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QUALITATIVE STUDY OF A FIXED POINT FOR A FEEDBACK CONTROL PROBLEM OF CAPUTO-VIA RIEMANN-LIOUVILLE FRACTIONAL ORDER DIFFERENTIAL EQUATION

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Abstract. In this research, we investigate the qualitative study of a fixed point for a feedback control problem of Caputo-Via Riemann-Liouville fractional order differential equation, in two classes $L_1(I)$ and $C(I)$ in a bounded interval $I = [0,T]$. The main tool applied in this work is the technique Schauder fixed point Theorem. In both cases we present a sufficient conditions for a unique solution and the continuous dependence on some functions. Additionally, we delve into the study of Hyers-Ulam stability. Finally, some examples are provided to verify our investigation.

Keywords: differential equation; existence of solution; Hyers–Ulam stability; continuous dependence; fractional order.

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1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. There has been a significant development in the study of fractional differential equations and inclusions in recent

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years; see the monographs of Kilbas et al. [1], Lakshmikantham et al. [16], Podlubny [11] and the survey by Agarwal et al. [15].

The feedback control have a wide range of practical applications across various disciplines, including but not limited to: Viscoelasticity, electrochemistry, control systems, porous media, electromagnetism, etc. (see [18, 14, 10, 11, 12]). These diverse applications highlight the importance of constrained problems in accurately capturing and addressing real-world phenomena across various scientific and engineering disciplines.

In this study, our focus is on examining of the fractional order differential equation,

$$ {}^{R}D^{\alpha}(x(t)-\eta(t)) = f(t,x(t)),\ t\in I,\ x(0)=\eta(0), \tag{1} $$

with the feedback control

$$ \eta(t) = g(t,x(t).\int_0^t h(s,\ x(s))ds), \tag{2} $$

where ${}^{R}D^{\alpha}$ is the refers to the fractional derivative of Riemann–Liouville of order $\alpha\in(0,\ 1)$. Our aim here is study the existence of solution $x\in L_1(I)$ and $x\in C(I)$ of the problem (1)-(2). The main tools in our study is applying Schauder fixed point Theorem [11]. Furthermore, the continuous dependence of the unique solution on the functions f and g will be proved. The Hyers – Ulam stability of the problem (1)-(2) will be given.

2. SOLVABILITY IN $L_1(I)$

Let $L_1 = L_1(I)$, be the class of Lebesgue integrable functions, with the standard norm

$$ \|x\|_1 = \int_0^T |x(t)|dt. $$

Take into account the following assumptions:

(i) $g : I\times R\to R$ is Carathéodory function [13] and there exist a bounded measurable function $a : I\to R$ and nonnegative constant b_1 such that

$$ |g(t,x)| \le |a(t)| + b_1|x|\ \forall\ t\in I,\ x\in R. $$

(ii) $f : I\times R\to R$ is Carathéodory function [13] and there exist a bounded measurable function $m : I\to R$ and nonnegative constant b_2 such that

$$ |f(t,x)| \le |m(t)| + b_2|x|\ \forall\ t\in I,\ x\in R. $$

(iii) $h : I \times R \to R$ is Carathéodory function [13] and there exist a bounded measurable function $v : I \to R$ and nonnegative constant b_3 such that

$$|h(t,x)| \leq |v(t)| + b_3|x| \ \forall\, t \in I,\ x \in R.$$

(iv) There exists a positive root r of the algebraic equation

$$b_1\, b_3\, r^2 + ((\frac{b_2 T^\alpha}{\Gamma(\alpha+1)} + \ b_1\|v\|_1) - 1)\ r + \frac{T^\alpha\ \|m\|_1}{\Gamma(\alpha+1)} + \|a\|_1 \ = \ 0.$$

Now, we have the following lemma.

Lemma 1. *The problem (1)-(2) is equivalent to the integral equation*

$$x(t) = g(t,x(t)).\int_0^t h(s,\ x(s))ds) + I^\alpha f(t,x(t)),\ t \in I. \tag{3}$$

Proof. Let $x \in L_1(I)$ be a solution of the problem (1)-(2), then we have

$$\begin{aligned} \frac{d}{dt} I^{1-\alpha}(x(t) - \eta(t)) &= f(t,x(t)) \\ I^{1-\alpha}(x(t) - \eta(t)) &= I\, f(t,x(t)), \end{aligned}$$

then from the properties of the fractional calculus and $x(0) = \eta(0)$, we obtain

$$\begin{aligned} I(x(t) - \eta(t)) &= I^{\alpha+1}\, f(t,x(t)) \\ x(t) - \eta(t) &= I^\alpha\, f(t,x(t)), \end{aligned}$$

then

$$x(t) = \eta(t) + I^\alpha f(t,x(t)), \tag{4}$$

substituting by (2) in (4), we obtain (3).

Conversely, let $x \in L_1(I)$ be a solution of (3). Substituting by (2) in (3), we obtain

$$\begin{aligned} x(t) - \eta(t) &= I^\alpha f(t,x(t)) \\ I^{1-\alpha}(x(t) - \eta(t)) &= I^{1-\alpha} I^\alpha f(t,x(t)). \end{aligned}$$

By differentiation, we get

$$\frac{d}{dt} I^{1-\alpha}(x(t) - \eta(t)) = \frac{d}{dt} I\, f(t,x(t)),$$

then

$$^{R}D^{\alpha}(x(t)-\eta(t)) = f(t,x(\phi(t))).$$

Now, we have the following existences Theorem.

Theorem 1. *Assume that $(i)-(iv)$ be satisfied, then the integral equation (3) has at least one solution $x \in L_1(I)$.*

Proof. Let the set

$$Q_r = \{x \in L_1(I) : \|x\|_1 \leq r\}.$$

Define the operator F by

$$Fx(t) = g(t,x(t).\int_0^t h(s,\ x(s))ds) + I^{\alpha} f(t,x(t)).$$

Now, let $x \in Q_r$, then

$$\begin{aligned} |Fx(t)| &= \left| g(t,x(t).\int_0^t h(s,\ x(s))ds) + I^{\alpha} f(t,x(t)) \right| \\ &\leq \left| g(t,x(t).\int_0^t h(s,\ x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \right| \\ &\leq |a(t)| + b_1 \left(|x(t)|.\int_0^t (|v(s)| + b_3|x(s)|)ds \right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|m(s)| + b_2|x(s)|)ds, \end{aligned}$$

then

$$\begin{aligned} \int_0^T |Fx(t)|dt &\leq \int_0^T |a(t)|dt + b_1\ (\|v\|_1 + b_3\ r).\int_0^T |x(t)|dt \\ &+ \int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|m(s)| + b_2|x(s)|)dsdt \\ &\leq \|a\|_1 + b_1\ (\|v\|_1 + b_3\ r).r + \int_0^T (|m(s)| + b_2|x(s)|) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dtds \\ &\leq \|a\|_1 + b_1\ (\|v\|_1\ r + b_3\ r^2) + \int_0^T (|m(s)| + b_2|x(s)|) \frac{T^{\alpha}}{\Gamma(\alpha+1)} ds \\ &\leq \|a\|_1 + b_1\ \|v\|_1\ r + b_1\ b_3\ r^2 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|m\|_1 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} b_2\ r\ =\ r, \end{aligned}$$

that is

$$\|Fx\|_1 \leq \|a\|_1 + b_1\ \|v\|_1\ r + b_1\ b_3\ r^2 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|m\|_1 + \frac{T^{\alpha}}{\Gamma(\alpha+1)} b_2\ r\ =\ r.$$

Hence the operator F maps the ball Q_r into itself and the class of functions $\{Fx\}$ is uniformly bounded on Q_r.

Now, let $x \in Q_r$, then

$$\begin{aligned}
\|(Fx)_h - (Fx)\|_1 &= \int_0^T |(Fx(s))_h - (Fx(s))| ds \\
&= \int_0^T \frac{1}{h}\left| \int_t^{t+h} |(Fx(\theta))d\theta - (Fx(s))\right| ds \\
&\leq \int_0^T \frac{1}{h}\int_t^{t+h} |Fx(\theta) - Fx(s)|\, d\theta ds \\
&\leq \int_0^T \frac{1}{h}\int_t^{t+h} \left| \left(g(\theta, x(\theta). \int_0^t x(\theta)d\theta) + I^\alpha f(\theta, x(\theta)) \right) \right. \\
&- \left. \left(g(s, x(s). \int_0^t x(s)ds) + I^\alpha f(s, x(s)) \right) \right| d\theta ds.
\end{aligned}$$

Since $F \in L_1(I)$, then

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h} \left| \left(g(\theta, x(\theta). \int_0^t x(\theta)d\theta) - g(s, x(s). \int_0^t x(s)ds) \right) \right. \\
&- \left. \left(I^\alpha f(\theta, x(\theta)) - I^\alpha f(s, x(s)) \right) \right| d\theta ds \to 0,\ as h \to 0.
\end{aligned}$$

This means that $Fx(t)_h \to (Fx)$ uniformly in $L_1(I)$. Thus the class of functions $\{Fx\}$ is relatively compact [11]. Hence F is compact operator.

Now, let $\{x_n\} \subset Q_r$, and $x_n \to x$, then

$$Fx_n(t) = g(t, x_n(t). \int_0^t h(s,\ x_n(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s))ds$$

and

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty} g(t, x_n(t). \int_0^t h(s, x_n(s))ds) + \lim_{n\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s))ds.$$

Applying Lebesgue dominated convergence Theorem [11], then from our assumptions we get

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= g(t,\ \lim_{n\to\infty} x_n(t). \int_0^t \lim_{n\to\infty} h(s, x_n(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\ \lim_{n\to\infty} x_n(s))ds \\
&= g(t, x(t). \int_0^t h(s,\ x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds \ = \ Fx(t).
\end{aligned}$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous. Now, by Schauder fixed point Theorem [11] there exists at least one solution $x \in L_1(I)$ of (3). Consequently there exists at least one solution $x \in L_1(I)$ of the problem (1)-(2).

2.1. Uniqueness of the solution. Now, replace the assumption (i) and (ii) by $(i)^*$ and $(ii)^*$ as follows:

$(i)^*$ $g : I \times R \to R$ is measurable in $t \in I$ $\forall x \in R$ and satisfies Lipschitz condition,

$$|g(t,x) - g(t,y)| \leq b_1 \, |x-y| \; \forall \, t \in I, \; x, \; y \in R. \tag{5}$$

$(ii)^*$ $f : I \times R \to R$ is measurable in $t \in I$ $\forall x \in R$ and satisfies Lipschitz condition,

$$|f(t,x) - f(t,y)| \leq b_2 \, |x-y| \; \forall \, t \in I, \; x,y \in R. \tag{6}$$

$(iii)^*$ $f : I \times R \to R$ is measurable in $t \in I$ $\forall x \in R$ and satisfies Lipschitz condition,

$$|h(t,x) - h(t,y)| \leq b_3 \, |x-y| \; \forall \, t \in I, \; x,y \in R. \tag{7}$$

So, we have the following Lemma.

Lemma 2. *The assumption $(i)^*$, $(ii)^*$ and $(iii)^*$ implies the assumption (i), (ii) and (iii).*

Proof. From (5), let $y = 0$, then we have

$$|g(t,x)| - |g(t,0)| \leq |g(t,x) - g(t,0)| \leq b_1|x|,$$

$$|g(t,x)| \leq |g(t,0)| + b_1|x|$$

and

$$|g(t,x)| \leq |a(t)| + b_1|x|, \; where \; |a(t)| = \sup_{t \in I} |g(t,0)|.$$

Also, from (6) and (7), we get

$$|f(t,x)| \leq |m(t)| + b_2|x|, \; where \; |m(t)| = \sup_{t \in I} |f(t,0)|$$

$$|h(t,x)| \leq |v(t)| + b_3|x|, \; where \; |v(t)| = \sup_{t \in I} |f(t,0)|$$

Theorem 2. *Let the assumptions $(i)^*$, $(ii)^*$ and $(iii)^*$ be satisfied. If*

$$\left(b_1 \; \|v\|_1 + 2 \; b_1 b_3 \; r + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} \right) < 1,$$

then the solution of the problem (1)-(2) is unique.

Proof. Let $x_1,\ x_2$ be two solutions in Q_r of (3), then

$$\begin{aligned}
|x_2(t)-x_1(t)| &= \Big|g(t,x_2(t).\int_0^t h(s,x_2(s))ds)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_2(s))ds \\
&- g(t,x_1(t).\int_0^t h(s,x_1(s))ds)-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_1(s))ds\Big| \\
&\leq \Big|g(t,x_2(t).\int_0^t h(s,x_2(s))ds)-g(t,x_1(t).\int_0^t h(s,\ x_2(s))ds)\Big| \\
&+ \Big|g(t,x_1(t).\int_0^t h(s,x_2(s))ds)-g(t,x_1(t).\int_0^t h(s,x_1(s))ds)\Big| \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f(s,x_2(s))-f(s,x_1(s))\Big|ds \\
&\leq b_1|x_2(t)-x_1(t)|.\int_0^t |h(s,x_2(s))|ds+b_1\ b_3\ |x_1(t)|.\int_0^t |x_2(s)-x_1(s)|ds \\
&+ b_2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_2(s)-x_1(s)|ds \\
&\leq b_1|x_2(t)-x_1(t)|.(\|v\|_1+b_3\ r)+b_1\ b_3\ |x_1(t)|\|x_2-x_1\|_1 \\
&+ b_2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_2(s)-x_1(s)|ds,
\end{aligned}$$

then

$$\begin{aligned}
\int_0^T |x_2(t)-x_1(t)|dt &\leq b_1\ (\|v\|_1+b_3\ r)\int_0^T |x_2(t)-x_1(t)|dt+b_1\ b_3\ \|x_2-x_1\|_1\int_0^T |x_1(t)|dt \\
&+ b_2\int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_2(s)-x_1(s)|ds \\
&\leq b_1\ (\|v\|_1+b_3\ r)\|x_2-x_1\|_1+b_1\ b_3\ \|x_2-x_1\|_1\ r \\
&+ b_2\int_0^T (|x_2(s)-x_1(s)|)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dtds \\
&\leq b_1\ (\|v\|_1+b_3\ r)\ \|x_2-x_1\|_1+b_1b_3\ \|x_2-x_1\|_1\ r \\
&+ b_2\int_0^T |x_2(s)-x_1(s)|\frac{T^\alpha}{\Gamma(\alpha+1)}ds \\
&\leq b_1\ (\|v\|_1+b_3\ r)\|x_2-x_1\|_1+b_1b_3\ \|x_2-x_1\|_1\ r \\
&+ b_2\ \frac{T^\alpha}{\Gamma(\alpha+1)}\|x_2-x_1\|_1,
\end{aligned}$$

Hence

$$\|x_2-x_1\|_1\left(1-\left(b_1\ \|v\|_1+2\ b_1b_3\ r+\frac{b_2T^\alpha}{\Gamma(\alpha+1)}\right)\right) \leq 0,$$

then $x_1 = x_2$ and the solution of (3) is unique. Consequently the problem (1)-(2) is unique.

2.2. Continuous dependence.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied for f, f^*, g and g^*. Then the unique solution $x \in L_1(I)$ depends continuously on the functions f and g in the sense that*

$$\forall \varepsilon > 0,\ \exists\ \delta(\varepsilon) \text{ such that}$$

$$max\{\|g(t,x(t)) - g^*(t,x(t))\|_1,\ \|f(t,x(t)) - f^*(t,x(t))\|_1\} < \delta, then\ \|x - x^*\|_1 < \varepsilon.$$

where x^* be a solution of

$$x^*(t) = g^*(t,x^*(t). \int_0^t h(s,x^*(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s,x^*(s))ds.$$

Proof.

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big| g(t,x(t). \int_0^t h(s,x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- g^*(t,x^*(t). \int_0^t h(s,x^*(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s,x^*(s))ds \Big| \\
&\leq \Big| g(t,x(t). \int_0^t h(s,x(s))ds) - g^*(t,x(t). \int_0^t h(s,x(s))ds) \Big| \\
&+ \Big| g^*(t,x(t). \int_0^t h(s,x(s))ds) - g^*(t,x^*(t). \int_0^t h(s,x(s))ds) \Big| \\
&+ \Big| g^*(t,x^*(t). \int_0^t h(s,x(s))ds) - g^*(t,x^*(t). \int_0^t h(s,x^*(s))ds) \Big| \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f(s,x(s)) - f^*(s,x(s)) \Big| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f^*(s,x(s)) - f^*(s,x^*(s)) \Big| ds,
\end{aligned}$$

then

$$\begin{aligned}
\int_0^T |x(t) - x^*(t)|dt &\leq \int_0^T \Big| g(t,x(t). \int_0^t h(s,x(s))ds) - g^*(t,x(t). \int_0^t h(s,x(s))ds) \Big| dt \\
&+ b_1\ (\|v\|_1 + b_3\ r) \int_0^T |x(t) - x^*(t)|dt + b_1\ b_3 \|x - x^*\| \int_0^T |x^*(t)|dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f(s,x(s)) - f^*(s,x(s)) \Big| dsdt \\
&+ \int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f^*(s,x(s)) - f^*(s,x^*(s)) \Big| dsdt \\
&\leq \delta + b_1 \, (\|v\|_1 + b_3 \, r) \, \|x - x^*\|_1 + b_1 \, b_3 \, r \, \|x - x^*\|_1 \\
&+ \delta \, \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} \, \|x - x^*\|_1.
\end{aligned}
$$

Hence

$$
\|x - x^*\|_1 \leq \frac{\left(1 + \frac{T^\alpha}{\Gamma(\alpha+1)}\right) \delta}{1 - \left(b_1 \, \|v\|_1 + 2 \, b_1 b_3 \, r + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)} = \varepsilon.
$$

2.3. Hyers-Ulam stability.

Definition 1. *Let the solution $x \in L_1(I)$ of the problem (1)-(2) be exists, then the problem (1)-(2) is Hyers - Ulam stable if $\forall \varepsilon > 0$, $\exists \, \delta(\varepsilon)$ such that for any $\delta - approximate$ solution x_s satisfies,*

$$
\left| \, {}^R D^\alpha (x_s(t) - \eta_s(t)) - f(t, x_s(t)) \, \right| < \delta, \tag{8}
$$

implies $\|x - x_s\|_1 < \varepsilon$.

Theorem 4. *Let the assumptions of Theorem 2 be satisfied, then the problem (1)-(2) is Hyers - Ulam stable.*

Proof. From (8), we have

$$
\begin{aligned}
-\delta &\leq {}^R D^\alpha (x_s(t) - \eta_s(t)) - f(t, x_s(t)) \leq \delta \\
-\delta^* = -\delta \, I^\alpha &\leq x_s(t) - \eta_s(t) - I^\alpha f(t, x_s(t)) \leq \delta \, I^\alpha = \delta^* \\
-\delta^* &\leq x_s(t) - \left(g(t, x_s(t). \int_0^t h(s, x_s(s)) ds) + I^\alpha f(t, x_s(t)) \right) \leq \delta^*.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|x(t) - x_s(t)| &= \left| g(t, x(t). \int_0^t h(s, x(s)) ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds - x_s(t) \right| \\
&\leq \left| g(t, x(t). \int_0^t h(s, x(s)) ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds \right.
\end{aligned}
$$

$$
\begin{aligned}
&- \quad g(t,x_s(t).\int_0^t h(s,x_s(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(s))ds\Big| \\
&+ \quad \Big|x_s(t) - (g(t,x_s(t).\int_0^t h(s,x_s(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(s))ds)\Big| \\
&\leq \quad \Big|g(t,x(t).\int_0^t h(s,x(s))ds) - g(t,x_s(t).\int_0^t h(s,x(s))ds)\Big| \\
&+ \quad \Big|g(t,x_s(t).\int_0^t h(s,x(s))ds) - g(t,x_s(t).\int_0^t h(s,x_s(s))ds)\Big| \\
&+ \quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(s)) - f(s,x_s(s))|ds \\
&+ \quad \Big|x_s(t) - (g(t,x_s(t).\int_0^t h(s,x_s(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(s))ds)\Big| \\
&\leq \quad b_1\ (\|v\|_1 + b_3\ r)|x(t) - x_s(t)|.\int_0^t |x(s)|ds + b_1\ b_3\ |x_s(t)|.\int_0^t |x(s) - x_s(s)|ds \\
&+ \quad b_2\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - x_s(s)|ds + \delta^*,
\end{aligned}
$$

then

$$
\begin{aligned}
\int_0^T |x(t) - x_s(t)|dt \quad &\leq \quad b_1\ (\|v\|_1 + b_3\ r) \int_0^T |x(t) - x_s(t)|dt + b_1\ b_3\ \|x - x_s\|_1 \int_0^T |x_s(t)|ds \\
&+ \quad b_2\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - x_s(s)|ds + \ \delta^*\ \int_0^T dt
\end{aligned}
$$

then

$$
\|x - x_s\|_1 \quad \leq \quad b_1\ (\|v\|_1 + b_3\ r)\|x - x_s\|_1 + b_1 b_3\ r\|x - x_s\|_1\ +\ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\|x - x_s\|_1 +\ \delta^*\ T.
$$

Hence

$$
\|x - x_s\|_1 \quad \leq \quad \frac{\delta^*\ T}{1 - \left(b_1\ \|v\|_1 + 2\ b_1 b_3\ r + \ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)} = \varepsilon.
$$

3. SOLVABILITY IN $C(I)$

Let $C = C(I)$, be the class of continuous functions with the standard norm

$$
\|x\| = \sup_{t \in I} |x(t)|.
$$

Take into account the following assumptions:

(iiv) There exists a positive root r of the algebraic equation

$$b_1\, b_3\, T\, r^2 + (b_1\, \|v\|\, T\, + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} - 1)\, r + \frac{T^\alpha\, \|m\|}{\Gamma(\alpha+1)} + \|a\| \;=\; 0.$$

Now, we have the following existences theorem.

Theorem 5. *Assume that* $(i)^*, (ii), (iii)$ *and* (iiv) *be satisfied, then the integral equation (3) has at least one solution* $x \in C(I)$.

Proof. Let the set

$$Q_r = \{x \in C(I) : \|x\| \le r\}$$

and define the operator F by

$$Fx(t) = g(t,x(t).\int_0^t h(s,x(s))ds) + I^\alpha f(t,x(t)).$$

Now, let $x \in Q_r$, then

$$\begin{aligned}
|Fx(t)| &= \left| g(t,x(t).\int_0^t h(s,x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \right| \\
&\le |a(t)| + b_1 |x(t).\int_0^t h(s,x(s))ds| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|m(s)| + b_2 |x(s)|)ds \\
&\le \|a\| + b_1 \|x\|.\int_0^t (|v(s)| + \; b_3 |x(s)|)ds \; + \frac{\|m\| T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\; \|x\| \\
&\le \|a\| + b_1\, r(\|v\|\; T + b_3\, r\, T) + \frac{\|m\| T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\; r \;=\; r,
\end{aligned}$$

then

$$\|Fx\| \;\le\; \|a\| + b_1\, \|v\|\, T\, r + b_1\, b_3\, T\, r^2 + \frac{\|m\| T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\; r \;=\; r.$$

Hence the operator F maps the ball Q_r into itself and the class of functions $\{Fx\}$ is uniformly bounded on Q_r.

Now, let $x \in Q_r$ and $t_1,\, t_2 \in I$ such that $t_1 \le t_2$, $|t_2 - t_1| < \delta$ and defined $\theta_g(\delta)$ [11, 5] as

$\theta_g(\delta) = \sup_{x \in Q_r} \{|g(t_2,x(t)) - g(t_1,x(t))| : \; t_1,t_2 \in I,\; t_1 < t_2,\; |t_2 - t_1| < \delta,\; ||x|| \le r\}$, then we have

$$|Fx(t_2) - Fx(t_1)| \;=\; \left| g(t_2,x(t_2).\int_0^{t_2} h(s,x(s))ds) + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \right.$$

$$
\begin{aligned}
&- \quad g(t_1,x(t_1)).\int_0^{t_1} h(s,x(s))ds) - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\Big| \\
&\leq \quad |g(t_2,x(t_2)).\int_0^{t_2} h(s,x(s))ds) - g(t_1,x(t_1)).\int_0^{t_1} h(s,x(s))ds)| \\
&+ \quad \Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\Big| \\
&\leq \quad |g(t_2,x(t_2)).\int_0^{t_2} h(s,x(s))ds) - g(t_1,x(t_2)).\int_0^{t_2} h(s,x(s))ds)| \\
&+ \quad |g(t_1,x(t_2)).\int_0^{t_2} h(s,x(s))ds) - g(t_1,x(t_1)).\int_0^{t_2} h(s,x(s))ds)| \\
&+ \quad |g(t_1,x(t_1)).\int_0^{t_2} h(s,x(s))ds) - g(t_1,x(t_1)).\int_0^{t_1} h(s,x(s))ds)| \\
&+ \quad \Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \quad \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\Big| \\
&\leq \quad \theta_g(\delta) + b_1|x(t_2)-x(t_1)|.\int_0^{t_2} |h(s,x(s))|ds + b_1|x(t_1)|.\int_{t_1}^{t_2} |h(s,x(s))|ds \\
&+ \quad \int_0^{t_1} \frac{(t_2-s)^{1-\alpha}-(t_1-s)^{1-\alpha}}{\Gamma(\alpha)(t_1-s)^{1-\alpha}(t_2-s)^{1-\alpha}} |f(s,x(s))|ds \\
&+ \quad \int_{t_1}^{t_2} \frac{1}{\Gamma(\alpha)(t_2-s)^{1-\alpha}} |f(s,x(s))|ds.
\end{aligned}
$$

This means that the class of functions $\{Fx\}$ is equicontinuous on Q_r and by Arzela-Ascoli Theorem [11] the class of functions $\{Fx\}$ is relatively compact, then the operator F is compact. Now, let $\{x_n\} \subset Q_r$ and $x_n \to x$, then

$$
\lim_{n\to\infty} Fx_n(t) \quad = \quad \lim_{n\to\infty} g(t,x_n(t)).\int_0^t h(s,x_n(s))ds) + \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_n(s))ds.
$$

Applying Lebesgue dominated convergence Theorem [11], then from our assumptions we get

$$
\begin{aligned}
\lim_{n\to\infty} Fx_n(t) \quad &= \quad g(t,\ \lim_{n\to\infty} x_n(t).\int_0^t \lim_{n\to\infty} h(s,x_n(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\ \lim_{n\to\infty} x_n(s))ds \\
&= \quad g(t,x(t)).\int_0^t h(s,x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \ = \ Fx(t).
\end{aligned}
$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous. Now, by Schauder fixed point Theorem [11] there exists at least one solution $x \in C(I)$ of (3). Consequently there exists at least one solution $x \in C(I)$ of the problem (1)-(2).

3.1. Uniqueness of the solution.

Theorem 6. *Let the assumptions $(i)^*, (ii)^*$ and $(iii)^*$ be satisfied. If*

$$\left(b_1\ \|v\|\ T + 2\ b_1\ b_3\ r\ T + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right) < 1,$$

then the solution of the problem (1)-(2) is unique.

Proof. Let $x_1,\ x_2$ be two solutions in Q_r of (3), then

$$\begin{aligned} |x_2(t) - x_1(t)| &= \Big| g(t, x_2(t). \int_0^t h(s, x_2(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_2(s))ds \\ &- g(t, x_1(t). \int_0^t h(s, x_1(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_1(s))ds \Big| \\ &\leq b_1 \|x_2 - x_1\|\ (\|v\|\ T + b_3\ r\ T) + b_1\ r(b_3\ \|x_2 - x_1\|\ T) + b_2\ \frac{T^\alpha}{\Gamma(\alpha+1)} \|x_2 - x_1\|. \end{aligned}$$

Hence

$$\|x_2 - x_1\| \left(1 - \left(b_1\ \|v\|\ T + 2\ b_1\ b_3\ r\ T + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)\right) \quad \leq \quad 0,$$

then $x_1 = x_2$ and the solution of (3) is unique. Consequently the problem (1)-(2) is unique.

3.2. Continuous dependence.

Theorem 7. *Let the assumptions of Theorem 6 be satisfied for $f,\ f^*,\ g$ and g^*. Then the unique solution $x \in C(I)$ depends continuously on the functions f and g in the sense that*

$$\forall \varepsilon > 0,\ \exists\ \delta(\varepsilon) \text{ such that}$$

$$max\{|g(t, x(t)) - g^*(t, x(t))|,\ |f(t, x(t)) - f^*(t, x(t))|\} < \delta,$$

$$\text{then } \|x - x^*\| < \varepsilon.$$

where x^* is the solution of

$$x^*(t) = g^*(t, x^*(t). \int_0^t h(s, x^*(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(s))ds.$$

Proof.

$$|x(t) - x^*(t)| \quad = \quad \Big| g(t, x(t). \int_0^t h(s, x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds$$

$$
\begin{aligned}
- &\quad g^*(t,x^*(t).\int_0^t h(s,x^*(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s,x^*(s))ds \Big| \\
\leq &\quad \delta + b_1 \|x-x^*\| \, (\|v\| \, T + b_3 \, r \, T) \, + b_1 \, b_3 \, r \, \|x-x^*\| \, T \\
+ &\quad \delta \, \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} \, \|x-x^*\|.
\end{aligned}
$$

Hence

$$
\|x-x^*\| \quad \leq \quad \frac{\left(2+\frac{T^\alpha}{\Gamma(\alpha+1)}\right) \delta}{1-\left(b_1 \, \|v\| \, T + 2 \, b_1 \, b_3 \, r \, T + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)} = \varepsilon.
$$

3.3. Hyers-Ulam stability.

Definition 2. [6, 9, 7] *Let the solution $x \in C(I)$ of the problem (1)-(2) be exists, then the problem (1)-(2) is Hyers - Ulam stable if $\forall \varepsilon > 0$, $\exists \, \delta(\varepsilon)$ such that for any $\delta -$approximate solution x_s satisfies,*

(9)
$$
\left| {}^R D^\alpha (x_s(t) - \eta_s(t)) - f(t, x_s(t)) \right| < \delta,
$$

implies $\|x - x_s\| < \varepsilon$.

Theorem 8. *Let the assumptions of Theorem 6 be satisfied, then the problem (1)-(2) is Hyers - Ulam stable.*

Proof. From (9), we have

$$
\begin{aligned}
-\delta &\leq {}^R D^\alpha (x_s(t) - \eta_s(0)) - f(t, x_s(t)) \leq \delta \\
-\delta^* = -\delta \, I^\alpha &\leq x_s(t) - \eta_s(0) - I^\alpha f(t, x_s(t)) \leq \delta \, I^\alpha = \delta^* \\
-\delta^* &\leq x_s(t) - \left(g(t, x_s(t).\int_0^t h(s, x_s(s))ds) + I^\alpha f(t, x_s(t)) \right) \leq \delta^*.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|x(t) - x_s(t)| &= \left| g(t,x(t).\int_0^t h(s,x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - x_s(t) \right| \\
&\leq \Big| g(t,x(t).\int_0^t h(s,x(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \quad g(t,x_s(t).\int_0^t h(s,x_s(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(s))ds \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left| x_s(t) - (g(t, x_s(t). \int_0^t h(s, x_s(s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_s(s))ds)) \right| \\
&\leq \quad \left| g(t, x(t). \int_0^t h(s, x(s))ds) - g(t, x_s(t). \int_0^t h(s, x(s))ds) \right| \\
&+ \quad \left| g(t, x_s(t). \int_0^t h(s, x(s))ds) - g(t, x_s(t). \int_0^t h(s, x_s(s))ds) \right| \\
&+ \quad \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, x_s(s))| ds + \delta^* \\
&\leq \quad b_1 |x(t) - x_s(t)|. \int_0^t |h(s, x(s))| ds + b_1 \, b_3 |x_s(t)|. \int_0^t |x(s) - x_s(s)| ds \\
&+ \quad b_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - x_s(s)| ds + \delta^*,
\end{aligned}
$$

then

$$
\|x - x_s\| \quad \leq \quad b_1 \|x - x_s\| \; (\|v\| \; T + b_3 \; r \; T) \; + b_1 \; b_3 \|x - x_s\| \; r \; T \; + \; \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} \|x - x_s\| + \; \delta^*.
$$

Hence

$$
\|x - x_s\| \quad \leq \quad \frac{\delta^*}{1 - \left(b_1 \; \|v\| \; T + 2 \; b_1 \; b_3 \; r \; T + \; \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} \right)} = \varepsilon.
$$

4. EXAMPLES

Example 1.

Consider the following

$$
{}^{R}D^{\alpha}(x(t) - \eta(t)) = \sqrt{\frac{t}{144}} + \frac{1}{2}|x(t)|, \; x(0) = \eta(0), \; t \in [0, \frac{1}{2}], \tag{10}
$$

where

$$
\eta(t) = \frac{t}{6} + \frac{x(t)}{12} \int_0^t (\frac{s}{2} + \frac{x(s)}{3}) ds. \tag{11}
$$

Set

$$
\begin{aligned}
f(t, x) &= \sqrt{\frac{t}{144}} + \frac{1}{2}|x(t)| \\
g(t, x) &= \frac{t}{6} + \frac{x(t)}{12} \int_0^t (\frac{s}{2} + \frac{x(s)}{3}) ds \\
h(t, x) &= \frac{t}{2} + \frac{x(t)}{3}.
\end{aligned}
$$

Putting

$$\|a\| = \frac{1}{6},\ \|m\| \ = \frac{1}{96},\ b_1 = \frac{1}{12},\ b_2 = \frac{1}{2},\ \alpha = \ \frac{1}{2},\ T = \frac{1}{2}$$
$$\|v\| = \frac{1}{2},\ b_3 = \frac{1}{3}$$

and r satisfied

$$b_1\ b_3\ T\ r^2 + (b_1\ \|v\|\ T\ + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} - 1)\ r + \frac{T^\alpha\ \|m\|}{\Gamma(\alpha+1)} + \|a\|\ =\ 0.$$
$$r\ =\ 0.303606193,$$

then the problem (10)-(11) has at least one solution $x \in C[0, \frac{1}{2}]$

and we can find that

$$b_1\ \|v\|\ T + 2\ b_1\ b_3\ r\ T + \ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\ = 0.4282091191 < 1,$$

then the problem (10)-(11) has a unique solution.

Example 2.

Taking into account the equation

$$ {}^R D^\alpha (x(t) - \eta(t)) = \frac{t}{4} + \frac{1}{8}|x(t)|,\ x(0) = \eta(0),\ t \in [0, \frac{1}{3}], \tag{12}$$

where

$$\eta(t) = \frac{t}{5} + \frac{x(t)}{12} \int_0^t (\frac{s}{5} + \frac{x(s)}{3}) ds. \tag{13}$$

Set

$$f(t,x) = \frac{t}{4} + \frac{1}{8}|x(t)|$$
$$g(t,x) = \frac{t}{5} + \frac{x(t)}{12} \int_0^t (\frac{s}{5} + \frac{x(s)}{3}) ds$$
$$h(t,x) = \frac{t}{5} + \frac{x(t)}{3}$$

Putting

$$\|a\| = \frac{1}{5},\ \|m\| \ = \frac{1}{4},\ b_1 = \frac{1}{12},\ b_2 = \frac{1}{8},\ \alpha = \ \frac{1}{2},\ T = \frac{1}{3}$$
$$\|v\| \ = \frac{1}{45},\ b_3 = \frac{1}{3}$$

and r satisfied

$$b_1\ b_3\ r^2 + ((\frac{b_2 T^\alpha}{\Gamma(\alpha+1)} + \ b_1\|v\|_1) - 1)\ r + \frac{T^\alpha\ \|m\|_1}{\Gamma(\alpha+1)} + \|a\|_1\ =\ 0.$$

$$r\ =\ 0.400700060,$$

then the problem (12)-(13) has at least one solution $x \in L_1[0, \frac{1}{3}]$

and we can find that

$$b_1\ \|v\|_1 + 2\ b_1 b_3\ r + \ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} = 0.1055467186 < 1,$$

then the problem (12)-(13) has a unique solution.

5. Conclusions

In this investigation, the qualitative study of a fixed point for a feedback control problem of Caputo-Via Riemann-Liouville fractional order differential equation. We discussed two cases: In the first case, we studied the existence of solution for the constraint problem (1)-(2) in the class $L_1(I)$, then we studied the existence of solution in the class $C(I)$. In two cases, we proved the continuous dependence of the unique solution on the functions f and g. Moreover, we thoroughly investigated the Hyers–Ulam stability of our problem. Finally, we given an examples are provided to illustrate our results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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