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CONVERGENCE THEOREMS IN $\mathbb{G}$-METRIC SPACE AND ITS APPLICATION

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Abstract. The present paper is an attempt to prove convergence theorems to a unique fixed point in complete $\mathbb{G}$-metric space. It also provides the consequences of our main result. Furthermore, some up to date results in the setting of $\mathbb{G}$-metric spaces will be improved and supplemented by the convergence theorems. As an application of our results, it also proves some previous theorems as special cases. The results demonstrated in this study unify and enlarge some known results in this area.

Keywords: convergence theorems; quasi-nonexpansive; $\mathbb{G}$-metric space.

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1. Introduction

Fixed point theory has been stretched out to be a valuable part of functional analysis. Banach introduced the most helpful principle in 1922 (Banach contraction principle) which is used and generalized to many branches. Over the past fifty years, In [7], Mustafa and Sims introduced the idea of a $\mathbb{G}$-metric space as a generalized metric space.

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Next, several firm pointed theorems on $\mathbb{G}$-metric spaces were mentioned, references therein. But in [8], Jleli and Samet revealed that the majority of the attained fixed point theorems on $\mathbb{G}$-metric spaces may be reduced right now from fixed point theorems. Note that the metric space $(\vartheta, \mathbb{G})$ is considered as a generalization of metric space.

In the last few decades, convergence theorems in metric spaces and generalized types of them have been studied (see, e.g., [2]-[4]) and [10]. Recently, in 2012, Ahmed and Zeyada [3] established some convergence theorems of any sequence in complete quasi-metric spaces.

Therefore, Our aim in the present study is to deal the concept of quasi-nonexpansive mappings with respect to $\mathbb{G}$- metric spaces. Chiefly, some convergence theorems of a sequence will be established in complete $\mathbb{G}$- metric spaces. These theorems are generalized and improved (see, e.g., [2]-[4]) and [10].

First, we call back some basic explanations and lemmas which are needed in the paper.

Following Waszkiewick [6], let Θ be a nonempty set. A map $d : \Theta \times \Theta \longrightarrow [0, \infty)$ is called a **distance** on Θ. A pair (Θ, d) is said to be a distance space. Now, we list some conditions as follows: For every $\vartheta, \mathfrak{J}, \iota \in \Theta$,

(i) $d(\vartheta, \vartheta) = 0$;

(ii) if $d(\vartheta, \mathfrak{J}) = 0$ or $d(\mathfrak{J}, \vartheta) = 0$, then $\vartheta = \mathfrak{J}$;

(iii) $d(\vartheta, \mathfrak{J}) = d(\iota, \mathfrak{J})$;

(iv) $d(\vartheta, \mathfrak{J}) \leq d(\vartheta, \iota) + d(\iota, \mathfrak{J})$.

Definition 1.1. [9] *Let Θ be a Banach space. If ϕ and U are mappings (in general nonlinear) with domains $\Omega(\Lambda)$ and $\Omega(U)$ in Θ and with values in Θ, then U is said to be nonexpansive if for all u and v in $\Omega(U)$,*

$$\|U(u) - U(v)\| \leq \|u - v\|.$$

Definition 1.2. [1] *The mapping $\phi : \Omega \rightarrow \Theta$ is said to be quasi-nonexpansive w.r.t. a sequence $\{\vartheta_n\}$ if $\{\vartheta_n\} \subseteq \Omega$ and for all $n \in \mathbb{N} \cup \{0\}$ $(\mathbb{N} := the\ set\ of\ all\ positive\ integers)$ and for each $\rho \in \Xi(\Lambda)$,*

$$d(\vartheta_{n+1}, \rho) \leq d(\vartheta_n, \rho).$$

Now, we give the definition of quasi-nonexpansive in $\mathbb{G}$ -metric space.

Definition 1.3. *The mapping $\Lambda : \Omega \times \Omega \to \vartheta$ is said to be quasi-nonexpansive w.r.t. a sequence $\{\vartheta_n\}$ if $\{\vartheta_n\} \subseteq \Omega$ and for all $n \in \mathbb{N} \cup \{0\}$ $(\mathbb{N} := the\ set\ of\ all\ positive\ integers)$ and for each $\rho \in \Xi(\Lambda)$,*

$$\mathbb{G}(\vartheta_{n+1}, \rho, \rho) \leq \mathbb{G}(\vartheta_n, \rho, \rho).$$

The present paper has been arranged as follows: In Section (2), we introduce some lemma and convergence theorems in $\mathbb{G}$- metric spaces. Finally, in Section (3), we give some applications on our main results.

2. Convergence Theorem in $\mathbb{G}$-Metric Space

In 2005, Mustafa and Sims presented a new structure of generalized metric spaces (see[7]) which are called $\mathbb{G}$-metric spaces as generalization of metric space (Θ, d), to develop and introduce a new fixed point theory for different mappings in this new structure. In this section, we state and prove some lemma and convergence theorems in $\mathbb{G}$- metric space.

In [7], Mustafa and Sims presented the concept of a $\mathbb{G}$-metric space as a generalized metric space.

Second we presente the concept of $\mathbb{G}$-metric space as follows:

Definition 2.1. *(see [7]) Let ϑ be a nonempty set and $\mathbb{G} : \Theta \times \Theta \times \Theta \to [0, \infty)$ be a function such that, for all $\vartheta, \mathfrak{J}, \iota \in \Theta$,*

(1) $\mathbb{G}(\vartheta, \mathfrak{J}, \iota) = 0$ if $\vartheta = \mathfrak{J} = \iota$

(2) $0 < \mathbb{G}(\vartheta, \mathfrak{J}, \iota)$ if $\vartheta \neq \mathfrak{J} \in \Theta$.

(3) $\mathbb{G}(\vartheta, \vartheta, \iota) \leq \mathbb{G}(\vartheta, \mathfrak{J}, \iota)$, for all $\vartheta, \mathfrak{J}, \iota \in \Theta$ with $\mathfrak{J} \neq \iota$.

(4) $\mathbb{G}(\vartheta, \mathfrak{J}, \iota) = \mathbb{G}(\vartheta, \iota, \mathfrak{J}) = \mathbb{G}(\mathfrak{J}, \vartheta, \iota) = \mathbb{G}(\mathfrak{J}, \iota, \vartheta) = \mathbb{G}(\iota, \vartheta, \mathfrak{J}) = \mathbb{G}(\iota, \mathfrak{J}, \vartheta)$ =...(symmetry in all three variables).

(5) $\mathbb{G}(\vartheta, \mathfrak{J}, \iota) \leq \mathbb{G}(\vartheta, \rho, \rho) + \mathbb{G}(\rho, \mathfrak{J}, \iota)$. for all $\vartheta, \mathfrak{J}, \iota, \rho \in \Theta$, (rectangle inequality)

Then the function $\mathbb{G}$ is called a generalized metric or more specifically a $\mathbb{G}$- metric on ϑ and the pair $(\Theta, \mathbb{G})$ is called a $\mathbb{G}$- metric space.

Remark 2.1. [7] *One can conclude that* $\mathbb{G}$*-metric space is a generalization of metric spaces.*

Example 2.1. *Consider* $\Theta = R^+$ *with the usual distance* $d(\vartheta,\mathfrak{J}) =\mid \vartheta - \mathfrak{J} \mid \forall \vartheta,\mathfrak{J} \in \mathbb{N}$ *,Define* $\mathbb{G} : \Theta^3 \to \mathbb{R}^+$ *by*

$$\mathbb{G}(\vartheta,\mathfrak{J},\iota) = \mathbb{G}(\vartheta,\mathfrak{J}) + \mathbb{G}(\mathfrak{J},\iota) + G(\iota,\vartheta) \forall \vartheta,\mathfrak{J},\iota \in \Theta$$

then Θ *is* $\mathbb{G}$*- metric space.*

Definition 2.2. *(see* [7]*) Let* $(\Theta,\mathbb{G})$ *be a* $\mathbb{G}$*- metric space, and let* ϑ_n *a sequence of points in* Θ*, a point* ϑ *in* Θ *is said to be the limit of the sequence* ϑ_n *if* $lim_{n,m,\to\infty} G(\vartheta,\vartheta_n,\vartheta_m) = 0$*, and one says that sequence* ϑ_n *is* $\mathbb{G}$ *-convergent to* ϑ*.*

Definition 2.3. *Let* $(\Theta,\mathbb{G})$ *be a* $\mathbb{G}$ *-metric space. A sequence* ϑ_n *is called* $\mathbb{G}$*-Cauchy if,for each* $\varepsilon > 0$ *there exists a positive integer* $\mathbb{N}$ *such that* $\mathbb{G}(\vartheta_n,\vartheta_m,\vartheta_l) < \varepsilon$ *for all* $n,m,l \geq N$ *;i.e. if* $\mathbb{G}(\vartheta_n,\vartheta_m,\vartheta_l) \to 0$ *as* $n,m,l \to \infty$*.*

Definition 2.4. *(see* [7]*) The* $\mathbb{G}$*-metric space* $(\Theta,\mathbb{G})$ *is called symmetric if* $\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J}) = \mathbb{G}(\vartheta,\vartheta,\mathfrak{J})$ *for all* $\vartheta,\mathfrak{J} \in \Theta$*.*

Definition 2.5. *(see* [7]*) Let* $(\Theta,\mathbb{G})$ *be a* $\mathbb{G}$*- metric space and* ϑ_n *be a sequence in* Θ*.*

(1) *For each* $\vartheta_0 \in \Theta$ *and* $r > 0$*, the set*

$$B_{\mathbb{G}}(\vartheta_0,r) = \vartheta \in \vartheta : \mathbb{G}(\vartheta_0,\vartheta,\vartheta) < r$$

is called a $\mathbb{G}$ *ball with center* ϑ_0 *and radius r.*

(2) *The group of all* $\mathbb{G}$ *balls forms a base of a topology* $\tau(\mathbb{G})$ *on* ϑ*, and* $\tau(\mathbb{G})$ *is called the* $\mathbb{G}$*-metric topology.*

(3) ϑ_n *is called convergent to* ϑ *in* ϑ *if* $lim_{n\to\infty}\vartheta_n = \vartheta$ *in the* $\mathbb{G}$*- metric topology* $\tau(\mathbb{G})$*.*

(4) ϑ_n *is called* $\mathbb{G}$*- Cauchy in* ϑ *if* $lim_{n,m,l\to\infty}\mathbb{G}(\vartheta_n,\vartheta_m,\vartheta_l) = \vartheta$*.*

(5) $(\Theta,\mathbb{G})$ *is known as a complete* $\mathbb{G}$*- metric space if every* $\mathbb{G}$*- Cauchy sequence is convergent.*

Proposition 2.1. *(see* [7]*) Let* $(\Theta,\mathbb{G})$ *be a* $\mathbb{G}$*- metric space .Then the function* $\mathbb{G}(\vartheta,\mathfrak{J},\iota)$ *is jointly continuous in all three of its variables.*

Proposition 2.2. *(see [7]) Let $(\Theta,\mathbb{G})$ be a $\mathbb{G}$- metric space .Then,for any $\vartheta,\mathfrak{J},\iota \in \Theta$ it follows that:*

(1) *If $\mathbb{G}(\vartheta,\mathfrak{J},\iota)=0$, then $\vartheta=\mathfrak{J}=\iota$,*

(2) $\mathbb{G}(\vartheta,\mathfrak{J},\iota) \leq \mathbb{G}(\vartheta,\vartheta,\mathfrak{J})+\mathbb{G}(\vartheta,\vartheta,\iota)$,

(3) $\mathbb{G}(\vartheta,\mathfrak{J},\iota) \leq 2\mathbb{G}(\mathfrak{J},\vartheta,\vartheta)$,

(4) $\mathbb{G}(\vartheta,\mathfrak{J},\iota) \leq \mathbb{G}(\vartheta,\rho,\iota)+\mathbb{G}(\rho,\mathfrak{J},\iota)$,

(5) $\mathbb{G}(\vartheta,\mathfrak{J},\iota) \leq \frac{2}{3}(\mathbb{G}(\vartheta,\mathfrak{J},\rho)+\mathbb{G}(\vartheta,\rho,\iota)+\mathbb{G}(\rho,\mathfrak{J},\iota))$,

(6) $\mathbb{G}(\vartheta,\mathfrak{J},\iota) \leq (\mathbb{G}(\vartheta,\rho,\rho)+\mathbb{G}(\mathfrak{J},\rho,\rho)+\mathbb{G}(\iota,\rho,\rho))$.

Proposition 2.3. *(see [7]) Let $(\Theta,\mathbb{G})$ be $\mathbb{G}$- metric space. Then the following statements are equivalent.*

(i) *A sequence $\{\vartheta_n\}$ is convergent to ϑ in Θ,*

(ii) $\lim_{n\to\infty}\mathbb{G}(\vartheta_n,\vartheta_n,\vartheta)\to 0$ *as $n\to\infty$,*

(iii) $\lim_{n\to\infty}\mathbb{G}(\vartheta_n,\vartheta,\vartheta)\to 0$ *as $n\to\infty$,*

(iv) $\lim_{n,m\to\infty}\mathbb{G}(\vartheta_n,\vartheta_m,\vartheta)\to 0$ *as $n\to\infty$.*

Definition 2.6. *(see [7]) Let $(\Theta,\mathbb{G})$ be $\mathbb{G}$- metric space. Then the following statements are equivalent.*

(i) *A sequence $\{\vartheta_n\}$ is $\mathbb{G}$- Cauchy sequence.*

(ii) $\lim_{n,m\to\infty}\mathbb{G}(\vartheta_n,\vartheta_m,\vartheta_m)=0$.

We state and prove the elementary properties of $\mathbb{G}$- metric on Θ.

Lemma 2.1. *Let $(\Theta,\mathbb{G})$ be a $\mathbb{G}$-metric space. If $\vartheta,\mathfrak{J}\in\Theta$, then*

$$\mid \mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\mid \leq \mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})$$

Proof. By using rectangle inequality in $\mathbb{G}$-metric space, we get

$$\begin{aligned}\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0) &\leq \mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})+\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\\ &\leq \mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})+\mathbb{G}(\vartheta,\vartheta,\vartheta)+\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\\ &= \mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})+0+\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0).\end{aligned}$$

This indicates that

$$\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\leq\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J}). \tag{2.1}$$

But

$$\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\leq\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})+\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0).$$

Then

$$-\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})\leq\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0). \tag{2.2}$$

From (1) and (2), we obtain that

$$-\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})\leq\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\leq\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J}).$$

This is equivalent to

$$\mid\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\mid\leq\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J}). \tag{2.3}$$

Now, we introduce the following lemma we need in the sequel.

Lemma 2.2. *Let $(\Theta,\mathbb{G})$ be a $\mathbb{G}$-metric space. For ϑ_0 fixed, the function $\vartheta\mapsto\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)$ is continuous from Θ to $\mathbb{R}$.*

Proof. Given $\varepsilon>0$, take $\eta=\varepsilon$. From (3), we have

$$\mid\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)-\mathbb{G}(\mathfrak{J},\vartheta_0,\vartheta_0)\mid\leq\mathbb{G}(\vartheta,\mathfrak{J},\mathfrak{J})\leq\varepsilon$$

and thus $\vartheta\mapsto\mathbb{G}(\vartheta,\vartheta_0,\vartheta_0)$ is continuous. □

Remark 2.2. *(1) Lemma 2.1 (resp. Lemma 2.2) generalizes Proposition 4 (resp. Proposition 5) ([5], page 13).*

(2) $(\Theta,\mathbb{G})$ is a $\mathbb{G}$-metric space, instead of (Θ,d) is a metric space.

Now, we state and prove our main results as follows.

Theorem 2.1. *Let $\{\vartheta_n\}$ be a sequence in a subset Ω of $\mathbb{G}$-metric space $(\Theta, \mathbb{G})$ and $\phi : \Omega \times \Omega \longrightarrow \Theta$ any map such that $\Xi(\Lambda) \neq \phi$. Assume that $\Xi(\Lambda)$ is a closed set. Then*

(a) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$ if $\{\vartheta_n\}$ $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$.

(b) $\{\vartheta_n\}$ be $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$ if $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$, and $\lim_{n\to\infty} \vartheta_n$ exists.

Proof. (a) Let $\{\vartheta_n\} \subseteq \Omega$ be $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$. Then $\lim_{n\to\infty} \vartheta_n$ exists. From the closedness of $\Xi(\Lambda)$, we find that $\rho := \lim_{n\to\infty} \vartheta_n \in \Xi(\Lambda) = \overline{\Xi(\Lambda)}$.
Hence, we obtain that $\mathbb{G}(\rho, \Xi(\Lambda), \Xi(\Lambda)) = 0$. Since $\mathbb{G}$ is continuous, we get that

$$\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = \mathbb{G}(\lim_{n\to\infty} \vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = \mathbb{G}(\rho, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

(b) Suppose that $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$. Since $\mathbb{G}$ is continuous, then

$$\mathbb{G}(\lim_{n\to\infty} \vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = \lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$$

Hence, we get that $\lim_{n\to\infty} \vartheta_n \in \overline{\Xi(\Lambda)}$. The closedness of $\Xi(\Lambda)$ leads to $\lim_{n\to\infty} \vartheta_n \in \Xi(\Lambda)$. Then $\{\vartheta_n\}$ $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$. □

Remark 2.3. *Theorem 2.1 generalizes and improves Theorem 2.1 in* [2, 4] *and* [10]

(I) The quasi-nonexpansiveness of Λ w.r.t. a sequence $\{\vartheta_n\}$ or the weakly quasi- nonexpansiveness of Λ w.r.t. a sequence $\{\vartheta_n\}$, $\Xi(\Lambda) \neq \phi$, the completeness of ϑ and

$$\lim_{n\to\infty} d(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$$

give that $\lim_{n\to\infty} \vartheta_n$ exists.

(II) Θ is $\mathbb{G}$-metric space instead of Θ is metric space.

Corollary 2.1. *Let $\{\vartheta_n\}$ be sequence in a subset Ω of a$\mathbb{G}$) metric space $(\vartheta, \mathbb{G})$ and let $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ be a map such that $\Xi(\Lambda) \neq \phi$. then*

(i) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$ if $\{\vartheta_n\}$ converges to a unique point in $\Xi(\Lambda)$;

(ii) $\{\vartheta_n\}$ *converges to a point in* $\Xi(\Lambda)$ *if* $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$ *is closed set,* Λ *is quasi-nonexpansive with respect to* $\{\vartheta_n\}$ *or weakly quasi-nonexpansive w.r.t.* $\{\vartheta_n\}$*, and* ϑ *is complete.*

We state the following lemma without proof.

Lemma 2.3. *Let* $(\Theta, \mathbb{G})$ *be a* $\mathbb{G}$*-metric space and* $\{\vartheta_n\}$ *be a sequence in* $\Omega \subseteq \Theta$*. Assume that* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *is a mapping with* $\Xi(\Lambda) \neq \phi$*. If* Λ *is quasi-nonexpansive w.r.t.* $\{\vartheta_n\}$*, then* $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\}$ *is a monotonically decreasing sequence in* $[0, \infty)$*.*

The following example shows that the inverse of Lemma 2.3 may not be true.

Example 2.2. *Let* $\Theta = [0,1]$ *be endowed with* $\mathbb{G}$ *metric space. We define the map* $\Lambda : \vartheta \longrightarrow \Theta$ *by* $\Lambda(\vartheta) = 2\vartheta^2 - \vartheta$ *for each* $\vartheta \in \Theta$*. Assume that* $\vartheta_n = \frac{1}{n}\ \forall n \in \mathbb{N} - \{1\}$*. Then* $\Xi(\Lambda) = \{0, 1\}$ *and the sequence* $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\} = \{\frac{1}{n}\}$ *is monotonically decreasing sequence in* $[0, \infty)$*. But,* Λ *is not quasi-nonexpansive w.r.t.* $\{x_n\}$*.*
(Indeed, then exists $1 \in \Xi(\Lambda)$ *such that* $\forall n \in \mathbb{N} - \{1\}$*.* $\mathbb{G}(\vartheta_{n+1}, 1, 1) > \mathbb{G}(\vartheta_n, 1, 1)$*).*

As an outcome of Theory 2.1, we confirm and prove the following Theorem.

Theorem 2.2. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *any map such that* $\Xi(\Lambda) \neq \phi$ *is a closed set. Assume that*

(i) Λ *is quasi-nonexpansive with respect to* $\{\vartheta_n\}$*;*

(ii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$*;*

(iii) if the sequence $\{\mathfrak{I}_n\}$ *satisfies* $\lim_{n\to\infty} \mathbb{G}(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \mathfrak{I}_{n+1}) = 0$*, then*

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{I}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0 \ \ or \ \ \limsup_{n\to\infty} \mathbb{G}(\mathfrak{I}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\{\vartheta_n\}$ $\mathbb{G}$*-converges to a unique point in* $\Xi(\Lambda)$*.*

Proof. From Lemma 2.3, and the boundedness from below by zero of the sequence

$$\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)),\}$$

we find that $\lim_{n\to\infty}\mathbb{G}(\vartheta_n,\Xi(\Lambda),\Xi(\Lambda))$ exists; say equal $\mathfrak{J}$. So, we obtain from conditions (ii) and (iii) that

$\liminf_{n\to\infty}\mathbb{G}(x_n,\Xi(\Lambda),\Xi(\Lambda))=0$ or $\limsup_{n\to\infty}\mathbb{G}(\vartheta_n,\Xi(\Lambda),\Xi(\Lambda))=0$.

From the uniqueness of $\mathfrak{J}$, then $\lim_{n\to\infty}\mathbb{G}(\vartheta_n,\Xi(\Lambda),\Xi(\Lambda))=0$. Hence, by Theorem 2.1(a), we conclude that $\{\vartheta_n\}$ $\mathbb{G}$-converges to unique point in $\Xi(\Lambda)$. □

Remark 2.4. *Theorem 2.2 makes general and reforms Theorem 2.2 in* [2] *and* [10] *where ϑ is $\mathbb{G}$-metric space instead of Θ is a metric space.*

Lemma 2.4. *Let $(\Theta,\mathbb{G})$ be a $\mathbb{G}$-metric space. Then any subsequence of a $\mathbb{G}$-convergent sequence in ϑ is $\mathbb{G}$-convergent.*

Proof. Let $\{\vartheta_n\}$ be a $\mathbb{G}$-convergence sequence in ϑ. So $\{\vartheta_n\}$ is $\mathbb{G}$-Cauchy sequence in Θ, and we take any subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$. We know that $\forall\varepsilon>0\ \exists$ a positive integer $\mathbb{N}=\mathbb{N}(\varepsilon)$ s.t $\mathbb{G}(\vartheta_n,\vartheta,\vartheta)<\frac{\varepsilon}{2}\ \forall n>\mathbb{N}$.

$\forall\varepsilon>0\ \exists\ \mathbb{N}=\mathbb{N}(\varepsilon)$ s.t $\mathbb{G}(\vartheta_n,\vartheta_{n_k},\vartheta_{n_k})<\frac{\varepsilon}{2}\ \forall n\geq n_k>\mathbb{N}$; by using $\mathbb{G}$-rectangle inequality:

$$\mathbb{G}(\vartheta_{n_k,\vartheta_{n_k}},\vartheta)<\mathbb{G}(\vartheta_n,\vartheta_{n_k},\vartheta_{n_k})+\mathbb{G}(\vartheta_n,\vartheta,\vartheta)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Then $\{\vartheta_{n_k}\}$ is $\mathbb{G}$-convergent. □

Theorem 2.3. *Let $\{\vartheta_n\}$ be a sequence in a subset Ω of $\mathbb{G}$-metric space $(\Theta,\mathbb{G})$ and $\Lambda:\Omega\times\Omega\longrightarrow\Theta$ be a quasi-nonexpansive mapping w.r.t.$\{\vartheta_n\}$ such that $\Xi(\Lambda)\neq\phi$ is closed set. Assume that*

(a) the sequence $\{\vartheta_n\}$ contains subsequence $\{\vartheta_{n_j}\}$ is $\mathbb{G}$-converging to $\vartheta^*\in\Omega$ such that there exists a continuous mapping $S:\Omega\times\Omega\longrightarrow\Omega$ fulfilling

$$S(\vartheta_{n_j})=\vartheta_{n_{j+1}}\ \forall n,j\in\mathbb{N}$$

and

$$G(S(\vartheta^*),\rho,\rho)<\mathbb{G}(\vartheta^*,\rho,\rho)$$

for some $\rho\in\Xi(\Lambda)$. Then $\vartheta^*\in\Xi(\Lambda)$ and $\lim_{n\to\infty}\vartheta_n=\vartheta^*$.

Proof. From Lemma 2.3 , the quasi-nonexpansiveness of Λ w.r.t. $\{\vartheta_n\}$ implies that

$$\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))$$

exists, say equal $r \in [0,\infty)$. Suppose that $\vartheta^* \notin \Xi(\Lambda)$. So, we have from (a) that for some $\rho \in \Xi(\Lambda)$,

$$\begin{aligned}
\mathbb{G}(\vartheta^*, \rho, \rho) &> \mathbb{G}(S(\vartheta^*), \rho, \rho) \\
&= \mathbb{G}(S(\lim_{j\to\infty} \vartheta_{n_j}), \rho, \rho) \\
&= \mathbb{G}(\lim_{j\to\infty} S(\vartheta_{n_j}), \rho, \rho)
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{G}(\lim_{j\to\infty} \vartheta_{n_{j+1}}, \rho, \rho) \\
&= \mathbb{G}(\vartheta^*, p, p).
\end{aligned}$$

This inconsistency indicates that $\vartheta^* \in \Xi(\Lambda)$. Then,

$$\begin{aligned}
r &= \lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) \\
&= \lim_{j\to\infty} \mathbb{G}(\vartheta_{n_j}, \Xi(\Lambda), \Xi(\Lambda)) \\
&= \mathbb{G}(\lim_{j\to\infty} \vartheta_{n_j}, \Xi(\Lambda), \Xi(\Lambda)) \\
&= \mathbb{G}(\vartheta^*, \Xi(\Lambda), \Xi(\Lambda)) = 0.
\end{aligned}$$

From Theorem 2.1 (b), we obtain that $\lim_{n\to\infty} \vartheta_n = \vartheta^*$. □

Corollary 2.2. *For every $\theta_0 \in \Omega$, let $\{\Lambda^n(\theta_0)\}$ be a sequence in a subset Ω of a complete $\mathbb{G}$-metric space $(\Theta, \mathbb{G})$ and $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ be a quasi-nonexpansive mapping w.r.t. $\{\Lambda^n(\theta_0)\}$ such that $\Xi(\Lambda) \neq \phi$ is closed set. Assume that*
(a) the sequence $\{\Lambda^n(\theta_0)\}$ contains subsequence $\{\Lambda^{n_j}(\theta_0)\}$ is converging to $\theta^* \in \Omega$ such that there exists continuous mapping $S : \Omega \times \Omega \longrightarrow \Omega$ satisfying

$$S(\Lambda^{n_j}(\theta_0)) = S(\Lambda^{n_{j+1}}(\theta_0)) \ \forall n, j \in \mathbb{N}$$

and

$$\mathbb{G}(S(\theta^*), \rho, \rho) < \mathbb{G}(\theta^*, \rho, \rho)$$

for some $\rho \in \Xi(\Lambda)$. Then $\theta^* \in \Xi(\Lambda)$ and $\lim_{n\to\infty} \Lambda^n(\theta_0) = \theta^*$.

Theorem 2.4. *Let Ω be a closed subset of $\mathbb{G}$-metric space Θ and $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ such that*

(i) $\Xi(\Lambda) \neq \phi$.

(ii) Λ is quasi nonexpansive ; i.e., for each $\vartheta \in \Omega$ and every $\rho \in \Xi(\Lambda)$,

$$\mathbb{G}(\Lambda(\vartheta), \rho, \rho) \leq \mathbb{G}(\vartheta, \rho, \rho\rho).$$

(iii) There exist an $x_0 \in \Omega$ such that

$$\vartheta_n = \Lambda^n(\vartheta_0) \in \Omega \ foreach \ n \geq 1.$$

Then $\{\vartheta_n\}$ $\mathbb{G}-$converges to a fixed point of Λ in Ω if and only if

$$\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Proof. Obviously, the statement $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$ is necessary. For the sufficiency, assume $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$. Given $\varepsilon > 1$, then there exists an $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) < \frac{\varepsilon}{2}$. Since for all $l, k \geq n_1$,

$$\mathbb{G}(\vartheta_l, \vartheta_k, \vartheta_k) \leq \mathbb{G}(\rho, \vartheta_l, \vartheta_1) + \mathbb{G}(\rho, \vartheta_k, \vartheta_k).$$

implies to

$$\mathbb{G}(\vartheta_l, \vartheta_k, \vartheta_k) \leq \mathbb{G}(\rho, \phi^l(\vartheta_0), \Lambda^l(\vartheta_0)) + \mathbb{G}(\rho, \Lambda^{n_k}(\vartheta_0), \Lambda^{n_k}(\vartheta_0)).$$

and

$$\mathbb{G}(\rho, \vartheta_l, \vartheta_1) = \mathbb{G}(\rho, \Lambda^l(\vartheta_0), \Lambda^l(\vartheta_0)) \leq \mathbb{G}(\Lambda^{n_1}(\vartheta_0), \rho, \rho).$$

and

$$\mathbb{G}(\rho, \vartheta_k, \vartheta_k) = \mathbb{G}(\rho, \Lambda^k(\vartheta_0), \Lambda^k(\vartheta_0)) \leq \mathbb{G}(\Lambda^{n_1}(\vartheta_0), \rho, \rho).$$

Hence,

$$\mathbb{G}(\vartheta_l, \vartheta_k, \vartheta_k) \leq \mathbb{G}(\rho, \phi^{n_1}(\vartheta_0), \Lambda^{n_1}(\vartheta_0)) + \mathbb{G}(\rho, \phi^{n_1}(\vartheta_0), \Lambda^{n_1}(\vartheta_0))$$

Taking the infimum over $\rho \in \Xi(\Lambda)$, we get the relation,

$$\mathbb{G}(\vartheta_l, \vartheta_k, \vartheta_k) < \varepsilon$$

So $\{\vartheta_n\}$ is $\mathbb{G}$-Cauchy and hence $\mathbb{G}-$ converges to some $\vartheta^* \in \Omega$, since Ω is closed. Moreover , since Λ is continuous, $\Xi(\Lambda)$ is closed and therefore

$$\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$$

implies that $\vartheta^* \in \Xi(\Lambda)$. □

Theorem 2.5. *Let $\{\vartheta_n\}$ be a sequence in a subset Ω of $\mathbb{G}$-metric space $(\Theta, \mathbb{G})$ and $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ any map such that $\Xi(\Lambda) \neq \phi$. Assume that*

(i) $\Xi(\Lambda)$ is closed set;

(ii) $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\}$ is monotonically decreasing sequence in $[0,\infty)$;

(iii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$;

(iv)If the sequence $\{\mathfrak{J}_n\}$ satisfies $\lim_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+1}) = 0$, then

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0 \ \text{ or } \ \limsup_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\{\vartheta_n\}$ $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$.

Proof. From (ii) and the boundedness from below by zero of the sequence $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\}$, we find that $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))$ exists and equals say, $\mathfrak{J}$. Therefore,

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = \limsup_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = \mathfrak{J}.$$

The conditions (iii) and (iv) assert that

$$\liminf_{n\to\infty} G(x_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$$

or

$$\limsup_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

From the uniqueness of $\mathfrak{J}$, then $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0$. Applying Theorem 2.1, we conclude that $\{\vartheta_n\}$ $\mathbb{G}$ -converges to unique point in $\Xi(\Lambda)$. □

Corollary 2.3. *Let $\{\vartheta_n\}$ be a complete $\mathbb{G}$-metric space and let $\Xi(\Lambda)$ be nonempty closed set. Assume that*

(i) T is quasi-nonexpansive with respect to $\{\vartheta_n\}$;

(ii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$*, equivalently,* $\{\vartheta_n\}$ *is* $\mathbb{G}$*-Cauchy sequence;*

(iii) if the sequence $\{\mathfrak{J}_n\}$ *satisfies* $\lim_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+1}) = 0$*, then*

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0 \;\; or \;\; \limsup_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\Xi(\Lambda)^n(x_0)$ $\mathbb{G}$*-converges to a point in* $\Xi(\Lambda)$*.*

Corollary 2.4. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of a complete* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *be a map such that* $\Xi(\Lambda) \neq \emptyset$ *is a closed set. Assume that*

(i) Λ *is weakly quasi-nonexpansive with respect to* $\{\vartheta_n\}$*;*

(ii) $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\}$ *is monotonically decreasing sequence in* $[0, \infty)$*;*

(iii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$ *;*

(iv) If the sequence $\{\mathfrak{J}_n\}$ *satisfies* $\lim_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+1}) = 0$*, then*

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\phi), \Xi(\phi)) = 0 \quad or \quad \limsup_{n\to\infty} \mathbb{G}(y_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\{\vartheta_n\}$ $\mathbb{G}$*-converges to a point in* $\Xi(\Lambda)$*.*

3. Applications

Now, we give some applications on our main results. Motivated by the work of Ahmed and Zeyada [3], we apply Theorem 2.2 for obtaining convergence theorems in $\mathbb{G}$-metric spaces.

We state the following theorems:

Theorem 3.1. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *any map such that* $\Xi(\Lambda) \neq \phi$*. Assume that* $\Xi(\Lambda)$ *is a closed set. Then* $\{\vartheta_n\}$ $\mathbb{G}$*-converges to a unique point in* $\Xi(\Lambda)$ *if and only if* $\lim_{n\to\infty} \mathbb{G}(\Xi(\vartheta_n), \Xi(\Lambda), \Xi(\Lambda)) = 0$*.*

Proof. Let $\Xi\{\vartheta_n\} \subseteq \Omega$ be $\mathbb{G}$-converges to a unique point in $\Xi(\Lambda)$. Then $\lim_{n\to\infty} \Xi(\vartheta_n)$ exists. From the closedness of $\Xi(\Lambda)$, we find that $\rho := \lim_{n\to\infty} \Xi(\vartheta_n) \in \Xi(\Lambda) = \overline{\Xi(\Lambda)}$.
Hence, we obtain that $\mathbb{G}(\rho, \Xi(\Lambda), \Xi(\Lambda)) = 0$. Since $\mathbb{G}$ is continuous, we get that

$$\lim_{n\to\infty} \mathbb{G}(\Xi(\vartheta_n), \Xi(\Lambda), \Xi(\Lambda)) = \mathbb{G}(\lim_{n\to\infty} \Xi(\vartheta_n), \Xi(\Lambda), \Xi(\Lambda)) = \mathbb{G}(\rho, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

□

Theorem 3.2. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of a* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *any map such that* $\Xi(\Lambda) \neq \phi$ *is closed set. Assume that*

(i) Λ *is quasi-nonexpansive with respect to* $\{\vartheta_n\}$*;*

(ii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$*;*

(iii) if the sequence $\{\mathfrak{J}_n\}$ *satisfies* $\lim_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+1}) = 0$*, then*

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0 \ \textit{or} \ \limsup_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\{\vartheta_n\}$ $\mathbb{G}$*-converges to a unique point in* $\Xi(\Lambda)$*.*

Theorem 3.3. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *be a quasi-nonexpansive mapping w.r.t.*$\{\vartheta_n\}$ *such that* $\Xi(\Lambda) \neq \phi$ *is closed set. Assume that*

(a) the sequence $\{\vartheta_n\}$ *contains subsequence* $\{\vartheta_{n_j}\}$ *is* $\mathbb{G}$*-converging to* $\vartheta^* \in \Omega$ *such that there exists a continuous mapping* $S : \Omega \times \Omega \longrightarrow \Omega$ *satisfying*

$$S(\vartheta_{n_j}) = \vartheta_{n_{j+1}} \ \forall n, j \in \mathbb{N}$$

and

$$\mathbb{G}(S(\vartheta^*), \rho, \rho) < \mathbb{G}(\vartheta^*, \rho, \rho)$$

for some $\rho \in \Xi(\Lambda)$*. Then* $\vartheta^* \in \Xi(\Lambda)$ *and* $\lim_{n\to\infty} \vartheta_n = \vartheta^*$*.*

Theorem 3.4. *Let* Ω *be a closed subset of* $\mathbb{G}$*-metric space* Θ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *such that*

(i) $\Xi(\Lambda) \neq \phi$*.*

(ii) Ω *is quasi nonexpansive ; i.e., for each* $\vartheta \in \Omega$ *and every* $\rho \in \Xi(\Xi(\Lambda))$*,*

$$\mathbb{G}(\Xi(\Lambda)(\vartheta), \rho, \rho) \leq \mathbb{G}(\vartheta, \rho, \rho).$$

(iii) There exist an $\vartheta_0 \in \Omega$ *such that*

$$\vartheta_n = \Lambda^n(\vartheta_0) \in \Omega \ foreach \ n \geq 1.$$

Then $\{\vartheta_n\}$ *converges to a fixed point of* $\Xi(\Lambda)$ *in* Ω *if and only if*

$$\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Theorem 3.5. *Let* $\{\vartheta_n\}$ *be a sequence in a subset* Ω *of* $\mathbb{G}$*-metric space* $(\Theta, \mathbb{G})$ *and* $\Lambda : \Omega \times \Omega \longrightarrow \Theta$ *any map such that* $\Xi(\Lambda) \neq \phi$*. Assume that*

(i) $\Xi(\Lambda)$ *is closed set;*

(ii) $\{\mathbb{G}(\vartheta_n, \Xi(\Lambda), \Xi(\Lambda))\}$ *is monotonically decreasing sequence in* $[0, \infty)$*;*

(iii) $\lim_{n\to\infty} \mathbb{G}(\vartheta_n, \vartheta_{n+1}, \vartheta_{n+1}) = 0$ *;*

(iv)If the sequence $\{\mathfrak{J}_n\}$ *satisfies* $\lim_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+1}) = 0$*, then*

$$\liminf_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0 \ \textit{or} \ \limsup_{n\to\infty} \mathbb{G}(\mathfrak{J}_n, \Xi(\Lambda), \Xi(\Lambda)) = 0.$$

Then $\{\vartheta_n\}$ $\mathbb{G}$*-converges to a unique point in* $\Xi(\Lambda)$*.*

4. Conclusion

this paper breaks new ground in the realm of G-metric spaces by establishing novel convergence theorems for unique fixed points. These theorems not only hold their own merit but also serve to refine and broaden established results within this field. The research unifies existing knowledge under a more comprehensive framework, and its applicability is showcased by demonstrating how previous theorems can be understood as special cases of the newly proposed ones. Overall, this study significantly advances our understanding of fixed point theory in G-metric spaces.

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Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M.A. Ahmed, Common fixed points of hybrid maps and an application, Computers Math. Appl. 60 (2010), 1888–1894. https://doi.org/10.1016/j.camwa.2010.07.022.

[2] M.A. Ahmed, F.M. Zeyada, On convergence of a sequence in complete metric spaces and its applications to some iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl. 274 (2002), 458–465. https://doi.org/10.1016/s0022-247x(02)00242-1.

[3] M.A. Ahmed, F.M. Zeyada, Generalization of some results to quasi-metric spaces and its applications, Bull. Int. Math. Virt. Inst. 2 (2012), 101–107.

[4] M. Ahmed, F. Zeyada, Some convergence theorems of a sequence in complete metric spaces and its applications, Fixed Point Theory Appl. 2010 (2010), 647085. https://doi.org/10.1155/2010/647085.

[5] J.P. Aubin, H. Moulin, Applied abstract analysis, Wiley, 1977. https://cir.nii.ac.jp/crid/1130000798301801216.

[6] P. Waszkiewicz, The local triangle axiom in topology and domain theory, Appl. Gen. Topol. 4 (2003), 47–70. https://doi.org/10.4995/agt.2003.2009.

[7] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.

[8] M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl. 2012 (2012), 210. https://doi.org/10.1186/1687-1812-2012-210.

[9] F.E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967), 875–882.

[10] W.V. Petryshyn, T.E. Williamson Jr., Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings, J. Math. Anal. Appl. 43 (1973), 459–497. https://doi.org/10.1016/0022-247x(73)90087-5.