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CERTAIN APPLICATIONS OF SUZUKI TYPE CONTRACTION IN S–METRIC SPACES

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Abstract. The aim this article is to investigate the triple fixed point results via Suzuki type contraction in complete S-metric space with an example and also we discussed integral equations and homotopy theory as an applications.

Keywords: common tripled fixed point; Suzuki type contraction; ω-compatible; S-completeness.

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1. Introduction

Examining the presence and uniqueness of fixed points of certain mappings in the setting of metric spaces is one of the topics of interest in nonlinear functional analysis. The Banach contraction principle is the main achievement in this direction.Fixed point theory has applications in various fields such as approximation theory, homotopy theory, integral, integro-differential and impulsive differential equations. And several metric spaces have been studied in this regard. The idea of S-metric space, generalization of G-metric space and D-metric space, was presented by Sedghi et al.[1] in 2012 . A few fixed point theorems for a self-map on an S-metric space

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were demonstrated by them. They also demonstrated the S-metric space's properties. A generalization of the results was later established by Sedghi et al. (See.[2],[10],[11],[12],[13],[14]) in the case of generalized fixed point theorems in S-metric spaces.

Suzuki recently established extended versions of the basic results of Edelstein and Banach, which sparked a lot of work in this field (See. [15],[16],[17],[18],[19]).

In the context of partially ordered metric spaces, Berinde and Borcut extended the concept of a coupled fixed point to a tripled fixed point in 2011(See [3],[4]). Aydi et al., Borcut Karapnar et al., Radenovi, and others provided some circular theorems pertaining to tripled fixed point theorems under this space (See. [5],[6],[7],[8],[9]).

The purpose of this work is to show that given Two mappings that satisfy generalized contractive conditions in S-metric space, there exists a unique common tripled fixed point, and that these mappings need modification of the distance function. Additionally, applications to integral equations is provided.

2. Preliminaries

Definition 2.1. *([1])Let $\mathscr{Q}$ be a non-empty set, and $S : \mathscr{Q}\times\mathscr{Q}\times\mathscr{Q} \to [0,\infty)$ be a function satisfiying these conditions.*

- (S_1) $S(\rho,\nu,\sigma) \geq 0$;
- (S_2) $S(\rho,\nu,\sigma) = 0$ *if and only if* $\rho = \nu = \sigma$;
- (S_3) $S(\rho,\nu,\sigma) \leq S(\rho,\rho,\varsigma)+S(\nu,\nu,\varsigma)+S(\sigma,\sigma,\varsigma)$ *for all* $\rho,\nu,\sigma,\varsigma \in \mathscr{Q}$ *(rectangle inequality)*

then S is metric on $\mathscr{Q}$ and pair $(\mathscr{Q},S)$ is known as S-metric space.

Definition 2.2. *([1]) A S- metric space on $(\mathscr{Q},S)$ is said to be symmetric if*
$S(\rho,\rho,\nu) = S(\nu,\nu,\rho)$ *for all* $\rho,\nu \in \mathscr{Q}$.

Definition 2.3. *([1]) Let $(\mathscr{Q},S)$ is a S-metric space and a sequence $\{\rho_n\}$ in $\mathscr{Q}$ is called:*

- (i) *A sequence $\{\nu_n\}$ is said to be S-Cauchy sequence if for every $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{Z}^+$ such that $S(\nu_i,\nu_j,\nu_k) < \varepsilon$, for all $i, j,k \geq n_0$.*
- (ii) *A sequence $\{\nu_n\}$ is said to be S-convergent to a point $\nu \in \mathscr{Q}$ if for each $\varepsilon > 0$, there is an integer $n_0 \in \mathbb{Z}^+$ such that $S(\nu_i,\nu_j,\nu) < \varepsilon$, for all $i, j \geq n_0$.*

(iii) *If every S-Cauchy sequence in $\mathscr{Q}$ is S-convergent in $\mathscr{Q}$ then S-complete.*

Definition 2.4. *([3]) Let $\mathscr{Q}$ be a nonempty set and let $\Gamma : \mathscr{Q}^3 \to \mathscr{Q}$ be a mapping. An element (ρ,ν,σ) is tripled fixed point of Γ if for $\rho,\nu,\sigma \in \mathscr{Q}$*

$$\begin{bmatrix} \Gamma(\rho,\nu,\sigma) \\ \Gamma(\nu,\sigma,\rho) \\ \Gamma(\sigma,\rho,\nu) \end{bmatrix} = \begin{bmatrix} \rho \\ \nu \\ \sigma \end{bmatrix}$$

Definition 2.5. *([3]) Let $\Gamma : \mathscr{Q}^3 \to \mathscr{Q}$ and $\Lambda : \mathscr{Q} \to \mathscr{Q}$ be two mappings. An element (ρ,ν,σ) is said to be a tripled coincident point of Γ and Λ if*

$$\begin{bmatrix} \Gamma(\rho,\nu,\sigma) \\ \Gamma(\nu,\sigma,\rho) \\ \Gamma(\sigma,\rho,\nu) \end{bmatrix} = \begin{bmatrix} \Lambda\rho \\ \Lambda\nu \\ \Lambda\sigma \end{bmatrix}$$

Definition 2.6. *([3]) Let $\Gamma : \mathscr{Q}^3 \to \mathscr{Q}$ and $\Lambda : \mathscr{Q} \to \mathscr{Q}$ be two mappings. An element (ρ,ν,σ) is said to be a tripled common point of Γ and Λ if*

$$\begin{bmatrix} \Gamma(\rho,\nu,\sigma) \\ \Gamma(\nu,\sigma,\rho) \\ \Gamma(\sigma,\rho,\nu) \end{bmatrix} = \begin{bmatrix} \Lambda\rho \\ \Lambda\nu \\ \Lambda\sigma \end{bmatrix} = \begin{bmatrix} \rho \\ \nu \\ \sigma \end{bmatrix}$$

Definition 2.7. *([3]) Let $(\mathscr{Q},S)$ be a S metric space. A pair (Γ,Λ) is called weakly compatible if $\Lambda(\Gamma(\rho,\rho,\sigma)) = \Gamma(\Lambda\rho,\Lambda\nu,\Lambda\sigma)$ whenever for all $\rho,\nu,\sigma \in \mathscr{Q}$ such that*

$$\begin{bmatrix} \Gamma(\rho,\nu,\sigma) \\ \Gamma(\nu,\sigma,\rho) \\ \Gamma(\sigma,\rho,\nu) \end{bmatrix} = \begin{bmatrix} \Lambda\rho \\ \Lambda\nu \\ \Lambda\sigma \end{bmatrix}$$

Theorem 2.8. *([19]) Let $(\mathscr{Q};d)$ be a complete metric space, let $\Lambda : \mathscr{Q} \to \mathscr{Q}$ be a mapping and define a nonincreasing function*

$$\Theta : [0;1) \to (\tfrac{1}{2};1] \text{ by } \Theta(r) = \begin{cases} 1, & 0 \le r \le \frac{\sqrt{5}-1}{2} \\ (1-r)r^{-2}, & \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}} \\ (1+r)^{-1}, & \frac{1}{\sqrt{2}} \le r \le 1 \end{cases}$$

If that there exists $r \in [0;1)$ such that

$$\Theta(r)d(\rho,\Lambda\rho) \le d(\rho;\nu) \text{ implies } d(\Lambda\rho;\Lambda\nu) \le d(\rho;\nu)$$

for all $\rho,\nu\in\mathscr{Q}$. Then there exists a unique fixed point a of Λ. Moreover, $\lim_{n\to\infty}\Lambda^n\rho=a$ for all $\rho\in\mathscr{Q}$.

3. Main Results

Theorem 3.1. *Let $(\mathscr{Q},S)$ be a S-metric space. Suppose that $\Gamma:\mathscr{Q}\times\mathscr{Q}\times\mathscr{Q}\to\mathscr{Q}$ and $\Lambda:\mathscr{Q}\to\mathscr{Q}$ be two mappings.*

$$\Theta(r)S(\Lambda\xi,\Lambda\zeta,\Gamma(\xi,\zeta,\varpi))\leq\max\left\{\begin{array}{c}S(\Lambda\xi,\Lambda\xi,\Lambda\varkappa),S(\Lambda\zeta,\Lambda\zeta,\Lambda\varsigma),\\ S(\Lambda\varpi,\Lambda\varpi,\Lambda\vartheta),S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)),\\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta))\end{array}\right\}\ \textit{implies}$$

$$S(\Gamma(\xi,\zeta,\varpi),\Gamma(\xi,\zeta,\varpi),\Gamma(\varkappa,\varsigma,\vartheta))\leq r\max\left\{\begin{array}{c}S(\Lambda\xi,\Lambda\xi,\Lambda\varkappa),S(\Lambda\zeta,\Lambda\zeta,\Lambda\varsigma),\\ S(\Lambda\varpi,\Lambda\varpi,\Lambda\vartheta),S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)),\\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta))\\ S(\Lambda\varkappa,\Lambda\varkappa,\Gamma(\varkappa,\varsigma,\vartheta)),S(\Lambda\varsigma,\Lambda\varsigma,\Gamma(\varsigma,\vartheta,\varkappa)),\\ S(\Lambda\vartheta,\Lambda\vartheta,\Gamma(\vartheta,\varkappa,\varsigma),)S(\Lambda\varkappa,\Lambda\varkappa,\Gamma(\xi,\zeta,\varpi)),\\ S(\Lambda\varsigma,\Lambda\varsigma,\Gamma(\zeta,\varpi,\xi)),S(\Lambda\vartheta,\Lambda\vartheta,\Gamma(\varpi,\xi,\zeta))\end{array}\right\}.$$

(3.1)

for all $\xi,\zeta,\varpi,\varkappa,\varsigma,\vartheta\in\mathscr{Q}$, where $r\in[0,1)$ and $\Theta:[0,1)\to(\frac{1}{2},1]$ defined as $\Theta(r)=\frac{1}{2+r}$ is strictly decreasing function

a) $\Gamma(\mathscr{Q}^3)\subseteq\Lambda(\mathscr{Q})$ and $\Lambda(\mathscr{Q})$ is complete,

b) pair (Γ,Λ) is ω-compatible.

Then Γ and Λ has unique common tripled fixed point in $\mathscr{Q}$.

Proof. Let $\xi,\zeta,\varpi\in\mathscr{Q}$ be an arbitrary, and from (a), we can construct the sequences $\{\xi_n\},\{\zeta_n\}$, $\{\varpi_n\}$ in $\mathscr{Q}$ as $\Gamma(\xi_n,\zeta_n,\varpi_n)=\Lambda\xi_{n+1}$, $\Gamma(\zeta_n,\varpi_n,\xi_n)=\Lambda\zeta_{n+1}$,
$\Gamma(\varpi_n,\xi_n,\zeta_n)=\Lambda\varpi_{n+1}$, where $n=0,1,2,3,...$

case(i) $\Lambda\xi_n\neq\Lambda\xi_{n+1}$ or $\Lambda\zeta_n\neq\Lambda\zeta_{n+1}$ or $\Lambda\varpi_n\neq\Lambda\varpi_{n+1}\forall n$

$\Theta(r)S(\Lambda\xi_0,\Lambda\xi_0,\Gamma(\xi_0,\zeta_0,\varpi_0))=\Theta(r)S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1)$

$\leq S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1)$

$$\leq \max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), S(\Lambda\xi_0,\Lambda\xi_0,\Gamma(\xi_0,\zeta_0,\varpi_0)), \\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Gamma(\zeta_0,\varpi_0,\xi_0)), S(\Lambda\varpi_0,\Lambda\varpi_0,\Gamma(\varpi_0,\xi_0,\zeta_0)) \end{array}\right\}.$$

Then from eqn 3.1 we get

$S(\Gamma(\xi_0,\zeta_0,\varpi_0),\Gamma(\xi_0,\zeta_0,\varpi_0),\Gamma(\xi_1,\zeta_1,\varpi_1))$

$$\leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), S(\Lambda\xi_0,\Lambda\xi_0,\Gamma(\xi_0,\zeta_0,\varpi_0)), \\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Gamma(\zeta_0,\varpi_0,\xi_0)), S(\Lambda\varpi_0,\Lambda\varpi_0,\Gamma(\varpi_0,\xi_0,\zeta_0)), \\ S(\Lambda\xi_1,\Lambda\xi_1,\Gamma(\xi_1,\zeta_1,\varpi_1)), S(\Lambda\zeta_1,\Lambda\zeta_1,\Gamma(\zeta_1,\varpi_1,\xi_1), \\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Gamma(\varpi_1,\xi_1,\zeta_1), S(\Lambda\xi_1,\Lambda\xi_1,\Gamma(\xi_0,\zeta_0,\varpi_0)), \\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Gamma(\zeta_0,\varpi_0,\xi_0)), S(\Lambda\varpi_1,\Lambda\varpi_1,\Gamma(\varpi_0,\xi_0,\zeta_0)) \end{array}\right.$$

$$\leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), \\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), \\ S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2), S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2), \\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2), S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_1), \\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_1), S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_1) \end{array}\right\}$$

$$S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2) \leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2), \\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2), S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2) \end{array}\right\}$$

(3.2)

similarly we can write

$$S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2) \quad\leq\quad r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1), S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2), \\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2), S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2) \end{array}\right\}$$

(3.3)

and

$$S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2) \leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1),S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1),\\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1),S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\} \tag{3.4}$$

Then from eqn (3.2) to (3.4) we can write

$$\max\left\{\begin{array}{c} S((\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2)),\\ S((\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2)),\\ S((\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2))\end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1),S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1),\\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1),S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\} \tag{3.5}$$

if $\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1),\\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1),\\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1)\end{array}\right\} \leq \max\left\{\begin{array}{c} S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),\\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\}.$

Then from Eqn (3.5), we have

$\max\left\{\begin{array}{c} S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),\\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),\\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\}.$

Which is contradiction to $1 \leq r$

and hence $\max\left\{\begin{array}{c} S(\Lambda\xi_1,\Lambda\xi_1,\Lambda\xi_2),\\ S(\Lambda\zeta_1,\Lambda\zeta_1,\Lambda\zeta_2),\\ S(\Lambda\varpi_1,\Lambda\varpi_1,\Lambda\varpi_2)\end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1),\\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1),\\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1)\end{array}\right\}.$

In general we can write

$$\begin{aligned}\max\left\{\begin{array}{c} S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\xi_{n+1}),\\ S(\Lambda\zeta_n,\Lambda\zeta_n,\Lambda\zeta_{n+1}),\\ S(\Lambda\varpi_1,\Lambda\varpi_n,\Lambda\varpi_{n+1})\end{array}\right\} &\leq r\max\left\{\begin{array}{c} S(\Lambda\xi_{n-1},\Lambda\xi_{n-1},\Lambda\xi_n),\\ S(\Lambda\zeta_{n-1},\Lambda\zeta_{n-1},\Lambda\zeta_n),\\ S(\Lambda\varpi_{n-1},\Lambda\varpi_{n-1},\Lambda\varpi_n)\end{array}\right\}\\ &\leq r^2\max\left\{\begin{array}{c} S(\Lambda\xi_{n-2},\Lambda\xi_{n-2},\Lambda\xi_{n-1}),\\ S(\Lambda\zeta_{n-2},\Lambda\zeta_{n-2},\Lambda\zeta_{n-1}),\\ S(\Lambda\varpi_{n-2},\Lambda\varpi_{n-2},\Lambda\varpi_{n-1})\end{array}\right\}\end{aligned}$$

$$\leq r^3 \max\left\{\begin{array}{c} S(\Lambda\xi_{n-3},\Lambda\xi_{n-3},\Lambda\xi_{n-2}), \\ S(\Lambda\zeta_{n-3},\Lambda\zeta_{n-3},\Lambda\zeta_{n-2}), \\ S(\Lambda\varpi_{n-3},\Lambda\varpi_{n-3},\Lambda\varpi_{n-2}) \end{array}\right\}$$

$$\vdots$$

$$\max\left\{\begin{array}{c} S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}), \\ S(\Lambda\zeta_{n},\Lambda\zeta_{n},\Lambda\zeta_{n+1}), \\ S(\Lambda\varpi_{1},\Lambda\varpi_{n},\Lambda\varpi_{n+1}) \end{array}\right\} \leq r^n \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$$
(3.6)

from eqn 3.6, $S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) \leq r^n \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$

$S(\Lambda\zeta_{n},\Lambda\zeta_{n},\Lambda\zeta_{n+1}) \leq r^n \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$

$S(\Lambda\varpi_{n},\Lambda\varpi_{n},\Lambda\varpi_{n+1}) \leq r^n \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$

for $m > n$ and by rectangle inequality

$S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{m}) \leq S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + S(\Lambda\xi_{m},\Lambda\xi_{m},\Lambda\xi_{n+1})$

$\leq 2S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\xi_{m})$

$\leq 2S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\xi_{n+2}) + S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\xi_{n+2}) + S(\Lambda\xi_{m},\Lambda\xi_{m},\Lambda\xi_{n+2})$

$\leq 2S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + 2S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\xi_{n+2}) + S(\Lambda\xi_{n+2},\Lambda\xi_{n+2},\Lambda\xi_{m})$

$\leq 2(S(\Lambda\xi_{n},\Lambda\xi_{n},\Lambda\xi_{n+1}) + S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\xi_{n+2}) + S(\Lambda\xi_{n+2},\Lambda\xi_{n+2},\Lambda\xi_{n+3})$

$+ \ldots + S(\Lambda\xi_{m-1},\Lambda\xi_{m-1},\Lambda\xi_{m})) \leq 2(r^n + r^{n+1} + r^{n+2} + \ldots\ldots.. + r^{m-1}) \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$

$$\leq 2r^n(1 + r + r^2 + \ldots\ldots.. + r^{m-n-1}) \max\left\{\begin{array}{c} S(\Lambda\xi_{0},\Lambda\xi_{0},\Lambda\xi_{1}), \\ S(\Lambda\zeta_{0},\Lambda\zeta_{0},\Lambda\zeta_{1}), \\ S(\Lambda\varpi_{0},\Lambda\varpi_{0},\Lambda\varpi_{1}) \end{array}\right\}$$

$$\leq 2r^n(1+r+r^2+..........)\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), \\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1) \end{array}\right\}$$

$$S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\xi_m) \leq 2\frac{r^n}{1-r}\max\left\{\begin{array}{c} S(\Lambda\xi_0,\Lambda\xi_0,\Lambda\xi_1), \\ S(\Lambda\zeta_0,\Lambda\zeta_0,\Lambda\zeta_1), \\ S(\Lambda\varpi_0,\Lambda\varpi_0,\Lambda\varpi_1) \end{array}\right\} \to 0 \; as \; n \to \infty.$$

Hence $\{\Lambda\xi_n\}$ is a Cauchy sequence in $\Lambda(\mathscr{Q})$. Similarly we can prove that $\{\Lambda\zeta_n\}$ and $\{\Lambda\varpi_n\}$ are Cauchy sequences in $\Lambda(\mathscr{Q})$. Since $\Lambda(\mathscr{Q})$ is complete, there exists ι, æ, κ in $\mathscr{Q}$ and α,β,γ in $\Lambda(\mathscr{Q})$ such that

$$\lim_{n\to\infty}\Lambda\xi_n = \alpha = \Lambda\iota \quad \lim_{n\to\infty}\Lambda\zeta_n = \beta = \Lambda æ \quad \lim_{n\to\infty}\Lambda\varpi_n = \gamma = \Lambda\kappa$$

since $\Lambda\xi_n \to \alpha \quad \Lambda\zeta_n \to \beta \quad \Lambda\varpi_n \to \gamma \quad$ as $n \to \infty$

we may assume that for infinitely many n $\Lambda\xi_n \neq \alpha,\ \Lambda\zeta_n \neq \beta,\ \Lambda\varpi_n \neq \gamma$.

Now we claim that

$$\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda æ,\Lambda æ,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)), S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}$$

$\forall \xi,\zeta,\varpi \in \mathscr{Q}$ with $\Lambda\iota \neq \Lambda\xi$, $\Lambda æ \neq \Lambda\zeta$, $\Lambda\kappa \neq \Lambda\varpi$. Let $\xi,\zeta,\varpi \in \mathscr{Q}$ with $\Lambda\iota \neq \Lambda\xi$, $\Lambda æ \neq \Lambda\zeta$, $\Lambda\kappa \neq \Lambda\varpi$. Then there exists a possitive integer $n_0 \in \mathbb{N}$ such that $S(\Lambda\iota,\Lambda æ,\Lambda\xi_n) \leq \frac{1}{3}S(\Lambda\iota,\Lambda\iota,\Lambda\xi), S(\Lambda æ,\Lambda æ,\Lambda\zeta_n) \leq \frac{1}{3}S(\Lambda æ,\Lambda æ,\Lambda\zeta), S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi_n) \leq \frac{1}{3}S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi)$

$\Theta(r)S(\Lambda\xi_n,\Lambda\xi_n,\Gamma(\xi_n,\zeta_n,\varpi_n)) = \Theta(r)S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\xi_{n+1})$

$\leq S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\xi_{n+1})$

$\leq S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\iota) + S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\iota) + S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Lambda\iota)$

$\leq 2S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\iota) + S(\Lambda\iota,\Lambda\iota,\Lambda\xi_{n+1})$

$\leq 2S(\Lambda\xi_n,\Lambda\iota,\Lambda\iota) + S(\Lambda\iota,\Lambda\iota,\Lambda\xi_{n+1})$

$\leq \frac{2}{3}S(\Lambda\xi,\Lambda\iota,\Lambda\iota) + \frac{1}{3}S(\Lambda\iota,\Lambda\iota,\Lambda\xi)$

$\leq S(\Lambda\xi,\Lambda\iota,\Lambda\iota) - S(\Lambda\iota,\Lambda\iota,\Lambda\xi_n)$

$\leq S(\Lambda\xi,\Lambda\xi,\Lambda\xi_n)$

$$\leq \max\left\{\begin{array}{c} S(\Lambda\xi,\Lambda\xi,\Lambda\xi_n), S(\Lambda\zeta,\Lambda\zeta,\Lambda\zeta_n), \\ S(\Lambda\varpi,\Lambda\varpi,\Lambda\varpi_n), S(\Lambda\xi_n,\Lambda\xi_n,\Gamma(\xi_n,\zeta_n,\varpi_n)), \\ S(\Lambda\zeta_n,\Lambda\zeta_n,\Gamma(\zeta_n,\varpi_n,\xi_n)), S(\Lambda\varpi_n,\Lambda\varpi n,\Gamma(\varpi_n,\xi_n,\zeta_n)) \end{array}\right\}.$$

Then

$$\begin{aligned} & S(\Gamma(\xi_n,\zeta_n,\varpi_n),\Gamma(\xi_n,\zeta_n,\varpi_n),\Gamma(\xi,\zeta,\varpi)) \\ \leq \; & r\max\left\{\begin{array}{c} S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\xi), S(\Lambda\zeta_n,\Lambda\zeta_n,\Lambda\zeta), \\ S(\Lambda\varpi_n,\Lambda\varpi_n,\Lambda\varpi), S(\Lambda\xi_n,\Lambda\xi_n,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta_n,\Lambda\zeta_n,\Gamma(\zeta,\varpi,\xi)), S(\Lambda\varpi_n,\Lambda\varpi n,\Gamma(\varpi,\xi,\zeta)), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) S(\Lambda\xi,\Lambda\xi,\Gamma(\xi_n,\zeta_n,\varpi_n)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta_n,\varpi_n,\xi_n)), S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi_n,\xi_n,\zeta_n)) \end{array}\right\} \end{aligned}$$

as $n \to \infty$ we have

$$S(\Lambda\iota,\Lambda\iota,\Gamma(\xi,\zeta,\varpi)) \;\leq\; r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

Similarly we can prove that

$$S(\Lambda æ,\Lambda æ,\Gamma(\zeta,\varpi,\xi)) \;\leq\; r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

and

$$S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi,\xi,\zeta)) \;\leq\; r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

From above we conclude that

$$\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda æ,\Lambda æ,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi),S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi),S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

Hence the claim. Now consider,

$$S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)) \leq S(\Lambda\xi,\Lambda\xi,\Lambda\iota)+S(\Lambda\xi,\Lambda\xi,\Lambda\iota)+S(\Gamma(\xi,\zeta,\varpi),\Gamma(\xi,\zeta,\varpi),\Lambda\iota)$$

$$\leq 2S(\Lambda\xi,\Lambda\xi,\Lambda\iota)+S(\Lambda\iota,\Lambda\iota,\Gamma(\xi,\zeta,\varpi)) \leq 2S(\Lambda\xi,\Lambda\xi,\Lambda\iota)+r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}$$

$$\leq 2\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}+r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

$$=(2+r)\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

Thus,

$$\frac{1}{2+r}S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)) \leq \max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi),S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi),S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

$$\Longrightarrow \Theta(r)S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)) \leq \max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi),S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi),S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\xi,\varpi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.$$

Then from eqn 3.1

$$S(\Gamma(\iota,æ,\kappa),\Gamma(\iota,æ,\kappa),\Gamma(\xi,\zeta,\varpi)) \quad\leq\quad r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi),S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi),S(\Lambda\iota\Lambda\iota,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda æ,\Lambda æ,\Gamma(\zeta,\varpi,\xi)),S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi,\xi,\zeta)), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)),S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)),S(\Lambda\xi,\Lambda\xi,\Gamma(\iota,æ,\kappa)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(æ,\kappa,\iota)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\kappa,\iota,æ)) \end{array}\right\}.$$

$$S(\Gamma(\iota,æ,\kappa),\Gamma(\iota,æ,\kappa),\Gamma(\xi,\zeta,\varpi)) \quad\leq\quad r\max\left\{\begin{array}{c} S(\Lambda\iota,\Lambda\iota,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\iota\Lambda\iota,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda æ,\Lambda æ,\Gamma(\zeta,\varpi,\xi)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi,\xi,\zeta)), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)), \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\iota,æ,\kappa)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(æ,\kappa,\iota)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\kappa,\iota,æ)) \end{array}\right\}.$$

Now

$$\begin{aligned} S(\Lambda\iota,\Lambda\iota,\Gamma(\iota,æ,\kappa)) &= lim_{n\to\infty}S(\Lambda\xi_{n+1},\Lambda\xi_{n+1},\Gamma(\iota,æ,\kappa)) \\ &= lim_{n\to\infty}S(\Gamma(\xi_n,\zeta_n,\varpi_n),\Gamma(\xi_n,\zeta_n,\varpi_n),\Gamma(\iota,æ,\kappa)) \end{aligned}$$

$$= lim_{n\to\infty} r \max\left\{\begin{array}{c} S(\Lambda\xi_n,\Lambda\xi_n,\Lambda\imath), \\ S(\Lambda\zeta_n,\Lambda\zeta_n,\Lambda æ), \\ S(\Lambda\varpi_n,\Lambda\varpi_n,\Lambda\kappa), \\ S(\Lambda\xi_n,\Lambda\xi_n,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda\zeta_n,\Lambda\zeta_n,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\varpi_n,\Lambda\varpi_n,\Gamma(\kappa,\imath,æ)), \\ S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)), \\ S(\Lambda\imath,\Lambda\imath,\Gamma(\xi_n,\zeta_n,\varpi_n)), \\ S(\Lambda æ,\Lambda æ,\Gamma(\zeta_n,\varpi_n,\xi_n)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\varpi_n,\xi_n,\zeta_n)) \end{array}\right\}.$$

$$\leq r \max\left\{\begin{array}{c} S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\}.$$

Similarly we can prove that

$$S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)) \leq r \max\left\{\begin{array}{c} S(\Lambda æ,\Lambda æ,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\}.$$

and

$$S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \leq r \max\left\{\begin{array}{c} S(\Lambda æ,\Lambda æ,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\}.$$

Thus,

$$\max\left\{\begin{array}{c} S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\} \leq r \max\left\{\begin{array}{c} S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\}.$$

Which holds when $\Gamma(\iota, æ, \kappa) = \Lambda\iota, \Gamma(æ, \kappa, \iota) = \Lambda æ$ and $\Gamma(\kappa, \iota, æ) = \Lambda\kappa$ therefore $(\iota, æ, \kappa)$ tripled coincidence point of Γ and Λ. Since the pair (Γ, Λ) is weakly compatible.

$$\Lambda\alpha = \Lambda^2\iota = \Lambda(\Gamma(\iota, æ, \kappa)) = \Gamma(\Lambda\iota, \Lambda æ, \Lambda\kappa) = \Gamma(\alpha, \beta, \gamma)$$

$$\Lambda\beta = \Lambda^2 æ = \Lambda(\Gamma(æ, \kappa, \iota)) = \Gamma(\Lambda æ, \Lambda\kappa, \Lambda\iota) = \Gamma(\beta, \gamma, \alpha)$$

$$\Lambda\gamma = \Lambda^2\kappa = \Lambda(\Gamma(\kappa, \iota, æ)) = \Gamma(\Lambda\kappa, \Lambda\iota, \Lambda æ) = \Gamma(\gamma, \alpha, \beta)$$

now

$$\begin{aligned}\Theta(r)S(\Lambda\alpha, \Lambda\alpha, \Gamma(\alpha, \beta, \gamma)) &\leq S(\Lambda\alpha, \Lambda\alpha, \Gamma(\alpha, \beta, \gamma)) \\ = 0 &\leq \max\left\{\begin{array}{c} S(\Lambda\iota, \Lambda\iota, \Lambda\alpha), \\ S(\Lambda æ, \Lambda æ, \Lambda\beta), \\ S(\Lambda\kappa, \Lambda\kappa, \Lambda\gamma), \\ S(\Lambda\alpha, \Lambda\alpha, \Gamma(\alpha, \beta, \gamma)), \\ S(\Lambda\beta, \Lambda\beta, \Gamma(\beta, \gamma, \alpha)), \\ S(\Lambda\gamma, \Lambda\gamma, \Gamma(\gamma, \alpha, \beta)) \end{array}\right\}.\end{aligned}$$

Then from eqn 3.1,

$$\begin{aligned}S(\Gamma(\alpha, \beta, \gamma), \Gamma(\alpha, \beta, \gamma), \Gamma(\iota, æ, \kappa)) &= S(\Lambda\alpha, \Lambda\alpha, \Lambda\iota) \\ &\leq r\max\left\{\begin{array}{c} S(\Lambda\alpha, \Lambda\alpha, \Lambda\iota), \\ S(\Lambda\beta, \Lambda\beta, \Lambda æ), \\ S(\Lambda\gamma, \Lambda\gamma, \Lambda\kappa), \\ S(\Lambda\alpha, \Lambda\alpha, \Gamma(\alpha, \beta, \gamma)), \\ S(\Lambda\beta, \Lambda\beta, \Gamma(\beta, \gamma, \alpha)), \\ S(\Lambda\gamma, \Lambda\gamma, \Gamma(\gamma, \alpha, \beta)) \\ S(\Lambda\iota, \Lambda\iota, \Gamma(\iota, æ, \kappa)), \\ S(\Lambda æ, \Lambda æ, \Gamma(æ, \kappa, \iota)), \\ S(\Lambda\kappa, \Lambda\kappa, \Gamma(\kappa, \iota, æ)) \\ S(\Lambda\iota, \Lambda\iota, \Gamma(\alpha, \beta, \gamma)), \\ S(\Lambda æ, \Lambda æ, \Gamma(\beta, \gamma, \alpha)), \\ S(\Lambda\kappa, \Lambda\kappa, \Gamma(\gamma, \alpha, \beta)) \end{array}\right\}.\end{aligned}$$

$$S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota) \leq r\max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota), \\ S(\Lambda\beta,\Lambda\beta,\Lambda æ), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \end{array}\right\}.$$

Similarly we can prove that

$$S(\Lambda\beta,\Lambda\beta,\Lambda æ) \leq r\max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota), \\ S(\Lambda\beta,\Lambda\beta,\Lambda æ), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \end{array}\right\}$$

and

$$S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \leq r\max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota), \\ S(\Lambda\beta,\Lambda\beta,\Lambda æ), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \end{array}\right\}.$$

From above we conclude that

$$\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota), \\ S(\Lambda\beta,\Lambda\beta,\Lambda æ), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \end{array}\right\} \leq r\max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\iota), \\ S(\Lambda\beta,\Lambda\beta,\Lambda æ), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\kappa) \end{array}\right\}.$$

Which holds when $\Lambda\iota = \Lambda\alpha, \Lambda æ = \Lambda\beta, \Lambda\kappa = \Lambda\gamma$ Then from above, we will write

$$\alpha = \Lambda\alpha = \Gamma(\alpha,\beta,\gamma), \beta = \Lambda\beta = \Gamma(\beta,\gamma,\alpha), \gamma = \Lambda\gamma = \Gamma(\gamma,\alpha,\beta)$$

$\therefore$ (α,β,γ) is a tripled fixed point of Γ and Λ. Now we will uniqueness of tripled fixed point. If possible (α',β',γ') is another tripled fixed point of Γ and Λ. Then

$$\begin{aligned} \Theta(r)S(\Lambda\alpha,\Lambda\alpha,\Gamma(\alpha,\beta,\gamma)) &\leq S(\Lambda\alpha,\Lambda\alpha,\Gamma(\alpha,\beta,\gamma)) \\ = 0 &\leq \max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\iota,\Lambda\alpha'), \\ S(\Lambda\beta,\Lambda\beta,\Lambda\beta'), \\ S(\Lambda\gamma,\Lambda\gamma,\Lambda\gamma'), \\ S(\Lambda\alpha,\Lambda\alpha,\Gamma(\alpha,\beta,\gamma)), \\ S(\Lambda\beta,\Lambda\beta,\Gamma(\beta,\gamma,\alpha)), \\ S(\Lambda\gamma,\Lambda\gamma,\Gamma(\gamma,\alpha,\beta)) \end{array}\right\}. \end{aligned}$$

Then from eqn 3.1

$S(\Gamma(\alpha,\beta,\gamma),\Gamma(\alpha,\beta,\gamma),\Gamma(\alpha',\beta',\gamma')) = S(\alpha,\alpha,\alpha')$

$$\leq r\max\left\{\begin{array}{c} S(\Lambda\alpha,\Lambda\alpha,\Lambda\alpha'),S(\Lambda\beta,\Lambda\beta,\Lambda\beta'),S(\Lambda\gamma,\Lambda\gamma,\Lambda\gamma'), \\ S(\Lambda\alpha,\Lambda\alpha,\Gamma(\alpha,\beta,\gamma)),S(\Lambda\beta,\Lambda\beta,\Gamma(\beta,\gamma,\alpha)),S(\Lambda\gamma,\Lambda\gamma,\Gamma(\gamma,\alpha,\beta)) \\ S(\Lambda\alpha',\Lambda\alpha',\Gamma(\alpha,\beta,\gamma)),S(\Lambda\beta',\Lambda\beta',\Gamma(\beta,\gamma,\alpha)),S(\Lambda\gamma',\Lambda\gamma',\Gamma(\gamma,\alpha,\beta)) \\ S(\Lambda\alpha',\Lambda\alpha',\Gamma(\alpha',\beta',\gamma')),S(\Lambda\beta',\Lambda\beta',\Gamma(\beta',\gamma',\alpha')),S(\Lambda\gamma',\Lambda\gamma',\Gamma(\gamma',\alpha',\beta')) \end{array}\right\}$$

$$S(\alpha,\alpha,\alpha') \leq r\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha'), \\ S(\beta,\beta,\beta'), \\ S(\gamma,\gamma,\gamma') \end{array}\right\}$$

Similarly we can write,

$$S(\beta,\beta,\beta') \quad \leq \quad r\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha'), \\ S(\beta,\beta,\beta'), \\ S(\gamma,\gamma,\gamma') \end{array}\right\}$$

and

$$S(\gamma,\gamma,\gamma') \quad \leq \quad r\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha'), \\ S(\beta,\beta,\beta'), \\ S(\gamma,\gamma,\gamma') \end{array}\right\}.$$

From above equations we can write

$$\left\{\begin{array}{c} S(\alpha,\alpha,\alpha'), \\ S(\beta,\beta,\beta'), \\ S(\gamma,\gamma,\gamma') \end{array}\right\} \quad \leq \quad r\max\left\{\begin{array}{c} S(\alpha,\alpha,\alpha'), \\ S(\beta,\beta,\beta'), \\ S(\gamma,\gamma,\gamma') \end{array}\right\}$$

which holds for $\alpha = \alpha', \beta = \beta', \gamma = \gamma'$

$\therefore (\alpha,\beta,\gamma)$ is unique common tripled fixed point of Γ and Λ.

case(ii): If $\Lambda\xi_n = \Lambda\xi_{n+1}, \Lambda\zeta_n = \Lambda\zeta_{n+1}, \Lambda\varpi_n = \Lambda\varpi_{n+1}$ for some n then $\Lambda\xi_n = \Gamma(\xi_n,\zeta_n,\varpi_n)$, $\Lambda\zeta_n = \Gamma(\zeta_n,\varpi_n,\xi_n)$, $\Lambda\varpi_n = \Gamma(\varpi_n,\xi_n,\zeta_n)$ so that (ξ_n,ζ_n,ϖ_n) is a tripled coincidence point of Γ and Λ . Now proceeding as in case (i) with $\Lambda\xi_n = \alpha$, $\Lambda\zeta_n = \beta$, $\Lambda\varpi_n = \gamma$ we can show that (α,β,γ) is the unique common tripled fixed point of Γ and Λ. □

Example 3.2. *Let* $\mathscr{Q} = [0,\infty)$ *and* $\Lambda(\xi,\zeta,\varpi) = |\zeta+\varpi-2\xi| + |\zeta-\varpi|$ *on* $(\mathscr{Q},S)$ *is a complete S-metric spaces. Let* $\Gamma : \mathscr{Q}^3 \to \mathscr{Q}$ *and* $\Lambda : \mathscr{Q} \to \mathscr{Q}$ *be defined by* $\Gamma(\xi,\zeta,\varpi) = \sin(\frac{\xi+\zeta+\varpi}{16})$ *and*

$\Lambda(\xi)=10\xi$.*Then obviously,* $\Gamma(\mathscr{Q}^3)\subseteq\Lambda(\mathscr{Q})$ *and the pair*(Γ,Λ) *is* ω*-compatible.*

And for $\xi,\zeta,\varpi\in\mathscr{Q}$

$$\frac{2}{3}S(\Lambda\imath,\Lambda æ,\Gamma(\imath,æ,\kappa))\leq S(\Lambda\imath,\Lambda æ,\Gamma(\imath,æ,\kappa)) \quad\leq\quad \max\left\{\begin{array}{c} S(\Lambda\imath,\Lambda\imath,\Lambda\xi), \\ S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi), \\ S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \end{array}\right\}.$$

Now,

$$\begin{aligned}
& S(\Gamma(\imath,æ,\kappa),\Gamma(\imath,æ,\kappa),\Gamma(\xi,\zeta,\varpi)) \\
\leq\ & |\Gamma(\imath,æ,\kappa)-\Gamma(\xi,\zeta,\varpi)| \\
\leq\ & |\sin(\frac{\imath+æ+\kappa}{16})-\sin(\frac{\xi+\zeta+\varpi}{16})| \\
\leq\ & 4|\cos(\frac{\imath+æ+\kappa+\xi+\zeta+\varpi}{16})\sin(\frac{\imath+æ+\kappa-\xi-\zeta-\varpi}{16})| \\
\leq\ & \frac{1}{8}|\imath+æ+\kappa-\xi-\zeta-\varpi| \\
\leq\ & \frac{1}{16}|10\xi-\sin(\frac{\imath+æ+\kappa}{16})| \\
\leq\ & \frac{1}{16}S(\Lambda\xi,\Lambda\xi,\Gamma(\imath,æ,\kappa)) \\
\leq\ & \frac{1}{16}\max\left\{\begin{array}{c} S(\Lambda\imath,\Lambda\imath,\Lambda\xi),S(\Lambda æ,\Lambda æ,\Lambda\zeta), \\ S(\Lambda\kappa,\Lambda\kappa,\Lambda\varpi),S(\Lambda\imath,\Lambda\imath,\Gamma(\imath,æ,\kappa)), \\ S(\Lambda æ,\Lambda æ,\Gamma(æ,\kappa,\imath)),S(\Lambda\kappa,\Lambda\kappa,\Gamma(\kappa,\imath,æ)) \\ S(\Lambda\xi,\Lambda\xi,\Gamma(\imath,æ,\kappa)),S(\Lambda\zeta,\Lambda\zeta,\Gamma(æ,\kappa,\imath)), \\ S(\Lambda\varpi,\Lambda\varpi,\Gamma(\kappa,\imath,æ))S(\Lambda\xi,\Lambda\xi,\Gamma(\xi,\zeta,\varpi)), \\ S(\Lambda\zeta,\Lambda\zeta,\Gamma(\zeta,\varpi,\xi)),S(\Lambda\varpi,\Lambda\varpi,\Gamma(\varpi,\xi,\zeta)) \end{array}\right\}.
\end{aligned}$$

Hence from main result $(0,0,0)$ is the tripled fixed point of Γ and Λ.

Corollary 3.3. *Let $(\mathscr{Q},S)$ be a complete S-metric space. Suppose that $\Gamma:\mathscr{Q}^3\to\mathscr{Q}$ be a mapping satisfying:*

$$\Theta(r)S(\xi,\xi,\Gamma(\xi,\zeta,\varpi))\leq\max\left\{\begin{array}{c}S(\xi,\xi,\sigma),S(\zeta,\zeta,\rho),\\ S(\varpi,\varpi,\rho),S(\xi,\xi,\Gamma(\xi,\zeta,\varpi)),\\ S(\zeta,\zeta,\Gamma(\zeta,\varpi,\xi))S(\varpi,\varpi,\Gamma(\varpi,\xi,\zeta))\end{array}\right\}\text{ implies}$$

$$S(\Gamma(\xi,\zeta,\varpi),\Gamma(\xi,\zeta,\varpi),\Gamma(\sigma,\rho,\rho))\leq r\max\left\{\begin{array}{c}S(\xi,\xi,\sigma),S(\zeta,\zeta,\rho),\\ S(\varpi,\varpi,\rho),S(\xi,\xi,\Gamma(\xi,\zeta,\varpi)),\\ S(\zeta,\zeta,\Gamma(\zeta,\varpi,\xi)),S(\varpi,\varpi,\Gamma(\varpi,\xi,\zeta)),\\ S(\sigma,\sigma,\Gamma(\sigma,\rho,\rho)),S(\rho,\rho,\Gamma(\rho,\rho,\sigma)),\\ S(\rho,\rho,\Gamma(\rho,\sigma,\rho)),S(\sigma,\sigma,\Gamma(\xi,\zeta,\varpi)),\\ S(\rho,\rho,\Gamma(\zeta,\varpi,\xi)),S(\rho,\rho,\Gamma(\varpi,\xi,\zeta))\end{array}\right\}.$$

for all $\xi,\zeta,\varpi,\sigma,\rho,\rho\in\mathscr{Q}$, where $r\in[0,1)$ and $\Theta:[0,1)\to[\frac{1}{2},1)$ defined as $\Theta(r)=\frac{1}{2+r}$ is a strictly decreasing function. Then there is a unique tripled fixed point of Γ in $\mathscr{Q}$.

4. Integral Equations: Applicaton

Here we will discuss, as an application to Corollary 3.3, existence of an unique solution to an initial value problem.

Theorem 4.1. *Consider the initial value problem*
$\xi'(t)=\Gamma(t,(\xi,\zeta,\varpi)(t)),\ t\in I=[0,1],\ (\xi,\zeta,\varpi)(0)=(\xi_0,\zeta_0,\varpi_0)$—-*(4.1)*

where $\Gamma:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *with* $\int_0^t\Gamma(s,\xi(s),\zeta(s),\varpi(s))ds=\max\left\{\begin{array}{c}\int_0^t\Gamma(s,\xi(s))ds,\\ \int_0^t\Gamma(s,\zeta(s))ds,\\ \int_0^t\Gamma(s,\varpi(s))ds\end{array}\right\}$

and $\xi_0,\zeta_0,\varpi_0\in\mathbb{R}$.

Then there exists unique solution in $C(I,\mathbb{R})$ for the initial value problem (4.1).

Proof. The integral equation corresponding to initial value problem (**??**) is $\xi(t) = \xi_0 + 2\int_0^t \Gamma(s,(\xi,\zeta,\varpi)(s))ds$. Let $\mathscr{Q} = C(I,\mathbb{R})$ and $S(\xi,\zeta,\varpi) = |\xi-\varpi|+|\zeta-\varpi|$, for all $\xi,\zeta,\varpi \in \mathscr{Q}$ and define $\oplus : \mathscr{Q}\times\mathscr{Q}\times\mathscr{Q} \to \mathscr{Q}$ by $\oplus(\xi,\zeta,\varpi)(t) = \frac{\xi_0}{2} + \int_0^t \Gamma(s,(\xi,\zeta,\varpi)(s))ds$. Clearly for all $\xi,\zeta,\varpi \in \mathscr{Q}$, we have

$$\frac{2}{3}S(\xi,\xi,\Gamma(\xi,\zeta,\varpi)) \le \max\left\{\begin{array}{c}(\xi,\xi,\alpha),S(\zeta,\zeta,\beta),S(\varpi,\varpi,\gamma)\\ S(\xi,\xi,\Gamma(\xi,\zeta,\varpi)),S(\zeta,\zeta,\Gamma(\zeta,\varpi,\xi)),S(\varpi,\varpi,\Gamma(\varpi,\xi,\zeta))\end{array}\right\}.$$

Now, $S(\oplus(\xi,\zeta,\varpi)(t),\oplus(\xi,\zeta,\varpi)(t),\oplus(\alpha,\beta,\gamma)(t)) = 2|(\oplus(\xi,\zeta,\varpi)(t)-\oplus(\alpha,\beta,\gamma)(t))|$

$$= 2\left|\frac{\xi_0}{2}+\int_0^t\Gamma(s,(\xi,\zeta,\varpi)(s))-\frac{\alpha_0}{2}-\int_0^t\Gamma(s,(\alpha,\beta,\gamma)(s))\right|$$

$$= |\xi(t)-\alpha(t)| = \tfrac{1}{2}S(\xi,\xi,\alpha)$$

$$\le \tfrac{1}{2}\max\left\{\begin{array}{c}S(\xi,\xi,\alpha),S(\zeta,\zeta,\beta),S(\varpi,\varpi,\gamma)\\ S(\xi,\xi,\Gamma(\xi,\zeta,\varpi)),S(\zeta,\zeta,\Gamma(\zeta,\varpi,\xi)),S(\varpi,\varpi,\Gamma(\varpi,\xi,\zeta)),\\ S(\alpha,\alpha,\Gamma(\xi,\zeta,\varpi)),S(\beta,\beta,\Gamma(\zeta,\varpi,\xi)),S(\gamma,\gamma,\Gamma(\varpi,\xi,\zeta)),\\ S(\alpha,\alpha,\Gamma(\alpha,\beta,\gamma)),S(\beta,\beta,\Gamma(\beta,\gamma,\alpha)),S(\gamma,\gamma,\Gamma(\gamma,\alpha,\beta))\end{array}\right\}.$$

Then from Corollary we can conclude that $\oplus$ has unique fixed point in $\mathscr{Q}$.

□

5. Application to Homotopy

Now we discuss the existence of unique solution to homotopy theory.

Theorem 5.1. *Assume that $(\mathscr{Q},S)$ be a complete S-metric space, $\Im$ and $\overline{\Im}$ be an open and closed subset of $\mathscr{Q}$ such that $\Im \subseteq \overline{\Im}$. Suppose $H_F : \overline{\Im}^3 \times [0,1] \to \mathscr{Q}$ be an operator satisfies.*

$\tau_0)$ *$\xi \ne H_F(\xi,\zeta,\varpi,\kappa)$, $\zeta \ne H_F(\zeta,\varpi,\xi,\kappa)$,$\varpi \ne H_F(\varpi,\xi,\zeta,\kappa)$ for each $\xi,\zeta,\varpi \in \partial\Im$ and $\kappa \in [0,1]$ (Here $\partial\Im$ is boundary of $\Im$ in $\mathscr{Q}$);*

$\tau_1)$ *for all $\xi,\zeta,\varpi,\sigma,\rho,\rho \in \overline{\Im}$ and $\kappa \in [0,1]$ such that*

$$\Theta(r)S(\xi,\xi,H_F(\xi,\zeta,\varpi,\kappa)) \le \max\left\{\begin{array}{c}S(\xi,\xi,\sigma),S(\zeta,\zeta,\rho),S(\varpi,\varpi,\rho)\\ S(\xi,\xi,H_F(\xi,\zeta,\varpi,\kappa)),\\ S(\zeta,\zeta,H_F(\zeta,\varpi,\xi,\kappa)),\\ S(\varpi,\varpi,H_F(\varpi,\xi,\zeta,\kappa))\end{array}\right\} \text{ implies}$$

$$S\left(H_F(\xi,\zeta,\varpi,\kappa), H_F(\lambda,\zeta,\kappa), H_F(\sigma,\rho,\rho,\kappa) \right) \leq r\max\left\{ \begin{array}{c} S(\xi,\xi,\sigma), S(\zeta,\zeta,\rho), S(\varpi,\varpi,\rho) \\ S(\xi,\xi,H_F(\xi,\zeta,\varpi,\kappa)), \\ S(\zeta,\zeta,H_F(\zeta,\varpi,\xi,\kappa)), \\ S(\varpi,\varpi,H_F(\varpi,\xi,\zeta,\kappa)), \\ S(\sigma,\sigma,H_F(\sigma,\rho,\rho,\kappa)), \\ S(\rho,\rho,H_F(\rho,\rho,\sigma,\kappa)), \\ S(\rho,\rho,H_F(\rho,\sigma,\rho,\kappa)), \\ S(\sigma,\sigma,H_F(\xi,\zeta,\varpi,\kappa)), \\ S(\rho,\rho,H_F(\zeta,\varpi,\xi,\kappa)), \\ S(\rho,\rho,H_F(\varpi,\xi,\zeta,\kappa)) \end{array} \right\}$$

where $r \in [0,1)$ and $\Theta : [0,1) \to [\frac{1}{2},1)$ defined as $\Theta(r) = \frac{1}{2+r}$ is a strictly decreasing function.

τ_2) $\exists\, M \geq 0 \ni S(H_F(\xi,\zeta,\varpi,\kappa), H_F(\xi,\zeta,\varpi,\kappa), H_F(\xi,\zeta,\varpi,\nu) \leq M|\kappa-\nu|$

for every $\xi,\zeta,\varpi \in \overline{\Im}$ and $\kappa,\nu \in [0,1]$.

Then $H_F(.,0)$ has a tripled fixed point $\iff$ $H_F(.,1)$ has a tripled fixed point.

Proof. Let the set

$$\mathscr{A} = \left\{ \kappa \in [0,1] : H_F(\xi,\zeta,\varpi,\kappa) = \xi, H_F(\zeta,\varpi,\xi,\kappa) = \zeta, H_F(\varpi,\xi,\zeta,\kappa) \text{ for some } \xi,\zeta,\varpi \in \Im \right\}.$$

Let $H_F(.,0)$ has a tripled fixed point in $\Im^3$ then we have $(0,0,0) \in \mathscr{A}^3$. So that $\mathscr{A}$ is non-empty set. Now we can show that $\mathscr{A}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathscr{A} = [0,1]$. As a result, $H_F(.,1)$ has a tripled fixed point in $\Im^3$. First we show that $\mathscr{A}$ closed in $[0,1]$. To see this, Let $\{\kappa_s\}_{s=1}^{\infty} \subseteq \mathscr{A}$ with $\kappa_s \to \kappa \in [0,1]$ as $s \to \infty$. We must show that $\kappa \in \mathscr{A}$. Since $\kappa_s \in \mathscr{A}$ for $s = 0,1,2,3,\cdots$, there exists sequences $\{\xi_s\},\{\zeta_s\},\{\varpi_s\}$ with $\xi_{s+1} = H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s)$, $\zeta_{s+1} = H_F(\zeta_s,\varpi_s,\xi_s,\kappa_s)$, $\varpi_{s+1} = H_F(\varpi_s,\xi_s,\zeta_s,\kappa_s)$.

Consider

$$\begin{aligned} S(\xi_s,\xi_s,\xi_{s+1}) &= S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_{s+1})) \\ &\leq S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)) + \\ &\quad S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)) + \\ &\quad S(H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_{s+1}), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_{s+1}), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)) \\ &\leq 2S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s), H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)) + M|\kappa_{s+1}-\kappa_s| \end{aligned}$$

$$\lim_{s\to\infty} S(\xi_s,\xi_s,\xi_{s+1}) \leq 2S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa),H_F(\xi_s,\zeta_s,\varpi_s,\kappa),H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa))$$

since

$$\Theta(r)S(\xi_s,\xi_s,H_F(\xi_s,\zeta_s,\varpi_s,\kappa)) \leq \max\left\{\begin{array}{c} S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\xi_s,\xi_s,H_F(\xi_s,\zeta_s,\varpi_s,\kappa)), \\ S(\zeta_s,\zeta_s,H_F(\zeta_s,\varpi_s,\xi_s,\kappa)), \\ S(\varpi_s,\varpi_s,H_F(\varpi_s,\xi_s,\zeta_s,\kappa)) \end{array}\right\}.$$

Then from $\tau_1)$

$$\begin{aligned}\lim_{s\to\infty} S(\xi_s,\xi_s,\xi_{s+1}) &\leq \lim_{s\to\infty} S(H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s),H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s),H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)) \\ &\leq \lim_{s\to\infty} r\max\left\{\begin{array}{c} S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\zeta_s,\zeta_s,\zeta_{s+1}), \\ S(\varpi_s,\varpi_s,\varpi_{s+1}), \\ S(\xi_s,\xi_s,H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s)), \\ S(\zeta_s,\zeta_s,H_F(\zeta_s,\varpi_s,\xi_s,\kappa_s)), \\ S(\varpi_s,\varpi_s,H_F(\varpi_s,\xi_s,\zeta_s,\kappa_s)), \\ S(\xi_{s+1},\xi_{s+1},H_F(\xi_{s+1},\zeta_{s+1},\varpi_{s+1},\kappa_s)), \\ S(\zeta_{s+1},\zeta_{s+1},H_F(\zeta_{s+1},\varpi_{s+1},\xi_{s+1},\kappa_s)), \\ S(\varpi_{s+1},\varpi_{s+1},H_F(\varpi_{s+1},\xi_{s+1},\zeta_{s+1},\kappa_s)), \\ S(\xi_{s+1},\xi_{s+1},H_F(\xi_s,\zeta_s,\varpi_s,\kappa_s)), \\ S(\zeta_{s+1},\zeta_{s+1},H_F(\zeta_s,\varpi_s,\xi_s,\kappa_s)), \\ S(\varpi_{s+1},\varpi_{s+1},H_F(\varpi_s,\xi_s,\zeta_s,\kappa_s)) \end{array}\right\} \\ &\leq \lim_{s\to\infty} r\max\left\{\begin{array}{c} S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\zeta_s,\zeta_s,\zeta_{s+1}), \\ S(\varpi_s,\varpi_s,\varpi_{s+1}) \end{array}\right\}\end{aligned}$$

$$\therefore \lim_{s\to\infty} \max\left\{\begin{array}{c} S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\zeta_s,\zeta_s,\zeta_{s+1}), \\ S(\varpi_s,\varpi_s,\varpi_{s+1}) \end{array}\right\} \leq \lim_{s\to\infty} r\max\left\{\begin{array}{c} S(\xi_s,\xi_s,\xi_{s+1}), \\ S(\zeta_s,\zeta_s,\zeta_{s+1}), \\ S(\varpi_s,\varpi_s,\varpi_{s+1}) \end{array}\right\}.$$

It follows that $\lim_{s\to\infty} S(\xi_s,\xi_s,\xi_{s+1}) = 0$, $\lim_{s\to\infty} S(\zeta_s,\zeta_s,\zeta_{s+1}) = 0$, $\lim_{s\to\infty} S(\varpi_s,\varpi_s,\varpi_{s+1}) = 0$ Now we will show that $\{\xi_s\},\{\zeta_s\},\{\varpi_s\}$ are Cauchy sequences in $(\mathscr{Q},S)$. Assume that if possible $\{\xi_s\},\{\zeta_s\},\{\varpi_s\}$ are not Cauchy sequences in $(\mathscr{Q},S)$. Then there exists $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{p_k\}$ and $\{q_k\}$ such that $p_k > q_k$, $S\left(\xi_{p_k},\xi_{p_k},\xi_{q_k}\right) \geq \varepsilon$, $S\left(\zeta_{p_k},\zeta_{p_k},\zeta_{q_k}\right) \geq \varepsilon$,, $S\left(\varpi_{p_k},\varpi_{p_k},\varpi_{q_k}\right) \geq \varepsilon$
and $S\left(\xi_{p_k},\xi_{p_k},\xi_{q_{k-1}}\right) < \varepsilon$, $S\left(\zeta_{p_k},\zeta_{p_k},\zeta_{q_{k-1}}\right) < \varepsilon$,, $S\left(\varpi_{p_k},\varpi_{p_k},\varpi_{q_{k-1}}\right) < \varepsilon$.
By using the rectagular inequality
$\varepsilon \leq S\left(\xi_{q_k},\xi_{q_k},\xi_{p_k}\right) \leq 2S\left(\xi_{q_k},\xi_{q_k},\xi_{q_{k+1}}\right) + S\left(\xi_{p_k},\xi_{p_k},\xi_{q_{k+1}}\right)$
as $k\to\infty$, $\varepsilon \leq \lim_{k\to\infty} S\left(\xi_{q_{k+1}},\xi_{q_{k+1}},\xi_{p_k}\right)$
$\leq \lim_{k\to\infty} S\left(H_F(\xi_{q_{k+1}},\zeta_{q_{k+1}},\varpi_{q_{k+1}},\kappa_{q_{k+1}}), H_F(\xi_{q_{k+1}},\zeta_{q_{k+1}},\varpi_{q_{k+1}},\kappa_{q_{k+1}}), H_F(\xi_{p_k},\zeta_{p_k},\varpi_{p_k},\kappa_{p_k})\right)$

$$\leq \lim_{k\to\infty} r\max\left\{\begin{array}{c} S\left(\xi_{q_{k+1}},\xi_{q_{k+1}},\xi_{p_k}\right), \\ S\left(\zeta_{q_{k+1}},\zeta_{q_{k+1}},\zeta_{p_k}\right), \\ S\left(\varpi_{q_{k+1}},\varpi_{q_{k+1}},\varpi_{p_k}\right), \\ S\left(\xi_{q_{k+1}},\xi_{q_{k+1}},H_F(\xi_{q_{k+1}},\zeta_{q_{k+1}},\varpi_{q_{k+1}},\kappa_{q_{k+1}})\right), \\ S\left(\zeta_{q_{k+1}},\zeta_{q_{k+1}},H_F(\zeta_{q_{k+1}},\varpi_{q_{k+1}},\xi_{q_{k+1}},\kappa_{q_{k+1}})\right), \\ S\left(\varpi_{q_{k+1}},\varpi_{q_{k+1}},H_F(\varpi_{q_{k+1}},\xi_{q_{k+1}},\zeta_{q_{k+1}},\kappa_{q_{k+1}})\right), \\ S(\xi_{p_k},\xi_{p_k},H_F(\xi_{p_k},\zeta_{p_k},\varpi_{p_k},\kappa_{p_k})), \\ S(\zeta_{p_k},\zeta_{p_k},H_F(\zeta_{p_k},\varpi_{p_k},\xi_{p_k},\kappa_{p_k})), \\ S(\varpi_{p_k},\varpi_{p_k},H_F(\varpi_{p_k},\xi_{p_k},\zeta_{p_k},\kappa_{p_k})), \\ S\left(\xi_{p_k},\xi_{p_k},H_F(\xi_{q_{k+1}},\zeta_{q_{k+1}},\varpi_{q_{k+1}},\kappa_{q_{k+1}})\right), \\ S\left(\zeta_{p_k},\zeta_{p_k},H_F(\zeta_{q_{k+1}},\varpi_{q_{k+1}},\xi_{q_{k+1}},\kappa_{q_{k+1}})\right), \\ S\left(\varpi_{p_k},\varpi_{p_k},H_F(\varpi_{q_{k+1}},\xi_{q_{k+1}},\zeta_{q_{k+1}},\kappa_{q_{k+1}})\right) \end{array}\right\}$$

$$\leq \lim_{k\to\infty} r\max\left\{\begin{array}{c} S\left(\xi_{q_{k+1}},\xi_{q_{k+1}},\xi_{p_k}\right), \\ S\left(\zeta_{q_{k+1}},\zeta_{q_{k+1}},\zeta_{p_k}\right), \\ S\left(\varpi_{q_{k+1}},\varpi_{q_{k+1}},\varpi_{p_k}\right) \end{array}\right\}$$

as above we can conclude that

$\lim_{k\to\infty} S\left(\xi_{q_{k+1}}, \xi_{q_{k+1}}, \xi_{p_k}\right) = 0, S\left(\zeta_{q_{k+1}}, \zeta_{q_{k+1}}, \zeta_{p_k}\right) = 0, S\left(\varpi_{q_{k+1}}, \varpi_{q_{k+1}}, \varpi_{p_k}\right) = 0 \quad \therefore \ \varepsilon \ \leq \ 0$

which is a contradiction

$\therefore \{\xi_s\}, \{\zeta_s\}, \{\varpi_s\}$ are Cauchy sequences in $(\mathscr{Q}, S)$. and by completeness of $(\mathscr{Q}, S)$ there exists $\sigma, \rho, \rho \in \mathscr{Q}$ with $\lim_{s\to\infty} \xi_{s+1} = \sigma, \lim_{s\to\infty} \zeta_{s+1} = \rho, \lim_{s\to\infty} \varpi_{s+1} = \rho$ since

$$\Theta(r) S(\sigma, \rho, \rho, H_F(\sigma, \rho, \rho, \kappa)) \leq \max \left\{ \begin{array}{c} S(\sigma, \sigma, \xi_s), \\ S(\rho, \rho, \zeta_s), \\ S(\rho, \rho, \varpi_s), \\ S(\sigma, \sigma, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\rho, \rho, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\rho, \rho, H_F(\rho, \sigma, \rho, \kappa)) \end{array} \right\}.$$

Then $\lim_{s\to\infty} S(H_F(\xi_s, \zeta_s, \varpi_s, \kappa), H_F(\xi_s, \zeta_s, \varpi_s, \kappa), H_F(\sigma, \rho, \rho, \kappa))$

$$\leq \lim_{s\to\infty} r \max \left\{ \begin{array}{c} S(\xi_s, \xi_s, \sigma), \\ S(\zeta_s, \zeta_s, \rho), \\ S(\varpi_s, \varpi_s, \rho), \\ S(\sigma, \sigma, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\rho, \rho, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\rho, \rho, H_F(\rho, \sigma, \rho, \kappa)), \\ S(\xi_s, \xi_s, H_F(\xi_s, \zeta_s, \varpi_s, \kappa)), \\ S(\zeta_s, \zeta_s, H_F(\xi_s, \zeta_s, \varpi_s, \kappa)), \\ S(\varpi_s, \varpi_s, H_F(\xi_s, \zeta_s, \varpi_s, \kappa)), \\ S(\xi_s, \xi_s, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\zeta_s, \zeta_s, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\varpi_s, \varpi_s, H_F(\rho, \sigma, \rho, \kappa)) \end{array} \right\} \leq r \max \left\{ \begin{array}{c} S(\sigma, \sigma, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\rho, \rho, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\rho, \rho, H_F(\rho, \sigma, \rho, \kappa)) \end{array} \right\}$$

$$\therefore \max \left\{ \begin{array}{c} S(\sigma, \sigma, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\rho, \rho, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\rho, \rho, H_F(\rho, \sigma, \rho, \kappa)) \end{array} \right\} \leq r \max \left\{ \begin{array}{c} S(\sigma, \sigma, H_F(\sigma, \rho, \rho, \kappa)), \\ S(\rho, \rho, H_F(\rho, \rho, \sigma, \kappa)), \\ S(\rho, \rho, H_F(\rho, \sigma, \rho, \kappa)) \end{array} \right\}$$

It follows $H_F(\sigma, \rho, \rho, \kappa) = \sigma, H_F(\rho, \rho, \sigma, \kappa) = \rho, H_F(\rho, \sigma, \rho, \kappa) = \rho$.

$\therefore \kappa \in \mathscr{A}$, hence $\mathscr{A}$ is closed in [0,1]. Let $\kappa_0 \in \mathscr{A}$ then there exists $\xi_0, \zeta_0, \varpi_0 \in \mathfrak{I}$ with

$\xi_0 = H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0), \zeta_0 = H_F(\zeta_0,\varpi_0,\xi_0,\kappa_0), \varpi_0 = H_F(\varpi_0,\xi_0,\zeta_0,\kappa_0)$. Since $\mathfrak{I}$ is open then there exist $r>0$ such that

$B_S(\xi_0,r) \subseteq \mathfrak{I}$. Choose $\kappa \in (\kappa_0-\varepsilon,\kappa_0+\varepsilon)$ such that $|\kappa-\kappa_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}$.

$\xi \in \overline{B_S(\xi_0,r)} = \{\xi \in \mathscr{A}/S(\xi,\xi,\xi_0) \leq r + S(\xi_0,\xi_0,\xi_0)\}$. Also

$$\Theta(r)S(\xi,\xi,H_F(\xi_0,\zeta_0,\varpi_0,\kappa)) \leq \max\left\{\begin{array}{c} S(\xi,\xi,\xi_0),\\ S(\zeta,\zeta,\zeta_0),\\ S(\varpi,\varpi,\varpi_0),\\ S(\xi,\xi,H_F(\xi_0,\zeta_0,\varpi_0,\kappa)),\\ S(\zeta,\zeta,H_F(\zeta_0,\varpi_0,\xi_0,\kappa)),\\ S(\varpi,\varpi,H_F(\varpi_0,\xi_0,\zeta_0,\kappa)) \end{array}\right\}$$

Now $S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),\xi_0) = S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0))$

$\leq S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa_0)) + S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa_0)) +$

$S(H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0),H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0),H_F(\xi,\zeta,\varpi,\kappa_0))$

$\leq 2M|\kappa-\kappa_0| + S(H_F(\xi,\zeta,\varpi,\kappa_0),H_F(\xi,\zeta,\varpi,\kappa_0),H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0))$

$\leq \frac{2}{M^{n-1}} + S(H_F(\xi,\zeta,\varpi,\kappa_0),H_F(\xi,\zeta,\varpi,\kappa_0),H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0))$

as $n \in \infty$, we get

$$S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),\xi_0) = S(H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi,\zeta,\varpi,\kappa),H_F(\xi_0,\zeta_0,\varpi_0,\kappa_0))$$

$$\leq r\max\left\{\begin{array}{c} S(\xi,\xi,\xi_0),\\ S(\zeta,\zeta,\zeta_0),\\ S(\varpi,\varpi,\varpi_0),\\ S(\xi,\xi,H_F(\xi_0,\zeta_0,\varpi_0,\kappa)),\\ S(\zeta,\zeta,H_F(\zeta_0,\varpi_0,\xi_0,\kappa)),\\ S(\varpi,\varpi,H_F(\varpi_0,\xi_0,\zeta_0,\kappa)),\\ S(\xi_0,\xi_0,H_F(\xi_0,\zeta_0,\varpi_0,\kappa)),\\ S(\zeta_0,\zeta_0,H_F(\zeta_0,\varpi_0,\xi_0,\kappa)),\\ S(\varpi_0,\varpi_0,H_F(\varpi_0,\xi_0,\zeta_0,\kappa)),\\ S(\xi_0,\xi_0,H_F(\xi,\zeta,\varpi,\kappa)),\\ S(\zeta_0,\zeta_0,H_F(\zeta,\varpi,\xi,\kappa)),\\ S(\varpi_0,\varpi_0,H_F(\varpi,\xi,\zeta,\kappa)) \end{array}\right\}$$

$$\leq r\max\left\{\begin{array}{c} S(\xi,\xi,\xi_0),\\ S(\zeta,\zeta,\zeta_0),\\ S(\varpi,\varpi,\varpi_0)\end{array}\right\}$$

So,

$$\max\left\{\begin{array}{c} S(\xi,\xi,\xi_0),\\ S(\zeta,\zeta,\zeta_0),\\ S(\varpi,\varpi,\varpi_0)\end{array}\right\}\leq r\max\left\{\begin{array}{c} S(\xi,\xi,\xi_0),\\ S(\zeta,\zeta,\zeta_0),\\ S(\varpi,\varpi,\varpi_0)\end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} r+S(\xi_0,\xi_0,\xi_0),\\ r+S(\zeta_0,\zeta_0,\zeta_0),\\ r+S(\varpi_0,\varpi_0,\varpi_0)\end{array}\right\}.$$

As a result, for each fixed $\kappa\in(\kappa_0-\varepsilon,\kappa_0+\varepsilon)$, $H_F(.,\kappa):\overline{B_S(\xi_0,r)}\to\overline{B_S(\xi_0,r)}$, $H_F(.,\kappa):\overline{B_S(\zeta_0,r)}\to\overline{B_S(\zeta_0,r)}$, $H_F(.,\kappa):\overline{B_S(\varpi_0,r)}\to\overline{B_S(\varpi_0,r)}$. Then all conditions of Theorem 5.1 are satisfied. Thus we conclude that $H(.,\kappa)$ has a tripled fixed point in $\overline{\Im}^3$. But this must be in $\Im^3$. Since (τ_0) holds. Thus, $\kappa\in\mathscr{A}$ for any $\kappa\in(\kappa_0-\varepsilon,\kappa_0+\varepsilon)$. Hence $(\kappa_0-\varepsilon,\kappa_0+\varepsilon)\subseteq\mathscr{A}$. Clearly $\mathscr{A}$ is open in [0, 1]. For the reverse implication, we use the same strategy. □

6. Conclusion

For two mappings, we made sure a common tripled fixed point existed and was unique via generalized contractive condition in S-metric space. And applications have been provided.

Conflict of Interests

The authors declare that there is no conflict of interests.

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