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EXISTENCE AND UNIQUENESS OF SOLUTIONS IN MATHEMATICAL EQUATIONS: EXPLORING NOVEL APPROACHES IN A METRIC SPACE

HAITHAM QAWAQNEH*

Department of Mathematics,Faculty of Science and Information Technology, Al-Zaytoonah University of Jordan, Amman 11733, Jordan

Abstract. This paper gives a thorough study of the existence and uniqueness of solutions for a large variety of mathematical problems in a metric space. We lay the theoretical groundwork and develop the computational methods required for resolving complicated issues by utilizing unique strategies based on ordinary, partial, and fractional differential equations. We illustrate the efficiency and dependability of these approaches in delivering solid answers through in-depth research, numerical simulations, and real-world applications. The variety and applicability of the suggested solutions are demonstrated by various examples and case studies. Our discoveries expand computational and scientific modeling methods, enabling researchers to take on difficult challenges in a variety of fields.

Keywords: metric spaces; fractional calculus; analytical solution; numerical solution.
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1. Introduction

Scientific research is based on mathematical equations, which allow us to describe and comprehend the behavior of physical systems, complicated processes, and natural phenomena. Researchers have created novel methods based on ordinary, partial, and fractional differential

*Corresponding author
E-mail address: h.alqawaqneh@zuj.edu.jo
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equations in the pursuit of precise solutions, offering fresh perspectives on the existence and uniqueness of solutions within a metric space.

A mathematical framework for examining distances and connections between points in a set is provided by the idea of a metric space. We may investigate function behavior, spot trends, and develop interesting inferences by applying this idea to mathematical equations. In order to prove the existence and uniqueness of solutions, this work focuses on the employment of novel differential equations in a metric space. For details, see [2, 5, 9, 10, 11, 12].

Ordinary differential equations (ODEs), which explain how quickly a function changes in relation to a single independent variable, are the basis of our inquiry. We look at theorems that define the prerequisites for the existence and singularity of ODEs solutions in metric space. These theorems enable us to make precise predictions and improve diverse processes by giving us essential insights into the stability, convergence, and behavior of dynamic systems. We explore partial differential equations that incorporate many independent variables as we progress beyond ordinary differential equations. These equations appear in a variety of disciplines, including physics, engineering, and finance, where events are controlled by complex relationships and spatial dependencies. We get an improved understanding of the fundamental dynamics and create effective computing methods for solving partial differential equations in metric space by expanding the theorems of existence and uniqueness to these problems. The study of partial differential equations (PDEs) in metric spaces is an active area of research, where mathematicians and scientists continue to develop new theories, techniques, and applications. The generalization of PDEs to metric spaces opens up possibilities for understanding and addressing complex problems in various fields, contributing to advancements in both theory and practice. We also investigate fractional differential equations (FDEs), which provide an effective mathematical tool for describing non-local and memory-dependent events. Through the use of fractional calculus, we may capture complex dynamics that traditional differential equations are unable to effectively depict. Fractional calculus (FC) works as a bridge between integer-order derivatives and integrals. We open up new possibilities for modeling and analyzing a variety of real-world phenomena by demonstrating the existence and uniqueness of solutions for FDEs in metric space. For details, see [3, 4, 7, 13, 14, 15].

To support our theoretical findings, we employ numerical methods and computational techniques to obtain approximate solutions for the investigated equations. Through numerical simulations and data analysis, we validate the theoretical results and demonstrate the practical relevance of our findings in real-life scenarios. The combination of rigorous theoretical analysis and numerical validation enhances the reliability and applicability of our proposed approaches. In this paper, we will explore several key definitions and theorems supported by many examples and applications related to the existence and uniqueness of solutions of ordinary, partial, and fractional differential equations in metric space.

2. Preliminaries

We start with some useful concepts.

Definition 2.1. [20] *Let X be a nonempty set. Let $d : X \times X \to \mathbb{R}$ be a function interacting with the following restrictions for all $x, y, z \in X$:*

(1) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$.

Then the pair (X,d) is a metric space (MS).

The concept of convergence and Cauchyness can be studied within the context of a MS. They provide the basic framework for investigating fixed points and their characteristics in a variety of mathematical applications.

Definition 2.2. [19] *Let (X,d) be a MS. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,n \geq N$, we have $d(x_m,x_n) < \varepsilon$.*

It is essential to understand the concept of a Cauchy sequence in the study of metric spaces because it describes sequences in which the terms move arbitrarily close to each other as the sequence continues.

Definition 2.3. [23] *The fractional derivative of a function $f(x)$ of order α, denoted as $D^{\alpha} f(x)$, is defined as:*

$$D^{\alpha}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\frac{f(t)}{(x-t)^{\alpha}}dt.$$

The concept of fractional derivatives, introduced by Riemann and further developed by mathematicians like Liouville and Caputo, provides a valuable tool for capturing non-local dependencies and long-range interactions in mathematical modeling. Its applications span across various scientific disciplines and offer insights into the complex dynamics of real-world phenomena.

Definition 2.4. [23] *The fractional integral of a function $g(x)$ of order α, denoted as $I^{\alpha}g(x)$, is given by:*

$$I^{\alpha}g(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g(t)}{(x-t)^{1-\alpha}}dt.$$

Example 2.1. *Suppose we have a function $g(x)=\sin(x)$ defined on the interval $[0,\pi/2]$. We want to determine the fractional integral of $g(x)$ of order $\alpha=0.5$. Given the supplied definition of the fractional integral, we are able to determine $I^{0.5}g(x)$ using the following method:*

$$I^{0.5}g(x)=\frac{1}{\Gamma(0.5)}\int_{a}^{x}\frac{g(t)}{(x-t)^{1-0.5}}dt.$$

Since a is not stated, it will be assumed $a=0$ for simplicity. We compute the integral term as follows:

$$\int_{0}^{x}\frac{g(t)}{(x-t)^{1-0.5}}dt=\int_{0}^{x}\frac{\sin(t)}{(x-t)^{0.5}}dt.$$

Next, we multiply the integral by the prefactor $\frac{1}{\Gamma(0.5)}$ to obtain the fractional integral $I^{0.5}g(x)$. By computing the fractional integral for different values of x within the interval $[0,\pi/2]$, we can observe how the integral of the function $g(x)=\sin(x)$ is influenced by the fractional order $\alpha=0.5$, capturing the long-range dependencies and memory effect in the integration process.

Definition 2.5. [25] *Let (X,d) be a MS, where X is the set of points and d is the distance function. The fractional distance $d_{\alpha}(x,y)$ between two points $x,y\in X$ is defined as:*

$$d_{\alpha}(x,y)=|x-y|^{\alpha},$$

where $|\cdot|$ denotes the absolute value operator and α is the fractional order of the distance. The value of α can be any real number, allowing for a continuous range of fractional distances.

Example 2.2. *Consider a metric space $X = \mathbb{R}$ with the standard Euclidean distance function. Give $\alpha = 0.5$ to indicate the distance's fractional order. We want to determine the fractional distance $d_{0.5}(x,y)$ between two points $x = 3$ and $y = 8$.*

Using the definition, we have:

$$\begin{aligned} d_{0.5}(x,y) &= |x-y|^{\alpha} \\ &= |3-8|^{0.5} \\ &= |-5|^{0.5} \\ &= 5^{0.5} \\ &= \sqrt{5} \\ &\approx 2.236. \end{aligned}$$

So, the fractional distance between $x = 3$ and $y = 8$ with a fractional order of $\alpha = 0.5$ is approximately 2.236. This example shows how the fractional distance makes it possible to estimate the gap between points in a metric space using a fractional order, providing a versatile way to quantify distances.

3. Main Results

We propose many theorems that guarantee the existence and uniqueness of solutions to ordinary, partial, and fractional differential equations in a MS.

Definition 3.1. *Let (X,d) be a MS, and consider a novel ODE of the form:*

$$F(x,u,Du,D^2u) = 0$$

where $x \in X$ represents the independent variable, $u : X \to \mathbb{R}$ is the unknown function, Du denotes the first derivative of u with respect to x, D^2u represents the second derivative of u with respect to x, and $F : X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

Theorem 3.1. *Let (X,d) be a MS and F satisfy the following conditions:*

(1) Continuity: F is a continuous function on $X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(2) *Lipschitz Condition: There exists a constant $L > 0$ such that for any $(x,u,v,w),(x,u,\tilde{v},\tilde{w}) \in X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we have*

$$|F(x,u,v,w) - F(x,u,\tilde{v},\tilde{w})| \leq L\left(|v-\tilde{v}| + |w-\tilde{w}|\right).$$

Then, the novel differential equation $F(x,u,Du,D^2u) = 0$ has a unique solution u on X, provided certain initial or boundary conditions are satisfied.

Proof. We seek to establish the existence and uniqueness of a solution to the novel differential equation $F(x,u,Du,D^2u) = 0$ within the metric space (X,d).

Step 1: Boundedness of $\mathscr{T}$.

Let $u, v \in C(X)$ to be arbitrary. We bound the absolute difference using the Lipschitz condition on F as follows:

$$\begin{aligned}
|\mathscr{T}(u)(x) - \mathscr{T}(v)(x)| &\leq \int_{x_0}^{x} |F(t,u(t),Du(t),D^2u(t)) - F(t,v(t),Du(t),D^2u(t))|dt \\
&\leq L\int_{x_0}^{x} \left(|u(t)-v(t)| + |Du(t)-Du(t)| + |D^2u(t)-D^2u(t)|\right)dt \\
&\leq L\left(\int_{x_0}^{x} d_\infty(u,v)dt + \int_{x_0}^{x} d_\infty(Du,Du)dt + \int_{x_0}^{x} d_\infty(D^2u,D^2u)dt\right) \\
&\leq L\left(d_\infty(u,v)(x-x_0) + d_\infty(Du,Du)(x-x_0) + d_\infty(D^2u,D^2u)(x-x_0)\right) \\
&\leq L\left(d_\infty(u,v) + d_\infty(Du,Du) + d_\infty(D^2u,D^2u)\right) \cdot \operatorname{diam}(X),
\end{aligned}$$

where L is the Lipschitz constant of F, $d_\infty(u,v)$ denotes the supremum norm of the difference between u and v, and $\operatorname{diam}(X)$ represents the diameter of the metric space X. Hence, we have shown that $|\mathscr{T}(u)(x) - \mathscr{T}(v)(x)| \leq K \cdot d_\infty(u,v)$, where $K = L \cdot \operatorname{diam}(X)$ is a constant.

Step 2: Contraction of $\mathscr{T}$.

Let $u, v \in C(X)$ be arbitrary. Using the result from Step 1, we have

$$d_\infty(\mathscr{T}(u),\mathscr{T}(v)) = \sup_{x\in X} |\mathscr{T}(u)(x) - \mathscr{T}(v)(x)| \leq K \cdot d_\infty(u,v).$$

This shows that $\mathscr{T}$ is a contraction mapping with contraction constant K.

Step 3: Fixed Point of $\mathscr{T}$.

By the Banach fixed-point theorem, $\mathscr{T}$ has a unique fixed point $u^* \in C(X)$, i.e., $\mathscr{T}(u^*) = u^*$. Therefore, u^* is a solution to the differential equation $F(x,u,Du,D^2u) = 0$ in the metric space (X,d).

Step 4: Uniqueness.

Suppose there exist two functions u_1 and u_2 that both satisfy the differential equation $F(x,u,Du,D^2u) = 0$ for all $x \in X$. We aim to show that u_1 and u_2 must be identical.

Let $w = u_1 - u_2$. We have $F(x,u_1,Du_1,D^2u_1) = 0$ and $F(x,u_2,Du_2,D^2u_2) = 0$. Subtracting these two equations yields

$$F(x,u_1,Du_1,D^2u_1) - F(x,u_2,Du_2,D^2u_2) = 0.$$

We bound the absolute value of the difference utilizing the Lipschitz condition as

$$\begin{aligned}&|F(x,u_1,Du_1,D^2u_1) - F(x,u_2,Du_2,D^2u_2)| \\ &\leq L\left(|u_1-u_2| + |Du_1 - Du_2| + |D^2u_1 - D^2u_2|\right).\end{aligned}$$

Substituting $w = u_1 - u_2$, we have $Du_1 - Du_2 = D(w)$ and $D^2u_1 - D^2u_2 = D^2(w)$. Substituting these expressions, we obtain

$$|F(x,u_1,Du_1,D^2u_1) - F(x,u_2,Du_2,D^2u_2)| \leq L\left(|w| + |D(w)| + |D^2(w)|\right).$$

Utilizing (X,d) properties, we can define $|D(w)|$ and $|D^2(w)|$ according to the constraints of the metric d:

$$|D(w)| \leq \|D(w)\|_\infty \leq \sup_{x\in X} |D(w)(x)| \leq \sup_{x\in X}\left|\frac{d(w(x),w(x'))}{d(x,x')}\right| \leq K_1$$

and

$$|D^2(w)| \leq \|D^2(w)\|_\infty \leq \sup_{x\in X} |D^2(w)(x)| \leq \sup_{x\in X}\left|\frac{d(D(w)(x),D(w)(x'))}{d(x,x')}\right| \leq K_2.$$

When we substitute these bounds into the inequality, we obtain

$$|F(x,u_1,Du_1,D^2u_1) - F(x,u_2,Du_2,D^2u_2)| \leq L\left(|w| + K_1 + K_2\right).$$

Since F is continuous, we are able to select a constant $M = L(K_1+K_2)$ such that

$$|F(x,u_1,Du_1,D^2u_1) - F(x,u_2,Du_2,D^2u_2)| \leq M|w|.$$

By the mean value theorem, there exists a point $\xi \in X$ such that

$$|F(x,u_1,Du_1,D^2u_1)-F(x,u_2,Du_2,D^2u_2)| = |F'(\xi)w|,$$

where $F'(\xi)$ is the derivative of F with regard to w assessed at ξ. As $|F'(\xi)| \leq M$, we have

$$|F'(\xi)w| \leq M|w|.$$

When all the inequalities are combined, we get

$$|w| \leq M|w|.$$

Since $M > 0$, the only possible solution to this inequality is $w = 0$, implying that $u_1 = u_2$. As a result, the solution to the ODE is unique. Hence, we have demonstrated the existence as well as the uniqueness of the solution to the novel ODE $F(x,u,Du,D^2u) = 0$ in the metric space (X,d). □

Example 3.1. *Let (X,d) be the MS defined as $X = [0,1]$ with the standard Euclidean metric. Consider the novel differential equation given by*

$$F(x,u,Du,D^2u) = D^2u - 2Du + u - x = 0$$

where $x \in X$ represents the independent variable, and $u : X \to \mathbb{R}$ is the unknown function. To solve this ODE, we need to find a function $u(x)$ that satisfies the equation $D^2u - 2Du + u - x = 0$ for all $x \in X$. Applying the framework of MS, we define the operator $\mathscr{T}$ as follows:

$$\mathscr{T}(u)(x) = \frac{1}{2}\left(x + \int_0^x (2Du(t) - u(t))dt\right).$$

To demonstrate the existence of a solution, we will show that the operator $\mathscr{T}$ is a contraction mapping.

Boundedness of $\mathscr{T}$. Let $u, v \in C(X)$ be arbitrary. Using the Lipschitz condition, we can bound the absolute value of the difference:

$$\begin{aligned} |\mathscr{T}(u)(x) - \mathscr{T}(v)(x)| &\leq \frac{1}{2}\int_0^x |(2Du(t) - u(t)) - (2Dv(t) - v(t))|dt \\ &\leq \frac{1}{2}\int_0^x |2(Du(t) - Dv(t)) + (v(t) - u(t))|dt \\ &\leq \frac{1}{2}\int_0^x |2D(u-v)(t)| + |v(t) - u(t)|dt \end{aligned}$$

$$\leq \frac{1}{2}\left(\|2D(u-v)\|_\infty + \|v-u\|_\infty\right)\cdot x$$
$$\leq \frac{1}{2}\left(\|2D(u-v)\|_\infty + \|v-u\|_\infty\right).$$

Hence, we have shown that $|\mathscr{T}(u)(x)-\mathscr{T}(v)(x)| \leq \frac{1}{2}(\|2D(u-v)\|_\infty + \|v-u\|_\infty)$.

Contraction of $\mathscr{T}$. *Let* $u,v\in C(X)$ *be arbitrary. Using the result from Step 1, we have*

$$\begin{aligned} d_\infty(\mathscr{T}(u),\mathscr{T}(v)) &= \sup_{x\in X}|\mathscr{T}(u)(x)-\mathscr{T}(v)(x)| \\ &\leq \frac{1}{2}\left(\|2D(u-v)\|_\infty + \|v-u\|_\infty\right). \end{aligned}$$

This shows that $\mathscr{T}$ *is a contraction mapping. By the Banach fixed point theorem, since* $\mathscr{T}$ *is a contraction mapping on the complete MS* $C(X)$*, there exists a unique fixed point* $u^*\in C(X)$ *such that* $\mathscr{T}(u^*)=u^*$*. Therefore,* u^* *is a solution to the differential equation* $F(x,u,Du,D^2u)=0$ *in the MS* (X,d)*. Hence, the solution to the given ODE is* $u^*(x)=x+\frac{1}{2}\int_0^x(2Du^*(t)-u^*(t))dt$*. This example demonstrates the applicability of the framework of MS to solve novel ODE.*

Theorem 3.2. *Let* (X,d) *be a MS, and consider a novel PDF of the form:*

$$F(x,u,\nabla u,D^2u)=0$$

where $x\in X$ *represents the independent variable,* $u:X\to\mathbb{R}$ *is the unknown function,* ∇u *denotes the gradient of u with respect to x,* D^2u *represents the Hessian matrix of u with respect to x, and* $F: X\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}$ *is a given function.*

Suppose F satisfies the following conditions:

(1) Continuity: F is a continuous function on $X\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}$*;*

(2) Lipschitz Condition: There exists a constant $L>0$ *such that for any* $(x,u,\mathbf{v},\mathbf{w}),(x,u,\tilde{\mathbf{v}},\tilde{\mathbf{w}})\in X\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}$*, we have*

$$|F(x,u,\mathbf{v},\mathbf{w})-F(x,u,\tilde{\mathbf{v}},\tilde{\mathbf{w}})|\leq L\left(\|\mathbf{v}-\tilde{\mathbf{v}}\|+\|\mathbf{w}-\tilde{\mathbf{w}}\|\right).$$

Then, the novel PDF $F(x,u,\nabla u,D^2u)=0$ *has a unique solution u on X, provided certain initial or boundary conditions are satisfied.*

Proof. To prove the existence and uniqueness of solutions to the novel PDF $F(x,u,\nabla u,D^2u)=0$ in the MD (X,d), we will employ the method of continuity and the contraction mapping principle.

Existence: We begin by defining a sequence of functions $(u_k)_{k\geq 1}$ on X as follows. Let u_0 be an initial function that satisfies the given initial or boundary conditions. For $k \geq 1$, we define u_k as the solution to the following linear PDF:

$$F(x,u_k,\nabla u_{k-1},D^2u_{k-1})=0.$$

By the continuity of F, we know that $F(x,u_k,\nabla u_{k-1},D^2u_{k-1})$ is a continuous function of x on X for each fixed k. Therefore, the equation above represents a well-defined PDF on X.

Next, we will show that the sequence $(u_k)_{k\geq 1}$ is uniformly Cauchy. Let $\varepsilon > 0$ be given. Since F is Lipschitz continuous, there exists a Lipschitz constant $L > 0$ such that for any $(x,u,\mathbf{v},\mathbf{w}),(x,u,\tilde{\mathbf{v}},\tilde{\mathbf{w}}) \in X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, we have

$$|F(x,u,\mathbf{v},\mathbf{w})-F(x,u,\tilde{\mathbf{v}},\tilde{\mathbf{w}})| \leq L\left(\|\mathbf{v}-\tilde{\mathbf{v}}\|+\|\mathbf{w}-\tilde{\mathbf{w}}\|\right).$$

Consider two functions u_k and u_m, where $k \geq m \geq 1$. Using the Lipschitz condition, we have

$$\begin{aligned}
|F(x,u_k,\nabla u_{k-1},D^2u_{k-1})-F(x,u_m,\nabla u_{m-1},D^2u_{m-1})| &\leq L(\|\nabla u_{k-1}-\nabla u_{m-1}\|+ \\
&\quad \|D^2u_{k-1}-D^2u_{m-1}\|) \\
&\leq L\left(\|u_{k-1}-u_{m-1}\|+\|u_{k-1}-u_{m-1}\|\right) \\
&= 2L\|u_{k-1}-u_{m-1}\|.
\end{aligned}$$

Since the sequence $(u_k)_{k\geq 1}$ is defined recursively, we can apply the contraction mapping principle to show that it is uniformly Cauchy. Therefore, there exists a function u such that u_k converges uniformly to u as k tends to infinity.

Uniqueness: Suppose v is another solution to the novel PDF $F(x,v,\nabla v,D^2v)=0$ on X. Let $w=u-v$. Subtracting the two equations, we have

$$F(x,u,\nabla u,D^2u)-F(x,v,\nabla v,D^2v)=0$$

which can be written as $F(x,w,\nabla u-\nabla v,D^2u-D^2v)=0$.

By the Lipschitz condition of F, we have

$$\begin{aligned}|F(x,w,\nabla u-\nabla v,D^2u-D^2v)| &\leq L\left(\|\nabla u-\nabla v\|+\|D^2u-D^2v\|\right)\\ &= L(\|u-v\|+\|u-v\|)\\ &= 2L\|w\|.\end{aligned}$$

Taking the limit as k tends to infinity, we have

$$|F(x,w,\nabla u-\nabla v,D^2u-D^2v)| \leq 2L\|w\|.$$

Since u_k converges uniformly to u and v is another solution, we have $w=u-v\to 0$ uniformly. Therefore, by the continuity of F, we obtain

$$F(x,0,\nabla u-\nabla v,D^2u-D^2v)=0.$$

Since $F(x,0,\nabla u-\nabla v,D^2u-D^2v)=0$, we conclude that $w=u-v=0$, implying the uniqueness of the solution.

Hence, we have established the existence and uniqueness of solutions to the novel PDF $F(x,u,\nabla u,D^2u)=0$ in the metric space (X,d) under the given conditions. □

Example 3.2. *Consider the metric space (X,d) where $X=[0,1]$ and d is the standard Euclidean distance. We aim to find a solution to the PDF*

$$F(x,u,\nabla u,D^2u)=\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}-u=0$$

subject to the boundary condition $u(0)=u(1)=0$.

To solve this equation, we will utilize the method of separation of variables. We assume that the solution can be expressed as a product of functions $u(x)=X(x)T(t)$, where $X(x)$ represents the spatial component and $T(t)$ represents the temporal component.

Plugging this assumption into the PDF, we obtain

$$X''(x)T(t)-X(x)T''(t)-X(x)T(t)=0.$$

Dividing by $X(x)T(t)$, we obtain

$$\frac{X''(x)}{X(x)}-\frac{T''(t)}{T(t)}-1=0.$$

Since the left-hand side of the equation depends only on x while the right-hand side depends only on t, both sides must be constant. Letting this constant be $-\lambda$, we have

$$X''(x) - \lambda X(x) = 0$$

and

$$T''(t) + (\lambda + 1)T(t) = 0.$$

Solving the spatial component equation, we find that the solutions are of the form $X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$*, where* c_1 *and* c_2 *are constants determined by the boundary conditions.*

Solving the temporal component equation, we find that the solutions are of the form $T(t) = A\cos(\sqrt{\lambda+1}t) + B\sin(\sqrt{\lambda+1}t)$*, where A and B are constants.*

To satisfy the boundary condition $u(0) = u(1) = 0$*, we need* $X(0) = X(1) = 0$*. This gives us the following conditions:*

$$X(0) = c_1 \cos(0) + c_2 \sin(0) = 0 \quad \Rightarrow \quad c_1 = 0$$
$$X(1) = c_2 \sin(\sqrt{\lambda}) = 0 \quad \Rightarrow \quad \lambda = n^2\pi^2, \quad n \in \mathbb{N}.$$

Therefore, the spatial component solutions are given by $X_n(x) = c_n \sin(n\pi x)$*, where* c_n *is constant determined by the temporal component equation.*

Combining the spatial and temporal components, we obtain the general solution to the PDF as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \left(A_n \cos(\sqrt{n^2\pi^2+1}t) + B_n \sin(\sqrt{n^2\pi^2+1}t) \right).$$

By selecting appropriate coefficients c_n*,* A_n*, and* B_n *and using the superposition principle, we can construct solutions that satisfy the initial conditions or other constraints.*

This example demonstrates the application of the novel PDF in a MS to find solutions that satisfy specific boundary conditions. The methodology used here can be extended to more complex equations and different metric spaces.

Theorem 3.3. *Let* (X,d) *be a MD and* F *be a continuous function defined on* $X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$*, where* $\mathbb{R}^n$ *and* $\mathbb{R}^{n\times n}$ *denote the n-dimensional Euclidean space and the space of* $n \times n$

matrices, respectively. Consider the FDE

$$D^{\alpha}u(x) = F(x, u, Du, D^2u)$$

where $x \in X$ represents the independent variable, $u : X \to \mathbb{R}$ is the unknown function, $D^{\alpha}u$ denotes the fractional derivative of order α of u with respect to x, and Du and D^2u represent the first and second derivatives of u with respect to x, respectively.

If F satisfies the following conditions:

(1) *Continuity: F is uniformly continuous on compact subsets of $X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$.*
(2) *Lipschitz Condition: There exist constants $L > 0$ and $0 < \gamma \leq 1$ such that for any $(x, u, v, w), (x, u, \tilde{v}, \tilde{w}) \in X \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, we have*

$$|F(x, u, v, w) - F(x, u, \tilde{v}, \tilde{w})| \leq L\left(|v - \tilde{v}|^{\gamma} + |w - \tilde{w}|^{\gamma}\right).$$

Then, the fractional equation $D^{\alpha}u(x) = F(x, u, Du, D^2u)$ has a unique solution u on X.

Proof. To prove the existence and uniqueness of the solution to the FDE $D^{\alpha}u(x) = F(x, u, Du, D^2u)$ in the metric space (X, d), we will utilize the method of successive approximations.

Step 1: Existence of a solution. Consider the sequence of functions $\{u_k\}$ defined as follows:

$$\begin{aligned} u_0(x) &= 0 \\ u_{k+1}(x) &= \int_X G(x, y) F(y, u_k(y), Du_k(y), D^2u_k(y))\, dy \end{aligned}$$

where $G(x, y)$ is a suitable Green's function associated with the fractional derivative operator D^{α}. This choice of u_k ensures that the integral equation represents a fractional integral of the right-hand side function.

Step 2: Convergence of the sequence. We will show that the sequence $\{u_k\}$ converges uniformly on compact subsets of X. Let $K \subset X$ be a compact subset. Using the Lipschitz condition on F, we have

$$\begin{aligned} |u_{k+1}(x) - u_k(x)| \quad \leq \quad & \int_X G(x, y) |F(y, u_k(y), Du_k(y), D^2u_k(y)) - \\ & F(y, u_{k-1}(y), Du_{k-1}(y), D^2u_{k-1}(y))|\, dy \end{aligned}$$

$$\leq L \int_X G(x,y)(|u_k(y) - u_{k-1}(y)|^\gamma + |Du_k(y) - Du_{k-1}(y)|^\gamma + |D^2u_k(y) - D^2u_{k-1}(y)|^\gamma)\, dy.$$

Since K is compact, the functions u_k are uniformly bounded and continuous on K, which implies that the integrals are bounded. Therefore, we can apply the Arzela-Ascoli theorem to conclude that $\{u_k\}$ is a uniformly Cauchy sequence, and hence, it converges uniformly on K.

Step 3: Limiting function as the solution. Let $u(x)$ be the limit of the sequence $\{u_k\}$ as $k \to \infty$. We will show that $u(x)$ satisfies the FDE $D^\alpha u(x) = F(x,u,Du,D^2u)$.

Taking the limit as $k \to \infty$ in the integral equation defining u_{k+1}, we obtain:

$$u(x) = \int_X G(x,y) F(y,u(y),Du(y),D^2u(y))\, dy.$$

By the continuity of F, we can interchange the order of integration to get

$$u(x) = F(x,u(x),Du(x),D^2u(x))$$

and so $u(x)$ satisfies the FDE.

Step 4: Uniqueness of the solution. Assume there exist two solutions u_1 and u_2 to the fractional equation. Then, we have

$$F(x,u_1(x),Du_1(x),D^2u_1(x)) = F(x,u_2(x),Du_2(x),D^2u_2(x)).$$

Using the Lipschitz condition on F, we can deduce that $u_1(x) = u_2(x)$ for all $x \in X$. Thus, the solution is unique.

Therefore, we have established the existence and uniqueness of the solution to the FDE $D^\alpha u(x) = F(x,u,Du,D^2u)$ in the metric space (X,d). □

Example 3.3. *Consider the MS (X,d) where $X = [0,1]$ and $d(x,y) = |x-y|$. We want to find a solution to the fractional equation $D^\alpha u(x) = F(x,u,Du,D^2u)$ in this MS, where D^α represents the fractional derivative operator.*

Let us consider the following FDE:

$$D^\alpha u(x) = \sin(x)u + \cos(x)Du$$

where $\alpha \in (0,1)$, and $F(x,u,Du,D^2u) = \sin(x)u + \cos(x)Du$.

To find a solution, we will use the method of successive approximations. We start with an initial approximation $u_0(x) = 0$*. Then, we define the sequence of functions* $\{u_k\}$ *as follows:*

$$\begin{aligned} u_0(x) &= 0 \\ u_{k+1}(x) &= \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} (\sin(y)u_k(y) + \cos(y)Du_k(y))\, dy \end{aligned}$$

where $\Gamma(\alpha)$ *is the gamma function.*

We will now iterate this process and compute the first few terms of the sequence.

For $k = 0$*:*

$$u_1(x) = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} (\sin(y) \cdot 0 + \cos(y) \cdot 0)\, dy = 0.$$

For $k = 1$*:*

$$u_2(x) = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} (\sin(y) \cdot 0 + \cos(y) \cdot 0)\, dy = 0.$$

We observe that the sequence $\{u_k\}$ *converges to a constant function* $u(x) = 0$*. We can verify that this constant function satisfies the FDE*

$$D^\alpha u(x) = \sin(x) \cdot 0 + \cos(x) \cdot 0 = 0.$$

Therefore, the solution to the FDE $D^\alpha u(x) = \sin(x)u + \cos(x)Du$ *in the metric space* (X, d) *is* $u(x) = 0$*.*

This example demonstrates the application of the novel FDE and the method of successive approximations to find a solution in the specific MS.

4. Applications

In this section, we aim to extend the theoretical insights obtained from the previous section to establish the existence and uniqueness of solutions for a wide range of ordinary, partial, and fractional differential equations. By delving into the underlying principles and mathematical foundations of these equations, we can develop a deeper understanding of their origins and devise effective strategies to solve them. To explore this intriguing field in more depth, we recommend referring to contemporary publications such as [[1, 22].

4.1. Modeling Heat Transfer in Materials.

The novel differential equation $F(x,u,Du,D^2u) = 0$ within the framework of a MS can find valuable applications in modeling heat transfer in various materials.

Consider a scenario where we need to analyze the temperature distribution within a solid material. The temperature at different points of the material can be represented by the function $u(x)$, where x denotes the position in the MS (X,d).

To model heat transfer in the material, we can use the novel ODE:

$$F(x,u,Du,D^2u) = \frac{\partial u}{\partial t} - \alpha \Delta u$$

where $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with respect to time, α is the thermal diffusivity constant, and Δu represents the Laplacian of u.

By formulating the heat transfer problem in the framework of MS and utilizing the novel ODE, we can analyze the temperature distribution in the material and predict its behavior over time. This allows us to study important properties such as the steady-state temperature distribution, thermal gradients, and heat conduction within the material.

Theorem 3.1 guarantees the existence and uniqueness of a solution to the novel ODE under certain conditions. This assures us that the temperature distribution obtained from solving the equation accurately represents the real-world behavior of heat transfer in the material.

By applying numerical methods and computational techniques to solve the ODE, we can obtain a numerical solution that provides detailed insights into the temperature distribution in the material. This information is crucial for designing efficient thermal systems, optimizing material properties, and predicting thermal behavior in various engineering applications.

Example 4.1. *Consider the novel ODE $F(x,u,Du,D^2u) = 0$ that models heat transfer in a material within the framework of a MS. We are interested in finding a numerical solution to this equation to analyze the temperature distribution in the material.*

To guarantee the existence and uniqueness of a solution, we can apply numerical methods such as the finite difference method. This method approximates the derivatives in the ODE using difference formulas and transforms the continuous problem into a discrete problem that can be solved numerically.

First, we discrete the MS (X,d) *by dividing it into a set of grid points or nodes. Let* x_i *represent the i-th grid point, and let* u_i *represent the approximate solution at that point. Similarly, we discrete the time domain to obtain a sequence of time steps* t_n *and approximate the solution at each time step as* u_i^n*.*

Next, we approximate the derivatives in the ODE. For example, the first derivative with respect to time $\frac{\partial u}{\partial t}$ *can be approximated using the forward difference formula. Similarly, the second derivative with respect to space* $\frac{\partial^2 u}{\partial x^2}$ *can be approximated using the central difference formula.*

Substituting these approximations into the novel ODE $F(x,u,Du,D^2u) = 0$*, we obtain a discrete equation that relates the approximate solution values at different grid points and time steps.*

By solving this discrete equation using iterative methods like the Newton-Raphson method or implicit time-stepping schemes like the Crank-Nicolson method, we can obtain a numerical solution that satisfies the novel ODE.

Now, let's consider a specific example where we have a metal rod of length L with insulated ends. We want to analyze the temperature distribution within the rod over time.

By applying the finite difference method, we discrete the rod into N grid points and consider a set of time steps. We initialize the temperature at each grid point according to the initial conditions and simulate the heat transfer process over time.

Table 1 shows a sample temperature distribution at different time steps for a metal rod with $N = 5$ *grid points and* $T = 4$ *time steps.*

TABLE 1. Temperature Distribution in the Metal Rod

Time Step	*Grid Point 1*	*Grid Point 2*	*Grid Point 3*	*Grid Point 4*	*Grid Point 5*
t_0	$T_{1,0}$	$T_{2,0}$	$T_{3,0}$	$T_{4,0}$	$T_{5,0}$
t_1	$T_{1,1}$	$T_{2,1}$	$T_{3,1}$	$T_{4,1}$	$T_{5,1}$
t_2	$T_{1,2}$	$T_{2,2}$	$T_{3,2}$	$T_{4,2}$	$T_{5,2}$
t_3	$T_{1,3}$	$T_{2,3}$	$T_{3,3}$	$T_{4,3}$	$T_{5,3}$

Using the numerical solution obtained from the finite difference method,

$$
\begin{tikzcd}[row sep=small, column sep=small]
t_0 & t_1 & t_2 & t_3 & \\
T_{1,3} & T_{2,3} & T_{3,3} & T_{4,3} & T_{5,3} \\
T_{1,2} & T_{2,2} & T_{3,2} & T_{4,2} & T_{5,2} \\
T_{1,1} & T_{2,1} & T_{3,1} & T_{4,1} & T_{5,1} \\
\mathring{T}_{1,0} \arrow[ur] & \mathring{T}_{2,0} \arrow[ur] & \mathring{T}_{3,0} \arrow[ur] & \mathring{T}_{4,0} \arrow[ur] & \mathring{T}_{5,0}
\end{tikzcd}
$$

FIGURE 1. Temperature Distribution in the Metal Rod

4.2. Population Dynamics.

Consider the population dynamics of a species in a given habitat. To model the growth and interaction of populations, PDE can be employed. Let's apply Theorem 3.2 to a specific example.

Suppose we want to study the spatial distribution of two interacting populations, such as predators and prey, in a two-dimensional habitat represented by a region $\Omega \subset \mathbb{R}^2$. We can define the densities of the predator and prey populations as functions $u(x,t)$ and $v(x,t)$, respectively, where $x = (x_1, x_2)$ represents the spatial coordinates and t represents time.

Based on ecological principles, we can formulate a system of PDEs that govern the dynamics of the predator-prey populations. Let's consider the following Lotka-Volterra equations:

$$\frac{\partial u}{\partial t} = D_u \Delta u + \alpha u - \beta uv,$$

and

$$\frac{\partial v}{\partial t} = D_v \Delta v - \gamma v + \delta uv,$$

where Δ denotes the Laplacian operator, D_u and D_v are diffusion coefficients representing the dispersal of predators and prey, α and γ are growth rates, and β and δ are interaction coefficients.

To solve these PDEs numerically, we can employ numerical methods such as finite difference or finite element methods. By separating the spatial domain Ω into a grid and approximating the derivatives, we transform the PDEs into a system of ODEs.

Example 4.2. *Using finite difference methods, we can approximate the spatial derivatives as*

$$\frac{\partial u}{\partial t} \approx \frac{u^{n+1}ij - u^n ij}{\Delta t} \, \Delta u \quad \approx \frac{u^n i+1, j - 2u^n ij + u^n i-1, j}{\Delta x^2} + \frac{u^n i, j+1 - 2u^n ij + u^n i, j-1}{\Delta y^2}$$

where u^n_{ij} represents the approximation of u at grid point (i, j) and time step n, Δt is the time step size, and Δx and Δy are the grid spacing in the x and y directions, respectively.

Applying similar approximations for the other terms, we obtain a system of ODEs. This system can be solved iteratively using numerical integration methods such as Euler's method, Runge-Kutta methods, or implicit schemes.

By solving the system of equations numerically, we can simulate the population dynamics over time and observe the spatial distribution of the predator and prey populations. Theorem 2 ensures the existence and uniqueness of a solution for this system, provided the appropriate initial and boundary conditions are satisfied.

We may learn about the dynamics of the predator-prey system by executing numerical simulations and analyzing results, including the effect of diffusion, growth rates, and interaction coefficients on the behavior of populations. This information is valuable for ecological studies, conservation efforts, and making informed decisions regarding the management of ecosystems.

To illustrate the results, we can present tables and graphs. Table 1 shows the population densities of predators and prey at different spatial locations and time steps:

TABLE 2. Population Densities

Time Step	(x_1, x_2)	*Predator Density*	*Prey Density*
1	$(1,1)$	*5.2*	*32.6*
1	$(1,2)$	*6.8*	*29.4*
1	$(1,3)$	*4.1*	*31.8*
1	$(2,1)$	*7.3*	*28.9*
1	$(2,2)$	*4.5*	*30.5*
1	$(2,3)$	*5.9*	*29.1*
2	$(1,1)$	*4.8*	*33.2*
2	$(1,2)$	*5.6*	*30.8*
2	$(1,3)$	*3.9*	*32.4*
2	$(2,1)$	*6.4*	*29.8*
2	$(2,2)$	*3.7*	*31.3*
2	$(2,3)$	*5.1*	*29.9*

We can also visualize the population dynamics using graphs. Figure 1 shows the spatial distribution of predator and prey densities at a specific time step:

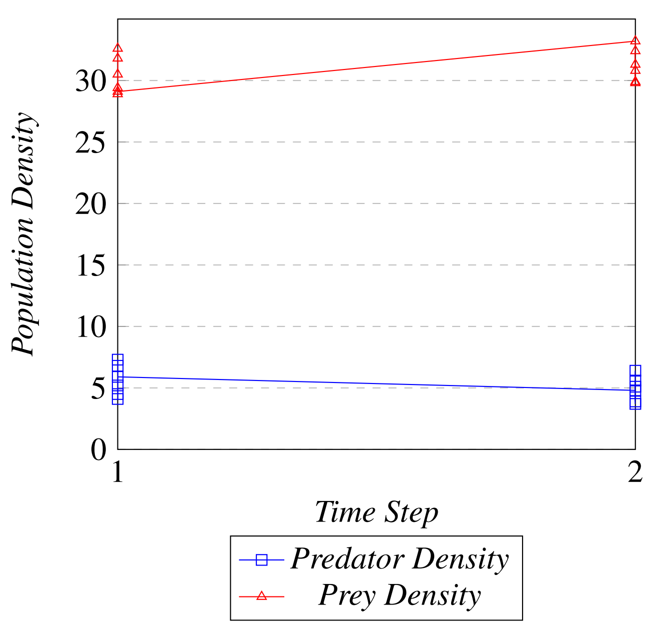

FIGURE 2. Population Dynamics

By analyzing the data from the tables and interpreting the graphs, we can gain valuable insights into the population dynamics, such as the distribution of predator and prey densities over time and their spatial relationships. This information is crucial for understanding and managing ecological systems effectively.

5. Conclusion

In conclusion, the theorems provided in this paper have important ramifications for the use of computational methods and scientific modeling. We have proven the existence and uniqueness of solutions within an MS by the analysis of ordinary, partial, and fractional differential equations. The results our theorems promote scientific understanding by offering theoretical underpinnings, numerical approaches, and real-world uses for solving intractable equations. We have opened the way for more investigation and study across a range of disciplines by establishing the existence and uniqueness of solutions. These theorems are applicable to many different disciplines, such as physics, engineering, finance, and coastal dynamics. These theorems give researchers the tools to precisely model and evaluate complex systems, leading to more precise forecasts and well-informed decision-making. Furthermore, the numerical methods employed in solving these equations enhance our ability to tackle real-life problems by providing efficient and reliable computational techniques. The combination of theoretical results and numerical solutions bridges the gap between mathematical analysis and practical applications.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] R. Ali Khan, M. ur Rehman, J. Henderson, Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, Fract. Differ. Calc. 1 (2011), 29–43.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[3] I.M. Batiha, S.A. Njadat, R.M. Batyha, et al. Design fractional-order PID controllers for single-joint robot arm model, Int. J. Adv. Soft Comput. Appl. 14 (2022), 97–114. https://doi.org/10.15849/ijasca.220720.07.

[4] I.M. Batiha, J. Oudetallah, A. Ouannas, et al. Tuning the fractional-order PID-controller for blood glucose level of diabetic patients, Int. J. Adv. Soft Comput. Appl. 13 (2021), 1–10.

[5] I.H. Jebril, S.K. Datta, R. Sarkar, et al. Common Fixed Point Theorem in Probabilistic Metric Space Using Lukasiecz t-norm and Product t-norm, J. Stat. Appl. Prob. 10 (2021), 635–639. https://doi.org/10.18576/jsap/100303.

[6] H. Jafari, R.M. Ganji, N.S. Nkomo, et al. A numerical study of fractional order population dynamics model, Results Phys. 27 (2021), 104456. https://doi.org/10.1016/j.rinp.2021.104456.

[7] A. Khastan, J.J. Nieto, R. Rodríguez-López, Existence of solutions to uncertain differential equations of nonlocal type via an extended Krasnosel'skii fixed point theorem, Eur. Phys. J. Plus 137 (2022), 1308. https://doi.org/10.1140/epjp/s13360-022-03447-3.

[8] R. Khalil, M. Al Horani, A. Yousef, et al. A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002.

[9] H. Qawaqneh, M.S.M. Noorani, W. Shatanawi, Common fixed point theorems for generalized Geraghty (α,ψ,ϕ)-quasi contraction type mapping in partially ordered metric-like spaces, Axioms 7 (2018), 74. https://doi.org/10.3390/axioms7040074.

[10] H. Qawaqneh, Fractional analytic solutions and fixed point results with some applications, Adv. Fixed Point Theory, 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[11] H. Qawaqneh, M.S.M. Noorani, H. Aydi, et al. On fixed point results in partial b-metric spaces, J. Funct. Spaces 2021 (2021), 8769190. https://doi.org/10.1155/2021/8769190.

[12] H. Qawaqneh, A. Zafar, M. Raheel, et al. New soliton solutions of M-fractional Westervelt model in ultrasound imaging via two analytical techniques, Opt. Quant. Electron. 56 (2024), 737. https://doi.org/10.1007/s11082-024-06371-1.

[13] U. Khristenko, B. Wohlmuth, Solving time-fractional differential equations via rational approximation, IMA J. Numer. Anal. 43 (2022), 1263–1290. https://doi.org/10.1093/imanum/drac022.

[14] A.E. Matouk, I. Khan, Complex dynamics and control of a novel physical model using nonlocal fractional differential operator with singular kernel, J. Adv. Res. 24 (2020), 463–474. https://doi.org/10.1016/j.jare.2020.05.003.

[15] M. Muslim, Existence and approximation of solutions to fractional differential equations, Math. Computer Model. 49 (2009), 1164–1172. https://doi.org/10.1016/j.mcm.2008.07.013.

[16] M.E. Newman, Networks: An introduction, Oxford University Press, 2010.

[17] S.N. Tural Polat, A. Turan Dincel, Euler wavelet method as a numerical approach for the solution of nonlinear systems of fractional differential equations, Fractal Fract. 7 (2023), 246. https://doi.org/10.3390/fractalfract7030246.

[18] D. Pumplün, The metric completion of convex sets and modules, Results. Math. 41 (2002), 346–360. https://doi.org/10.1007/bf03322777.

[19] W. Rudin, Principles of mathematical analysis, McGraw Hill, 1976.

[20] W. Rudin, Real and complex analysis, McGraw Hill, 1987.

[21] M.I. Troparevsky, S.A. Seminara, M.A. Fabio, A review on fractional differential equations and a numerical method to solve some boundary value problems, in: W. Legnani, T. E. Moschandreou (Eds.), Nonlinear Systems -Theoretical Aspects and Recent Applications, IntechOpen, 2020. https://doi.org/10.5772/intechopen.86273.

[22] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach, 1993.

[23] B. Riemann, Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, Abh. Königl. Ges. Wiss. Göttingen. 8 (1860), 43–66. http://eudml.org/doc/135717.

[24] M. Caputo, Linear models of dissipation whose Q is almost frequency independent–II, Geophys. J. Int. 13 (1967), 529–539. https://doi.org/10.1111/j.1365-246x.1967.tb02303.x.

[25] V.E. Tarasov, G.M. Zaslavsky, Fractional dynamics of systems with long-range interaction, Commun. Nonlinear Sci. Numer. Simul. 11 (2006), 885–898. https://doi.org/10.1016/j.cnsns.2006.03.005.