in \mathscr{X}$ such that $\delta(\mathscr{D}, \mathscr{T}, x_0) < \infty$. Then, $\mathscr{T}$ has unique fixed point ω and the sequence $(\mathscr{T}^n x_0)_{n\in\mathbb{N}}$ converges to ω.

Proof. Lemma 2.1-3 justifies that the linear map $\psi : u \mapsto aua^*$ defined on $\mathscr{A}$ is positive. Moreover $\mathscr{T}$ is ψ-Chatterjea contraction and $\|\psi\| = \|a\|^2 < \frac{1}{2}$. According to Theorem 3.1, $\mathscr{T}$ has unique fixed point of $\mathscr{X}$. □

Corollary 3.3. *Let $(\mathscr{X}, \mathbb{A}, \mathscr{D})$ be a C^*-algebra-valued generalized metric space and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ a $\|\psi\|$-Chatterjea contraction i.e. for every $(x,y) \in \mathscr{X}^2$,*

$$\|\mathscr{D}(\mathscr{T}x, \mathscr{T}y)\| \leqslant \|\psi\| \Big(\|\mathscr{D}(\mathscr{T}x, y) + \mathscr{D}(x, \mathscr{T}y)\| \Big). \tag{3.8}$$

with $0 < \|\psi\| < \dfrac{1}{2}$.
Suppose that there exists $x_0 \in \mathscr{X}$ such that $\delta(\mathscr{D}, \mathscr{T}, x_0) < \infty$. Then, $\mathscr{T}$ has unique fixed point ω and the sequence $(\mathscr{T}^n x_0)_{n\in\mathbb{N}}$ converges to ω.

Proof. We define $\mathscr{D}$ as follow

$$\mathscr{D}(x,y) = \|\mathscr{D}(x,y)\|, \text{ for all } x, y \in \mathscr{X}.$$

We know that $(\mathscr{X}, \mathscr{D})$ is generalized metric space, so $(\mathscr{X}, \mathbb{R}, \mathscr{D})$ is C^*-algebra-valued generalized metric space. For each $x, y \in X$,

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) \leqslant \|\psi\|(\|\mathscr{D}(\mathscr{T}x, y) + \mathscr{D}(x, \mathscr{T}y)\|) \leqslant \|\psi\|(\mathscr{D}(\mathscr{T}x, y) + \mathscr{D}(x, \mathscr{T}y)), \tag{3.9}$$

and $\delta(\mathscr{D}, \mathscr{T}, x_0) = \delta(\mathscr{D}, \mathscr{T}, x_0) < \infty$. According to Theorem 3.1, $\mathscr{T}$ has unique fixed point of $\mathscr{X}$. □

Definition 3.1. *Let $(\mathscr{X}, \preccurlyeq)$ be a partially ordered set.*

(i) *A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be nondecreasing or order preserving if $\mathscr{T}x \preccurlyeq \mathscr{T}y$ whenever $x \preccurlyeq y$.*
(ii) *We say that a partially ordered C^*-algebra valued generalized metric space $(\mathscr{X}, \mathbb{A}, \mathscr{D})$ satisfies the $(\mathscr{P})$-property, if for every nondecreasing sequence $(x_n)_{n\geqslant 0}$ that converges to x in $\mathscr{X}$ (with respect to $\mathbb{A}$), implies that $x_n \preccurlyeq x$, for all $n \in \mathbb{N}$.*

Theorem 3.2. *Let $(\mathscr{X},\mathbb{A},\mathscr{D},\preccurlyeq)$ be a partially ordered complete C^*-algebra valued generalized metric space with the $(\mathscr{P})$-property.*
Let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be an order preserving mapping such that

$$(x\preccurlyeq y\Rightarrow \mathscr{D}(\mathscr{T}x,\mathscr{T}y)\preceq\psi\left(\mathscr{D}(\mathscr{T}x,\mathscr{T}y)+\mathscr{D}(x,\mathscr{T}y)+\mathscr{D}(\mathscr{T}x,y)\right)), \qquad (11)$$

for all $(x,y)\in\mathscr{X}^2$, where $\psi:\mathbb{A}\to\mathbb{A}$ is a linear positive function with $\|\psi\|<\frac{1}{3}$.
Suppose that there exists $x_0\in\mathscr{X}$ such that $\delta(\mathscr{D},\mathscr{T},x_0)<\infty$, $x_0\preccurlyeq\mathscr{T}x_0$ and the set $\mathscr{F}$ of fixed points of $\mathscr{T}$ is totally ordered. Then, $\mathscr{T}$ has unique fixed point ω of $\mathscr{X}$ and the sequence $(\mathscr{T}^n x_0)_{n\in\mathbb{N}}$ converges to ω.

Proof. We define the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{X}$ by

$$x_0\in\mathscr{X} \text{ and } x_{n+1}=\mathscr{T}x_n, \text{ for all } n\in\mathbb{N}.$$

As $x_0\preccurlyeq\mathscr{T}x_0$ and $\mathscr{T}$ is order preserving, then $x_n\preccurlyeq x_{n+1}$ for all $n\in\mathbb{N}$.
Let $n,m\in\mathbb{N}^*$, assume that $n<m$. According to inequality (11), we have

$$(I_d-\psi)[\mathscr{D}(x_n,x_m)]\preceq\psi\left(\mathscr{D}(x_{n-1},x_m)+\mathscr{D}(x_n,x_{m-1})\right).$$

Since $\|\psi\|<1$, then $I_d-\psi$ is invertible in the normalized algebra $\mathscr{L}_c(\mathbb{A})$ of continuous endomorphisms of $\mathbb{A}$, so $\mathscr{D}(x_n,x_m)\preceq\Psi\left(\mathscr{D}(x_{n-1},x_m)+\mathscr{D}(x_n,x_{m-1})\right)$, where $\Psi=(I_d-\psi)^{-1}\circ\psi$.
We have Ψ is continuous linear from $\mathbb{A}$ into $\mathbb{A}$ and

$$\|\Psi\|\leqslant\|I_d-\psi\|^{-1}\|\psi\|<\frac{1}{3}\|(I_d-\psi)^{-1}\|.$$

As $(I_d-\psi)^{-1}=\sum_{k=0}^{+\infty}\psi^k$, then $\|\Psi\|\leqslant\frac{1}{3}\sum_{k=0}^{+\infty}\|\psi\|^k<\frac{1}{3}\sum_{k=0}^{+\infty}\left(\frac{1}{3}\right)^k=\frac{1}{3}\times\frac{3}{2}=\frac{1}{2}$.
Let $u\in\mathbb{A}_+$, we have $\Psi(u)=(I_d-\psi)^{-1}(\psi(u))=\sum_{k=0}^{+\infty}\psi^{k+1}(u)=\lim_{n\to+\infty}\sum_{k=0}^{n}\psi^{k+1}(u)$.

Since ϕ is positive , then $\sum_{k=0}^{n}\phi^{k+1}(u)\succeq 0$, for all $n\in\mathbb{N}$. Taking into account Proposition2.1 of [19], $\mathbb{A}_+$ is closed in a C^*-algebra $\mathbb{A}$, so $\Phi(u)\succeq 0$. Hence, Φ is positive.

According to lemma 3.2 and Theorem 3.1, $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-Cauchy sequence in the $\mathscr{D}$-complete space $\mathscr{X}$, there exists $x\in\mathscr{X}$ such that $\lim_{n\to+\infty}\|\mathscr{D}(x_n,x)\|=0$, and since $(x_n)_{n\in\mathbb{N}}$ is nondecreasing, and $(\mathscr{X},\mathbb{A},\mathscr{D})$ satisfies the $(\mathscr{P})$-property, then $x_n \preceq x$, for all $n\in\mathbb{N}$. Thus, for each $n\in\mathbb{N}$,

$$(I_d-\psi)[\mathscr{D}(\mathscr{T}(x_n),\mathscr{T}(x))] \preceq \mathscr{D}(x_n,\mathscr{T}(x))+\mathscr{D}(x,\mathscr{T}(x_n)).$$

And consequently,

$$\mathscr{D}(x_{n+1},\mathscr{T}(x)) \preceq \Psi\left(\mathscr{D}(x_n,\mathscr{T}(x))+\mathscr{D}(x,\mathscr{T}(x_n))\right).$$

We have

$$\begin{aligned}\|\mathscr{D}(x_{n+1},\mathscr{T}(x))\| &\leqslant \|\Psi\|\|\mathscr{D}(x_n,\mathscr{T}(x))+\mathscr{D}(x,\mathscr{T}(x_n))\|\\ &\leqslant \|\Psi\|\left(\|\mathscr{D}(x_n,\mathscr{T}(x))\|+\|\mathscr{D}(x,\mathscr{T}(x_n))\|\right),\end{aligned}$$

Thus, $\limsup\limits_{n\to+\infty}\|\mathscr{D}(x_n,\mathscr{T}(x))\| \leqslant \|\Psi\|\limsup\limits_{n\to+\infty}\|\mathscr{D}(x_n,\mathscr{T}(x))\|$ i.e.

$$\limsup_{n\to+\infty}\|\mathscr{D}(x_n,\mathscr{T}(x))\|(1-\|\Psi\|)\leqslant 0, \text{ and since } \|\Psi\|<\frac{1}{2},$$

we get

$$\limsup_{n\to+\infty}\|\mathscr{D}(x_n,\mathscr{T}(x))\|=0.$$

According to condition (D_4), we obtain

$$0\preceq \mathscr{D}(x,\mathscr{T}(x))\preceq \limsup_{n\to+\infty}\|\mathscr{D}(x_n,\mathscr{T}(x))\|.c=0$$

Hence, $x=\mathscr{T}(x)$.

Uniqueness: Let $y\in\mathscr{X}$ such that $\mathscr{T}(y)=y$, since $\mathscr{F}$ is totally ordered, we can assume that $x\preceq y$.

Then

$$(I_d-\psi)[\mathscr{D}(\mathscr{T}(x),\mathscr{T}(y)] \preceq \mathscr{D}(\mathscr{T}(x),y)+\mathscr{D}(\mathscr{T}(y),x)$$

in other words $\mathscr{D}(x,y)\preceq 3\psi(\mathscr{D}(x,y))$. Then $\|\mathscr{D}(x,y)\|\leqslant 3\|\psi\|\|\mathscr{D}(x,y)\|$. Therefore

$$\|\mathscr{D}(x,y)\|(1-3\|\psi\|)\leqslant 0. \text{ Since } \|\psi\|<\frac{1}{3}, \text{ then } \mathscr{D}(x,y)=0. \text{ Hence, } x=y.$$

□

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[2] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71–76.

[3] S.K. Chatterjea, Fixed point theorem. C. R. Acad. Bulgare Sci. 25 (1972), 727–730.

[4] D. Dumitrescu, A. Pitea, Fixed point theorems on almost (φ,θ)-contractions in Jleli-Samet generalized metric spaces, Mathematics. 10 (2022), 4239. https://doi.org/10.3390/math10224239.

[5] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), 61. https://doi.org/10.1186/s13663-015-0312-7.

[6] K. Chaira, A. Eladraoui, M. Kabil, et al. Kannan fixed point theorem on generalized metric space with a graph, Appl. Math. Sci. 13 (2019), 263–274. https://doi.org/10.12988/ams.2019.9226.

[7] Z. Ma, L. Jiang, C^*-Algebra-valued b-metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), 222. https://doi.org/10.1186/s13663-015-0471-6.

[8] R. Mustafa, S. Omran, Q.N. Nguyen, Fixed point theory using ψ contractive mapping in C^*-algebra valued b-metric space, Mathematics. 9 (2021), 92. https://doi.org/10.3390/math9010092.

[9] M. Asim, M. Imdad, C^*-algebra valued symmetric spaces and fixed point results with an application, Korean J. Math. 28 (2020), 17–30. https://doi.org/10.11568/KJM.2020.28.1.17.

[10] K. Chaira, A. Eladraoui, M. Kabil, A. Kamouss, Fisher fixed point results in generalized metric spaces with a graph, Int. J. Math. Math. Sci. 2020 (2020), 7253759. https://doi.org/10.1155/2020/7253759.

[11] K. Chaira, A. Eladraoui, M. Kabil, A. Kamouss, Common fixed point for some generalized contractive mappings in a modular metric space with a graph, Fixed Point Theory Algorithms Sci. Eng. 2021 (2021), 4. https://doi.org/10.1186/s13663-021-00690-8.

[12] K. Chaira, M. Kabil, A. Kamouss, Fixed point results for $\mathscr{C}$-contractive mappings in generalized metric spaces with a graph, J. Function Spaces 2021 (2021), 8840347. https://doi.org/10.1155/2021/8840347.

[13] K. Chaira, M. Kabil, A. Kamouss, Common fixed points in generalized metric spaces with a graph, J. Math. 2019 (2019), 2846315. https://doi.org/10.1155/2019/2846315.

[14] S. Chandok, D. Kumar, C. Park, C^*-algebra-valued partial metric space and fixed point theorems, Proc. Math. Sci. 129 (2019), 37. https://doi.org/10.1007/s12044-019-0481-0.

[15] M.E. Ege, C. Alaca, C^*-algebra-valued S-metric spaces, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. 67 (2018), 165–177. https://doi.org/10.1501/commua1_0000000871.

[16] C. Shen, L. Jiang, Z. Ma, C^*-algebra-valued G-metric spaces and related fixed-point theorems, J. Function Spaces 2018 (2018), 3257189. https://doi.org/10.1155/2018/3257189.

[17] G.J. Murphy, C^*-algebras and operator theory, Academic Press, 2014.

[18] K.R. Davidson, C^*-algebras by example, American Mathematical Society, Providence, 1996.

[19] W. Tapanyo, W. Ratiphaphongthon, A. Arunchai, Completion of C^*-algebra-valued metric spaces, Thai J. Math. 20 (2022), 1119–1136.