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COMMON FIXED POINTS OF TWO GENERALIZED QUASI-NONEXPANSIVE MAPPINGS BY A NEW THREE STEP ITERATION SCHEME

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Abstract. In this paper, we introduce a generalized iteration scheme for approximating the common fixed points of two generalized quasi-nonexpansive mappings and prove strong convergence results in uniformly convex Banach spaces. These results generalize and extend some known results.

Keywords: Generalized iteration scheme, uniformly convex Banach space, asymptotically and generalized quasi-nonexpansive mapping, common fixed point, strong convergence.

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1. Introduction

It is well known that the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [3] who proved that every asymptotically nonexpansive self mapping of nonempty closed bounded and convex subset of a uniformly convex Banach

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space has fixed point. Since 1972, the weak and strong convergence problems of iterative sequences with errors for asymptotically nonexpansive type mappings in the Hilbert spaces and Banach spaces setting have been studied by many authors.

Let C be a nonempty subset of a real Banach space E. A self mapping $T : C \to C$ is called uniformly $L-$Lipschitzian if there exists some positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

A self mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

A self mapping $T : C \to C$ is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [0, \infty)$, $k_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

Let $F(T)$ denotes the set of all fixed points of a mapping T. If $F(T) \neq \phi$, then T is called asymptotically quasi-nonexpansive, if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - p\| \leq (1 + k_n)\|x - p\|$$

for all $x \in C$, $p \in F(T)$ and all $n \geq 1$.

In 1995, Liu [5] introduced the concept of Ishikawa iteration scheme with errors by the sequence $\{x_n\}_{n=1}^{\infty}$ defined as follows:

$$x_1 \in C$$

$$(1) \qquad \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n + u_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n + v_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1] and $\{u_n\}$, $\{v_n\}$ are bounded sequences in E satisfying the following conditions

$$\sum_{n=0}^{\infty} \|u_n\| < \infty \text{ and} \sum_{n=0}^{\infty} \|v_n\| < \infty \tag{2}$$

If $\beta_n = 0$ and $v_n = 0$, then the Ishikawa iteration process (1) reduces to the Mann iteration procedure with errors in the sense of Liu which is defined recursively as follows:

$$x_1 \in C$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n \tag{3}$$

with $\{\alpha_n\} \subset [0,1]$ satisfying appropriate conditions and $\{u_n\}$ satisfying the following condition $\sum_{n=0}^{\infty} \|u_n\| < \infty$.

Clearly, the Ishikawa iteration process (1) with null sequences $\{u_n\}$and$\{v_n\}$ reduces to the usual Ishikawa iteration procedure and similarly the Mann iteration process (3) with null sequence $\{u_n\}$ reduces to the usual Mann iteration procedure.

A more satisfactory concept of Ishikawa and Mann iterative processes with errors was given by Xu[6] as follows.

Let C be a nonempty convex subset of a Banach space E and a mapping $T : C \to C$. The sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by

$$x_1 \in C$$

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n \\ y_n &= \alpha_n^{'} x_n + \beta_n^{'} T x_n + \gamma_n^{'} v_n, \quad n \geq 1 \end{aligned} \tag{4}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n^{'}\}, \{\beta_n^{'}\}$ and $\{\gamma_n^{'}\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'} = 1$ is called the Ishikawa iteration scheme with errors. If $\beta_n^{'} = 0 = \gamma_n^{'}$ and $\alpha_n^{'} = 1$, then the Ishikawa iteration scheme (4) reduces to the Mann iteration scheme with errors defined as follows:

$$x_1 \in C$$

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 1 \tag{5}$$

with $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1, \quad \alpha_n + \beta_n + \gamma_n = 1,$ and $\{u_n\}$ is bounded sequence in C.

In 2004, Fukhar-ud-din and Khan [2] studied an iterative process with errors in the sense of Liu for two asymptotically nonexpansive mappings in uniformly convex Banach spaces. In 2006, Jeong and Kim [4] studied the Ishikawa iterative scheme with error members for a pair of asymptotically nonexpansive mappings S, T defined as follows :

$$x_1 \in C$$

$$\begin{aligned} x_{n+1} &= \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n \\ y_n &= \alpha_n' T^n x_n + \beta_n' x_n + \gamma_n' v_n, \quad n \geq 1 \end{aligned} \tag{6}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}$ and $\{\gamma_n'\}$ are sequences in [0,1] such that $0 < \delta \leq \alpha_n, \alpha_n' \leq (1-\delta) < 1$ and $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1$.

Let C be a nonempty closed convex subset of E then a self mapping $f : C \to C$ is said to be a contraction on C, if there exists a constant $\alpha \in (0,1)$ such that

$$\|fx - fy\| \leq \alpha \|x - y\|$$

for all $x, y \in C$.

Now we introduce a new three step generalized iterative scheme $\{x_n\}_{n=1}^{\infty}$ associated with two asymptotically quasi-nonexpansive mappings $S, T : C \to C$ as follows:

$$x_1 = x \in C$$

$$\begin{aligned} z_n &= \alpha_n' S^n x_n + \beta_n' f(x_n) + \gamma_n' v_n \\ y_n &= \alpha_n T^n z_n + \beta_n g(x_n) + \gamma_n u_n \\ x_{n+1} &= a_n w + (1 - a_n) y_n, \quad n \in \mathbb{N} \end{aligned} \tag{7}$$

where x_1 and w are given arbitrary but fixed, usually different, elements of C and $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n' \quad a_n \in (0,1)$.

If $a_n = 0$ and $f = g = I$, then iterative scheme (7) reduces to the two step iterative scheme (6).

The aim of this paper is to prove the strong convergence of the iterative scheme (7) to a common fixed point of S and T under certain restrictions.

2. Preliminaries

To prove our main results we need the following definitions and lemmas in the sequel.

Definition 2.1 [11] *Let E be a real Banach space, C be a nonempty subset of E and $F(T)$ denotes the set of fixed point of T. A self mapping $T : C \to C$ is said to be generalized quasi-nonexpansive with respect to $\{k_n\}$, if there exists a sequence $\{k_n\} \subset [0, 1)$ with $\lim_{n\to\infty} k_n = 0$ such that*

$$\|T^n x - p\| \leq \|x - p\| + k_n\|x - T^n x\|$$

for all $x \in C$ and $p \in F(T)$ $\quad n \geq 1$.

Remark 2.1 *If $k_n = 0$ for all $n \geq 1$, then the generalized quasi-nonexpansive mapping reduces to the usual asymptotically quasi-nonexpansive mapping.*

Lemma 2.1 [12] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad n \geq 1$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

We can easily prove the following lemma.

Lemma 2.2 *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\sum_{n=1}^{\infty} a_n b_n < \infty$ and $\sum_{n=1}^{\infty} a_n b_n c_n < \infty$.*

3. Main results

We begin with the following lemma, which plays an important role to prove our main result.

Lemma 3.1. *Let C be a nonempty convex subset of a normed space E. Let $S, T : C \to C$ be generalized quasi-nonexpansive mappings and self maps $f, g : C \to C$ are contraction on C. Let for all $x \in C, p \in F(T) \cap F(S)$ and for all $n \geq 1$*

$$\|T^n x - p\| \leq \|x - p\| + k_n \|x - T^n x\|$$
$$\|S^n x - p\| \leq \|x - p\| + k_n^{'} \|x - S^n x\|$$

where $\{k_n\}, \{k_n^{'}\}$ are sequences in [0,1) with $\sum_{n=1}^{\infty} k_n < \infty, \sum_{n=1}^{\infty} k_n^{'} < \infty$. Let $\{x_n\}$ be the sequence defined iteratively by (7), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n^{'}\}, \{\beta_n^{'}\}$ and $\{\gamma_n^{'}\}$ are in [0,1] and $\{a_n\} \in (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'}$. Also assume that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$. If $F(T) \cap F(S) \neq \phi$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T) \cap F(S)$.

Proof. Since S and T are generalized quasi-nonexpansive mappings, then there exists sequences $\{k_n\}, \{k_n^{'}\}$ in [0,1) with $\sum_{n=1}^{\infty} k_n < \infty, \sum_{n=1}^{\infty} k_n^{'} < \infty$ such that for all $x \in C, p \in F(T) \cap F(S)$for all $n \geq 1$

$$\|T^n x - p\| \leq \|x - p\| + k_n \|x - T^n x\|$$
$$\|S^n x - p\| \leq \|x - p\| + k_n^{'} \|x - S^n x\|$$

Let $p \in F(T) \cap F(S)$. Since $\{u_n\}$and$\{v_n\}$ are bounded sequences in C, so without loss of generality we may assume that there exists $M > 0$ such that

$$\max\{\|w - p\|, \sup_{n\geq 1} \|u_n - p\|, \sup_{n\geq 1} \|v_n - p\|\} \leq M$$

Now

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|a_n w+(1-a_n)y_n-p\| \\
&\leq (1-a_n)\|y_n-p\|+a_n\|w-p\| \\
&\leq \|y_n-p\|+a_nM \\
&\leq \|\alpha_nT^nz_n+\beta_ng(x_n)+\gamma_nu_n-p\|+a_nM \\
&\leq \alpha_n\|T^nz_n-p\|+\beta_n\|g(x_n)-p\|+\gamma_n\|u_n-p\|+a_nM \\
&\leq \alpha_n[\|z_n-p\|+k_n\|z_n-T^nz_n\|]+\beta_n\|g(x_n)-g(p)\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(\frac{1+k_n}{1-k_n})\|z_n-p\|+k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+\frac{2k_n}{1-k_n})\|z_n-p\|+k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)\|z_n-p\|+k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)\|\alpha_n^{'}S^nx_n+\beta_n^{'}f(x_n)+\gamma_n^{'}v_n-p\| \\
&\quad +k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)[\alpha_n^{'}\|S^nx_n-p\|+\beta_n^{'}\|f(x_n)-p\|+\gamma_n^{'}\|v_n-p\|] \\
&\quad +k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)[\alpha_n^{'}\{\|x_n-p\|+k_n^{'}\|x_n-S^nx_n\|\}+\beta_n^{'}\|f(x_n)-f(p)\| \\
&\quad +\gamma_n^{'}\|v_n-p\|]+k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)[\alpha_n^{'}(\frac{1+k_n^{'}}{1-k_n^{'}})\|x_n-p\|+k^{'}\beta_n^{'}\|x_n-p\|+\gamma_n^{'}M] \\
&\quad +k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)[\alpha_n^{'}(1+\frac{2k_n^{'}}{1-k_n^{'}})\|x_n-p\|+k^{'}\beta_n^{'}\|x_n-p\|+\gamma_n^{'}M] \\
&\quad +k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n(1+l_n)[\alpha_n^{'}(1+l_n^{'})\|x_n-p\|+k^{'}\beta_n^{'}\|x_n-p\|+\gamma_n^{'}M] \\
&\quad +k\beta_n\|x_n-p\|+[\gamma_n+a_n]M \\
&\leq \alpha_n\alpha_n^{'}(1+l_n)(1+l_n^{'})\|x_n-p\|+k^{'}\alpha_n\beta_n^{'}(1+l_n)\|x_n-p\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_n \gamma_n^{'}(1+l_n)M + k\beta_n\|x_n - p\| + [\gamma_n + a_n]M \\
&\le [\alpha_n \alpha_n^{'}(1+l_n)(1+l_n^{'}) + k^{'}\alpha_n\beta_n^{'}(1+l_n) + k\beta_n]\|x_n - p\| \\
&+ [\gamma_n + a_n + \alpha_n\gamma_n^{'}(1+l_n)]M \\
&\le [\alpha_n\alpha_n^{'} + k^{'}\alpha_n\beta_n^{'} + k\beta_n](1+l_n)(1+l_n^{'})\|x_n - p\| \\
&+ [\gamma_n + a_n + \alpha_n\gamma_n^{'}(1+l_n)]M
\end{aligned}
$$

That is

$$
\begin{aligned}
\|x_{n+1} - p\| &\le (1+l_n)(1+l_n^{'})\|x_n - p\| + [a_n + \gamma_n + \gamma_n^{'}(1+l_n)]M \\
&\le (1+\delta_n)\|x_n - p\| + b_nM
\end{aligned} \tag{8}
$$

where $\delta_n = (1+l_n)(1+l_n^{'})$ and $b_n = a_n + \gamma_n + \gamma_n^{'}(1+l_n)$.
Using the fact that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty} k_n < \infty, \sum_{n=1}^{\infty} k_n^{'} < \infty$ and $\sum_{n=1}^{\infty}\gamma_n^{'} < \infty$ and applying Lemma 2.1 and Lemma 2.2, $\lim_{n\to\infty}\|x_n - p\|$ exists.

This completes the proof.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space E. Let S and T be generalized quasi-nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in [0,1) such that $\sum_{n=1}^{\infty} k_n < \infty$. Let $F(T)\cap F(S) \ne \phi$ and let $\{x_n\}$ be the sequence defined as in Lemma 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$, where $d(x, F(T)\cap F(S)) = \inf_{p\in F(T)\cap F(S)}\|x - p\|$.*

Proof. From Lemma 3.1, we have

$$\|x_{n+1} - p\| \le (1+l_n)(1+l_n^{'})\|x_n - p\| + [a_n + \gamma_n + \gamma_n^{'}(1+l_n)]M$$

which implies that

$$d(x_{n+1}, F(T)\cap F(S)) \le (1+l_n)(1+l_n^{'})d(x_n, F(T)\cap F(S)) + [a_n + \gamma_n + \gamma_n^{'}(1+l_n)]M$$

By Lemma 2.1, we know that $\lim_{n\to\infty} d(x_n, F(T)\cap F(S))$ exists.
Because $\liminf_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$, then $\lim_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$. Next we prove that $\{x_n\}$is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrary chosen. Since

$\lim_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$, there exists a positive integer n_0 such that $d(x_n, F(T) \cap F(S)) < \frac{\epsilon}{4}$, for all $n \geq n_0$. In particular, $\inf\{\|x_{n_0} - p\| : p \in F(T) \cap F(S)\} < \frac{\epsilon}{4}$. Thus, there must exists $p^* \in F(T) \cap F(S)$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$.

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< 2(\frac{\epsilon}{2}) = 2\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Thus the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $x_n \to q^*$ as $n \to \infty$. Now we prove that q^* is a common fixed point of T and S. Indeed, we know that the set $F(T) \cap F(S)$ is closed. From the continuity of $d(x, F(T) \cap F(S)) = 0$ with $\lim_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$ and $\lim_{n\to\infty} x_n = q^*$, we get

$$d(q^*, F(T) \cap F(S)) = 0$$

and so $q^* \in F(T) \cap F(S)$. This completes the proof of the theorem.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Banach space E. Let T and S be asymptotically quasi-nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in [0,1) such that $\sum_{n=1}^{\infty} k_n < \infty$. Let $F(T) \cap F(S) \neq \phi$, and let $\{x_n\}$ be the sequence defined as in Lemma 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\liminf_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$, where $d(x, F(T) \cap F(S)) = \inf_{p\in F(T)\cap F(S)} \|x - p\|$.*

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Banach space E. Let T and S be asymptotically nonexpansive mappings with a sequence $\{k_n\}_{n=1}^{\infty}$ in [0,1) such that $\sum_{n=1}^{\infty} k_n < \infty$. Let $F(T) \cap F(S) \neq \phi$, and let $\{x_n\}$ be the sequence defined as in Lemma 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\liminf_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$, where $d(x, F(T) \cap F(S)) = \inf_{p\in F(T)\cap F(S)} \|x - p\|$.*

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