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COUPLED FIXED POINT THEOREM ON ORTHOGONAL S-METRIC SPACES AND APPLICATION TO COUPLED SYSTEM OF NONLINEAR FRACTIONAL LANGEVIN EQUATIONS

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Abstract. In this article, we establish a coupled fixed point theorem within the framework of orthogonal S-metric spaces. Our results extend and generalize several well-known findings that are already present in the literature. Furthermore, we explore the practical applications of our primary results by examining their relevance to coupled systems of nonlinear fractional Langevin equations.

Keywords: Caputo-Hadamard fractional calculus; orthogonal metric space; coupled fixed point; S-metric space.

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1. Introduction

A fixed point theorem(FPT) in the environment of metric spaces defines the circumstances under which a self-mapping of a metric space has an invariant point. These theorems are foundational in various branches of mathematics, including functional analysis, topology, and dynamical systems theory. FPT's are basic tools in mathematics, providing a theoretical foundation for the existence & uniqueness for solutions to equations involving self-mappings on metric

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spaces. They have widespread applications across various fields and serve as the basis for many fundamental results in mathematical analysis and beyond.

Coupled fixed point (CFP) theorems are basic conclusions in the theory of metric spaces that specify the circumstances under which two or more self-mappings share a fixed point. This concept finds applications in various areas of mathematics, particularly in the investigation of differential equations, optimization problems & dynamic systems. CFP theorems provide powerful tools for establishing the existence & uniqueness of solutions to coupled mapping problems in metric spaces, making them indispensable in both theoretical and applied mathematics.

In 2006, Bhaskar and Lakshmikanthan [1] established the idea of CFP of a mapping & demonstrated certain CFP outcomes in partly ordered metric spaces. Lakshmikanthan et al. [2] introduced a mixed g-monotone mapping and demonstrated linked coincidence and common FPT's with nonlinear contractive mappings in partly ordered complete metric spaces. Sabetghadam et al. [3] proved multiple CFP theorems that satisfy distinct contractive criteria in cone metric spaces. Chuanzhi et al. [4] proposed CFP theorems for mapping meeting various contractive requirements on a $\mathscr{C}^*$-algebra. Bulbul et al. [5] demonstrated the existence & uniqueness of CFP theorems that meet novel rational contractive requirements for three mappings. In 2012, Sedghi et al. [6] developed S-metric spaces. Sedghi and Shobe [7] established a generalized FPT for S-metric spaces. Chouhan et al. [8] proposed a novel fixed point theory with expansive mappings on S-metric space. Ajay et al. [9] introduced CFP theorem in S-metric spaces.

In 2017, Gordji et al. [10] introduced the orthogonality in complete metric spaces. In orthogonal metric spaces ($\mathscr{O}\mathscr{M}S$) the existence & uniqueness of the first order ordinary differential equation solution was established by Gordji and Habibi [11]. Senapati et al. [12] established Banach's FPT in $\mathscr{O}\mathscr{M}S$ using ω-distance. Gordji et al. [13] established the existence & uniqueness of fixed points for mappings of ε-connected $\mathscr{O}\mathscr{M}S$. Gungor et al. [14] established certain FPT's for $\mathscr{O}\mathscr{M}S$ by adjusting distance functions. Yang et al. [15] presented an orthogonal (Q,ψ)-contraction. Sawangsup et al. [16] established orthogonal Q-contraction mapping. Sawangsup et al. [17] proposed the notion of orthogonal $\mathscr{Z}$-contraction mappings. Gunaseelan et al. [18] introduced an orthogonal Q-expanding type mappings. Arul Joseph et al. [19] proved FPT's using orthogonal triangular μ-admissibility on $\mathscr{O}\mathscr{M}S$. Ismat et al. [20] generalized orthogonal -Suzuki contraction mapping. Arul Joseph et al. [21] proposed the concept of orthogonally triangular μ-admissible contraction. Gunaseelan et al. [22] proved CFP theorem in $\mathscr{O}\mathscr{M}S$. Boyd-Wong and Matkowski type FPT's in $\mathscr{O}\mathscr{M}S$ was proved by Singh et al. [23]. In 2021, FPT's on orthogonal S-metric spaces is proved by Zeinab et al. [24] using Banach contraction.

In the past three decades, fractional calculus has become increasingly popular and important, mainly due to its demonstrated applications in a variety of seemingly diverse & extensive fields. Mainardi et al. [25] revisited Brownian motion through the lens of the fractional Langevin equation, which is revealed to be a specific instance of the generalized Langevin equation introduced by Kubo [26]. A coupled system of Langevin equations that incorporates both Hadamard-type & Riemann-Liouville fractional derivatives. We examine a coupled system of nonlinear Langevin equations with fractional components, represented by the following form

$$\begin{cases} \mathscr{D}^{\mu_1}(\phi_1\mathscr{D}^{\sigma_1}+\lambda_1)\wp_1(\pi)=\eta_1(\pi,\wp_1(\pi),\wp_2(\pi)), \pi\in\mathscr{F}=[1,\mathscr{A}],\\ \mathscr{D}^{\mu_2}(\phi_2\mathscr{D}^{\sigma_2}+\lambda_2)\wp_2(\pi)=\eta_2(\pi,\wp_1(\pi),\wp_2(\pi)), \pi\in\mathscr{F}=[1,\mathscr{A}],\\ \wp_1(1)=\xi_1\wp_1(\mathscr{A}), \wp_1^1(1)=0=\wp_1^1(\mathscr{A}), \xi_1\neq 1,\\ \wp_2(1)=\xi_2\wp_2(\mathscr{A}), \wp_2^1(1)=0=\wp_2^1(\mathscr{A}), \xi_2\neq 2, \end{cases} \tag{1}$$

where $\mu_1,\mu_2\in(0,1]$, $\sigma_1,\sigma_2\in(1,2]$, $\phi_1,\phi_2>0$, and $\mathscr{D}^{\vartheta}$ is the Caputo-Hadamard type fractional derivative where $\vartheta\in\{\mu_1,\mu_2,\sigma_1,\sigma_2\}$. It should be emphasized that the boundary conditions associated with these equations are of a non-periodic nature & the order of derivatives is specified across distinct intervals. Kilbas et al. [27] established the existence & uniqueness of solutions for Cauchy Type & Cauchy problems. They provided explicit solutions for linear differential equations & their corresponding initial-value problems by reducing them to Volterra integral equations and employing operational & compositional methods. Additionally, they applied one- and multidimensional Fourier integral transforms, Mellin & Laplace to derive differential equations. Furthermore, they developed a theory concerning ”sequential linear fractional differential equations”.

Based on the generality of CFP theorem on S-metric space and orthogonality condition, we are the first who establish CFP theorem on orthogonal S-metric space with supporting example. Finally, an application is given to prove the existence & uniqueness of the coupled system of nonlinear fractional Langevin equations.

2. Preliminaries

Definition 2.1. *[6] Let ∇ be a non-void set. An S-metric on ∇ is a mapping $S : \nabla^3 \to [0, \infty)$ that satisfies the following requirements, for each $\varpi, \xi, \rho, \mathfrak{c} \in \nabla$,*

(i) $S(\varpi, \xi, \rho) \geq 0$,

(ii) $S(\varpi, \xi, \rho) = 0 \quad iff \quad \varpi = \xi = \rho$,

(iii) $S(\varpi, \xi, \rho) \leq S(\varpi, \varpi, \mathfrak{c}) + S(\xi, \xi, \mathfrak{c}) + S(\rho, \rho, \mathfrak{c})$.

Then the pair (∇, S) is called an S-metric space.

Definition 2.2. *[1] Let $(\nabla, \leq)$ be a partially ordered set and $Q : \nabla \times \nabla \to \nabla$ be a function. The function Q is said to exhibit the mixed monotone property if $Q(\varpi, \xi)$ is non-decreasing in ϖ and non-increasing in ξ. (i.e), for any $\varpi, \xi \in \nabla$,*

$$\varpi_1 \leq \varpi_2 \Rightarrow Q(\varpi_1, \xi) \leq Q(\varpi_2, \xi), \quad for \quad \varpi_1, \varpi_2 \in \nabla \quad and$$

$$\xi_1 \leq \xi_2 \Rightarrow Q(\varpi, \xi_2) \leq Q(\varpi, \xi_1), \quad for \quad \xi_1, \xi_2 \in \nabla.$$

Definition 2.3. *[1] An element $(\varpi, \xi) \in \nabla \times \nabla$ is said to be a CFP of the mapping $Q : \nabla \times \nabla \to \nabla$ if $Q(\varpi, \xi) = \varpi$ and $Q(\xi, \varpi) = \xi$.*

Definition 2.4. *[1] Let $(\nabla, \leq)$ be a partially ordered set with a metric S, making (∇, S) a metric space. Additionally, define a partial ordering on the product space $\nabla \times \nabla$ as follows: for $(\varpi, \xi), (\omega, \vartheta) \in \nabla \times \nabla$, define*

$$(\omega, \vartheta) \leq (\varpi, \xi) \Leftrightarrow \varpi \leq \omega, \vartheta \leq \xi.$$

Definition 2.5. *[10] Consider a binary relation $\perp$ $(br_\perp)$ on a non-void set ∇. If $br_\perp$ adheres to the following condition:*

$$\exists \quad \varpi_0 \quad (\forall \quad \xi \in \nabla, \xi \perp \varpi_0) \quad or \quad (\forall \quad \xi \in \nabla, \varpi_0 \perp \xi),$$

then pair, $(\nabla,\perp)$ *known as an orthogonal set*$(\perp - set)$ *and element* ϖ_0 *is referred as an orthogonal element* $(\perp - element)$.

Example 2.1. *[10] Let* $\nabla = 2\mathbb{Z}$ *and define a* $br_\perp$ *on* $2\mathbb{Z}$ *such that* $\kappa\perp\ell$ *if* $\kappa\cdot\ell = 0$. *Then* $(2\mathbb{Z},\perp)$ *forms a* $\perp$*-set with 0 as a* $\perp$*-element.*

Definition 2.6. *[10] Let* $(\nabla,\perp)$ *be an* $\perp - set$. *A sequence* $\{\varpi_\ell\}_{\ell\in\mathbb{N}}$ *is referred as an orthogonal sequence*$(\perp - sequence)$ *if*

$$(\forall \quad \ell\in\mathbb{N}; \varpi_\ell \perp \varpi_{\ell+1}) \quad or \quad (\forall \quad \ell\in\mathbb{N}; \varpi_{\ell+1} \perp \varpi_\ell).$$

Definition 2.7. *[11] Consider a* $br_\perp$ *on a non-void set* ∇ *equipped with a metric* δ. *Then the triplet* $(\nabla,\perp,\delta)$ *is termed as an orthogonal metric space. The set* ∇ *is labeled as an orthogonal complete if every Cauchy* $\perp$*-sequence converges within* ∇.

Definition 2.8. *[11] Let* $(\nabla,\perp,\delta)$ *be an orthogonal metric space and* χ *be a self-map on* ∇. *If whenever an orthogonal sequence* $\{\varpi_\ell\}_{\ell\in\mathbb{N}} \to \varpi$ *implies* $\chi\varpi_\ell \to \chi\varpi$ *as* $\ell\to\infty$ *, then* χ *is termed as an orthogonal continuous* $(\perp - continuous)$ *at* ϖ.

Definition 2.9. *[11] Consider a* $br_\perp$ *on a non-void set* ∇ *and* $(\nabla,\perp)$ *be an* $\perp - set$. *A function* $\chi : \nabla \to \nabla$ *is termed as an orthogonal preserving* $(\perp - preserving)$ *if* $\chi\varpi\perp\chi\rho$ *whenever* $\varpi\perp\rho$.

Definition 2.10. *[27] Let* $\mathfrak{r} > 0$ *and* $\mathfrak{h} : [1,\infty) \to \mathbb{R}$. *The* $\mathfrak{r}$*th-order Hadamard fractional integral of* $\mathfrak{h}$ *is defined as*

$$\mathscr{I}^{\mathfrak{r}}\mathfrak{h}(\pi) = \frac{1}{\Gamma(\mathfrak{r})}\int_1^\pi (log\frac{\pi}{\mathfrak{t}})^{\mathfrak{q}-1}\frac{\mathfrak{g}(\mathfrak{t})}{\mathfrak{t}}d\mathfrak{t}, \quad 1 \le \pi < \infty,$$

provided that the integral exists.

Definition 2.11. *[27] Let* $\mathfrak{r} \ge 0$, $\ell = [\mathfrak{r}+1]$ *and* $\mathfrak{h} : [1,\infty) \to \mathbb{R}$. *The* $\mathfrak{r}$*th-order Hadamard fractional derivative of* $\mathfrak{h}$ *is defined as*

$$^{\mathscr{H}}\mathscr{D}^{\mathfrak{r}}\mathfrak{h}(\pi) = \frac{1}{\Gamma(\ell-\mathfrak{r})}(\pi\frac{d}{d\pi})^\ell \int_1^\pi (log\frac{\pi}{\mathfrak{t}})^{\ell-\mathfrak{r}-1}\frac{\mathfrak{h}(\mathfrak{t})}{\mathfrak{t}}d\mathfrak{t},$$

provided that the integral exists.

In this article, we establish a novel on CFP Theorem on orthogonal S-metric space. Finally, application of determining CFP Theorem for banach contractions of the Fractional differential equations from these results is provided.

3. Main Results

Definition 3.1. *Consider a $br_{\perp}$ defined on a non-void set ∇. An S-metric on ∇ is a function $S : \nabla^3 \to [0,\infty)$ that satisfies the following requirements, for each $\varpi, \xi, \rho, \mathfrak{c} \in \nabla$ with $\varpi \perp \xi \perp \rho \perp \mathfrak{c}$,*

(i) $S(\varpi, \xi, \rho) \geq 0$,

(ii) $S(\varpi, \xi, \rho) = 0 \quad iff \quad \varpi = \xi = \rho$,

(iii) $S(\varpi, \xi, \rho) \leq S(\varpi, \varpi, \mathfrak{c}) + S(\xi, \xi, \mathfrak{c}) + S(\rho, \rho, \mathfrak{c})$.

Then the pair $(\nabla, S, \perp)$ is referred as an orthogonal S-metric space.

Example 3.1. *Consider a $br_{\perp}$ defined on a non-void set ∇. Let $\nabla = \mathbb{R}^n$ and $\|.\|$ is a norm on ∇, then*

$$S(\varpi, \xi, \rho) = \|\xi + \rho - 2\varpi\| + \|\xi - \rho\|, \quad \forall \varpi, \xi, \rho \in \nabla \quad with \quad \varpi \perp \xi \perp \rho$$

is an orthogonal S-metric space on ∇.

Definition 3.2. *Let $(\nabla, S, \perp)$ be an orthogonal S-metric space and $\{\varpi_\ell\}_{\ell \geq 0}$ be an $\perp$-sequence in ∇.*

(i) An $\perp$-sequence $\{\varpi_\ell\}$ is convergent to $\varpi \in \nabla$ if, for each $\varepsilon > 0$, $\exists\ \mathscr{J} \in \mathbb{N}$ s.t $S(\varpi_\ell, \varpi_\ell, \varpi) < \varepsilon$, $\forall\ \ell \geq \mathscr{J}$.

(ii) An $\perp$-sequence $\{\varpi_\ell\}$ in ∇ is said to be Cauchy $\perp$-sequence if, for each $\varepsilon > 0$, $\exists\ \mathscr{J} \in \mathbb{N}$ s.t $S(\varpi_\ell, \varpi_\ell, \varpi_\phi) < \varepsilon$, $\forall\ \phi, \ell \geq \mathscr{J}$.

(iii) An orthogonal S-metric space $(\nabla, S, \perp)$ is said to be complete if every Cauchy $\perp$-sequence is convergent.

Definition 3.3. *Let $(\nabla, S, \perp)$ be an orthogonal S-metric space and a mapping $\chi : (\nabla, S, \perp) \to (\nabla, S, \perp)$, then*

(i) χ is said to be $\perp$-preserving if $\chi\varpi \perp \chi\xi$ whenever $\varpi \perp \xi$.

(ii) χ *is said to be* $\perp$*-continuous if* $\perp$*-sequence* $\{\varpi_\ell\}$ *in* ∇ *such that* $\varpi_\ell \to \varpi \Rightarrow \chi\varpi_\ell \to \chi\varpi$ *as* $\ell \to \infty$.

Lemma 3.2. *Let* $(\nabla, S, \perp)$ *be an orthogonal S-metric space. If* $\exists$ $\perp$*-sequences* $\{\varpi_\ell\}$ *and* $\{\xi_\ell\}$ *s.t* $\lim_{\ell\to\infty} \varpi_\ell = \varpi$ *and* $\lim_{\ell\to\infty} \xi_\ell = \xi$*, then*

$$\lim_{\ell\to\infty} S(\varpi_\ell, \varpi_\ell, \xi_\ell) = S(\varpi, \varpi, \xi).$$

Definition 3.4. *Consider an orthogonal S-metric space* $(\nabla, S, \perp)$*. A mapping* $\chi : (\nabla, S, \perp) \to (\nabla, S, \perp)$ *is said to be ICS if* χ *is injective,* $(S, \perp)$*-continuous, surjective and has the property: for every* $\perp$*-sequence* $\{\varpi_\ell\}$ *in* ∇*, if* $\{\chi\varpi_\ell\}$ *converges then* $\{\varpi_\ell\}$ *is also converges.*

Let Θ be the set of all functions $\theta : [0,1) \to [0,1)$ such that

(i) θ is non-decreasing.

(ii) $\theta(\pi) < \pi$ for all $\pi > 0$.

(iii) $\lim_{\wp\to\pi^+} \theta(\wp) < \pi$ for all $\pi > 0$.

Theorem 3.3. *Let* $(\nabla, \leq)$ *be a partially ordered set and* $(\nabla, S, \perp)$ *is an complete orthogonal S-metric space. Suppose* $\chi : \nabla \to \nabla$ *be* $\perp$*-preserving,* $(S, \perp)$*-continuous, ICS mappings, and* $Q : \nabla \times \nabla \to \nabla$ *is s.t* Q *has the mixed monotone property. Assume that* $\exists$ $\theta \in \Theta$ *s.t*

$$S(\chi Q(\varpi,\xi), \chi Q(\varpi,\xi), \chi Q(\omega,\vartheta)) \leq \theta(\max\{S(\chi\varpi, \chi\varpi, \chi\omega), S(\chi\xi, \chi\xi, \chi\vartheta)\}) \tag{2}$$

for any $\varpi, \xi, \omega, \vartheta \in \nabla$ *with* $\varpi \perp \omega$ *and* $\xi \perp \vartheta$ *for which* $\varpi \leq \omega, \vartheta \leq \xi$*. Suppose either*

(a) Q *is continuous or*

(b) ∇ *has the following property:*

(i) If non-decreasing sequence $\varpi_\ell \to \varpi$*, then* $\varpi_\ell \leq \varpi$ $\forall \ell$.

(ii) If non-decreasing sequence $\xi_\ell \to \xi$*, then* $\xi_\ell \geq \xi$ $\forall \ell$.

If $\exists$ $\varpi_0, \xi_0 \in \nabla$ *and* $\varpi_0 \perp \xi_0$ *such that* $\varpi_0 \leq Q(\varpi_0, \xi_0)$*,* $\xi_0 \geq Q(\xi_0, \varpi_0)$*, then* $\exists$ $\varpi, \xi \in \nabla$ *and* $\varpi \perp \xi$ *such that* $Q(\varpi, \xi) = \varpi$*,* $Q(\xi, \varpi) = \xi$ *that is* Q *has a CFP.*

Proof. The orthogonality of a non-void set implies that $\exists$ $\varpi_0 \in \nabla$, such that

$$(\forall \quad \xi \in \nabla, \varpi_0 \perp \xi) \quad or \quad (\forall \quad \xi \in \nabla, \xi \perp \varpi_0).$$

Since χ is $\perp$-preserving, $\varpi_0 \perp \chi\varpi_0$ or $\chi\varpi_0 \perp \varpi_0$.

Let $\varpi_0, \xi_0 \in \nabla$ and $\varpi_0 \perp \xi_0$ such that $\varpi_0 \leq Q(\varpi_0, \xi_0)$, $\xi_0 \geq Q(\xi_0, \varpi_0)$.

Define

$$\varpi_1 = Q(\varpi_0, \xi_0), \quad \xi_1 = Q(\xi_0, \varpi_0). \tag{3}$$

Consider the $\perp$-sequences $\{\varpi_\ell\}$ and $\{\xi_\ell\}$ in ∇ such that

$$\varpi_{\ell+1} = Q(\varpi_\ell, \xi_\ell), \quad \xi_{\ell+1} = Q(\xi_\ell, \varpi_\ell). \tag{4}$$

Since Q has the mixed monotone property,

$$\varpi_\ell \leq \varpi_{\ell+1}, \quad \xi_{\ell+1} \leq \xi_\ell \quad for \quad \ell = 0, 1, 2, ...$$

For some $\ell \in \mathscr{N}$,

$$\varpi_\ell = \varpi_{\ell+1}, \quad or \quad \xi_\ell = \xi_{\ell+1}$$

then from (4), Q has the coupled fixed point.

Suppose, for any $\ell \in \mathscr{N}$,

$$\varpi_\ell \neq \varpi_{\ell+1}, \quad or \quad \xi_\ell \neq \xi_{\ell+1} \tag{5}$$

Since χ is injective, then for any $\ell \in \mathscr{N}$, by (5)

$$0 < \max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\} \tag{6}$$

using (2) and (4), we get

$$\begin{aligned} S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}) &= S(\chi Q(\varpi_{\ell-1}, \xi_{\ell-1}), \chi Q(\varpi_{\ell-1}, \xi_{\ell-1}), \chi Q(\varpi_\ell, \xi_\ell)) \\ &\leq \theta(\max\{S(\chi\varpi_{\ell-1}, \chi\varpi_{\ell-1}, \chi\varpi_\ell), S(\chi\xi_{\ell-1}, \chi\xi_{\ell-1}, \chi\xi_\ell)\}) \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1}) &= S(\chi\xi_{\ell+1}, \chi\xi_{\ell+1}, \chi\xi_\ell) \\ &= S(\chi Q(\xi_\ell, \varpi_\ell), \chi Q(\xi_\ell, \varpi_\ell), \chi Q(\xi_{\ell-1}, \varpi_{\ell-1})) \\ &\leq \theta(\max\{S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell-1}), S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell-1})\}) \end{aligned} \tag{8}$$

Since $\theta(\pi) < \pi$, $\forall\ \pi > 0$, so from (7) and (8), we get

$$\begin{aligned} 0 &< \max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\} \\ &\leq \theta(\max\{S(\chi\varpi_{\ell-1}, \chi\varpi_{\ell-1}, \chi\varpi_\ell), S(\chi\xi_{\ell-1}, \chi\xi_{\ell-1}, \chi\xi_\ell)\}) \\ &< \max\{S(\chi\varpi_{\ell-1}, \chi\varpi_{\ell-1}, \chi\varpi_\ell), S(\chi\xi_{\ell-1}, \chi\xi_{\ell-1}, \chi\xi_\ell)\} \end{aligned} \tag{9}$$

$$\max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\} < \max\{S(\chi\varpi_{\ell-1}, \chi\varpi_{\ell-1}, \chi\varpi_\ell), \\ S(\chi\xi_{\ell-1}, \chi\xi_{\ell-1}, \chi\xi_\ell)\}$$

Thus $\max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\}$ is a positive non-increasing sequence, so $\exists\ \wp \geq 0$ s.t

$$\lim_{\ell\to\infty} \max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\} = \wp$$

Assume that $\wp > 0$. Letting $\ell \to \infty$ in (9), we get

$$\begin{aligned} 0 < \wp &\leq \lim_{\ell\to\infty} \theta(\max\{S(\chi\varpi_{\ell-1}, \chi\varpi_{\ell-1}, \chi\varpi_\ell), S(\chi\xi_{\ell-1}, \chi\xi_{\ell-1}, \chi\xi_\ell)\}) \\ &\leq \lim_{\pi\to\wp^+} \theta(\pi) < \wp \end{aligned} \tag{10}$$

which is a contradiction. so we get,

$$\lim_{\ell\to\infty} \max\{S(\chi\varpi_\ell, \chi\varpi_\ell, \chi\varpi_{\ell+1}), S(\chi\xi_\ell, \chi\xi_\ell, \chi\xi_{\ell+1})\} = 0 \tag{11}$$

Now, we prove that $\{\chi\varpi_\ell\}$ and $\{\chi\xi_\ell\}$ are cauchy $\perp$-sequences.

Suppose, on the contrary, $\{\chi\varpi_\ell\}$ or $\{\chi\xi_\ell\}$ is not a cauchy $\perp$-sequences,

$$\lim_{\ell,\phi\to\infty} S(\chi\varpi_\phi, \chi\varpi_\phi, \chi\varpi_\ell) \neq 0 \quad or \quad \lim_{\ell,\phi\to\infty} S(\chi\xi_\phi, \chi\xi_\phi, \chi\xi_\ell) \neq 0$$

then $\exists\ \varepsilon > 0$, where we may obtain subsequences of integers (ϕ_κ) and (ℓ_κ) with $\ell_\kappa > \phi_\kappa > \kappa$ s.t

$$\max\{S(\chi\varpi_{\phi_\kappa}, \chi\varpi_{\phi_\kappa}, \chi\varpi_{\ell_\kappa}), S(\chi\xi_{\phi_\kappa}, \chi\xi_{\phi_\kappa}, \chi\xi_{\ell_\kappa})\} \geq \varepsilon \tag{12}$$

For $\ell_\kappa > \phi_\kappa$ and satisfying (12). Then

$$\max\{S(\chi\varpi_{\phi_\kappa}, \chi\varpi_{\phi_\kappa}, \chi\varpi_{\ell_\kappa-1}), S(\chi\xi_{\phi_\kappa}, \chi\xi_{\phi_\kappa}, \chi\xi_{\ell_\kappa-1})\} < \varepsilon \tag{13}$$

By triangular inequality and (13), we get

$$\begin{aligned} S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\ell_\kappa}) &\leq S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa-1})+S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa-1}) \\ &\quad + S(\chi\varpi_{\ell_\kappa},\chi\varpi_{\ell_\kappa},\chi\varpi_{\phi_\kappa-1}) \\ &< 0+0+\varepsilon \end{aligned}$$

Also from (11), we get

$$\lim_{\kappa\to\infty} S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\ell_\kappa}) \leq \varepsilon \quad and \quad \lim_{\kappa\to\infty} S(\chi\xi_{\phi_\kappa},\chi\xi_{\phi_\kappa},\chi\xi_{\ell_\kappa}) \leq \varepsilon \tag{14}$$

Also,

$$\begin{aligned} S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-1}) \leq &\{S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-2}) \\ &+ S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-2}) \\ &+ S(\chi\varpi_{\ell_\kappa-1},\chi\varpi_{\ell_\kappa-1},\chi\varpi_{\phi_\kappa-2})\} \\ \leq &2S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-2})+\varepsilon \end{aligned}$$

using (11), we get

$$\lim_{\kappa\to\infty} S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-2}) \leq \varepsilon \quad and \quad \lim_{\kappa\to\infty} S(\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-2}) \leq \varepsilon \tag{15}$$

using (12), (14) and (15), we get

$$\begin{aligned} &\lim_{\ell\to\infty} \max\{S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\ell_\kappa}), S(\chi\xi_{\phi_\kappa},\chi\xi_{\phi_\kappa},\chi\xi_{\ell_\kappa})\} \\ &\qquad = \lim_{\ell\to\infty} \max\{S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-1}), \\ &\qquad\qquad S(\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-1},\chi\xi_{\ell_\kappa-1})\} \\ &\qquad = \varepsilon \end{aligned} \tag{16}$$

Now, by equation (2), we get

$$\begin{aligned} S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\ell_\kappa}) &= S(\chi Q(\varpi_{\phi_\kappa-1},\xi_{\phi_\kappa-1}),\chi Q(\varpi_{\phi_\kappa-1},\xi_{\phi_\kappa-1}),\chi Q(\varpi_{\ell_\kappa-1},\xi_{\ell_\kappa-1})) \\ &\leq \theta(\max\{S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-1}),S(\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-1},\chi\xi_{\ell_\kappa-1})\}) \end{aligned} \tag{17}$$

and

$$S(\chi\xi_{\phi_\kappa},\chi\xi_{\phi_\kappa},\chi\xi_{\ell_\kappa}) = S(\chi Q(\xi_{\phi_\kappa-1},\varpi_{\phi_\kappa-1}),\chi Q(\xi_{\phi_\kappa-1},\varpi_{\phi_\kappa-1}),\chi Q(\xi_{\ell_\kappa-1},\varpi_{\ell_\kappa-1}))$$
$$\leq \theta(\max\{S(\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-1},\chi\xi_{\ell_\kappa-1}),S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-1})\}) \tag{18}$$

Now, by (17) and (18), we obtain that

$$\max\{S(\chi\varpi_{\phi_\kappa},\chi\varpi_{\phi_\kappa},\chi\varpi_{\ell_\kappa}),S(\chi\xi_{\phi_\kappa},\chi\xi_{\phi_\kappa},\chi\xi_{\ell_\kappa})\} \leq \theta(\max\{S(\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\phi_\kappa-1},\chi\varpi_{\ell_\kappa-1}),$$
$$S(\chi\xi_{\phi_\kappa-1},\chi\xi_{\phi_\kappa-1},\chi\xi_{\ell_\kappa-1})\}) \tag{19}$$

Assume $\kappa \to \infty$, in (19) and using (16), we obtain

$$0 < \varepsilon \leq \lim_{\pi\to\varepsilon^+} \theta(\pi) < \varepsilon$$

which is a contradiction.

Hence $\{\chi\varpi_\ell\}$ and $\{\chi\xi_\ell\}$ are cauchy $\perp$-sequences in $(\nabla,S,\perp)$. Since ∇ is an complete orthogonal S-metric space, $\{\chi\varpi_\ell\}$ and $\{\chi\xi_\ell\}$ are convergent $\perp$-sequences.

Since χ is an ICS mapping, $\exists\ \varpi,\xi \in \nabla$ such that

$$\lim_{\ell\to\infty} \varpi_\ell = \varpi \quad \textit{and} \quad \lim_{\ell\to\infty} \xi_\ell = \xi. \tag{20}$$

Since χ is continuous, we get

$$\lim_{\ell\to\infty} \chi\varpi_\ell = \chi\varpi \quad \textit{and} \quad \lim_{\ell\to\infty} \chi\xi_\ell = \chi\xi. \tag{21}$$

Now, assume that the assumption (1) holds, (i.e) Q is continuous. By (4), (20) and (21) we have

$$\begin{aligned}\varpi = \lim_{\ell\to\infty} \varpi_{\ell+1} &= \lim_{\ell\to\infty} Q(\varpi_\ell,\xi_\ell)\\ &= Q[\lim_{\ell\to\infty} \varpi_\ell, \lim_{\ell\to\infty} \xi_\ell]\\ &= Q(\varpi,\xi)\end{aligned}$$

and

$$\begin{aligned}\xi = \lim_{\ell\to\infty} \xi_{\ell+1} &= \lim_{\ell\to\infty} Q(\xi_\ell,\varpi_\ell)\\ &= Q[\lim_{\ell\to\infty} \xi_\ell, \lim_{\ell\to\infty} \varpi_\ell]\\ &= Q(\xi,\varpi).\end{aligned}$$

Thus we have show that Q has a CFP. Suppose, now the assumption (2) holds. Since $\{\varpi_\ell\}$ is increasing with $\varpi_\ell \to \varpi$ and also $\{\xi_\ell\}$ is non increasing with $\xi_\ell \to \xi$. Then by assumption (2), we get $\varpi_\ell \leq \varpi$ and $\xi_\ell \geq \xi$ $\forall \ell$.

Now,

$$\begin{aligned}
S(\chi\varpi,\chi\varpi,\chi Q(\varpi,\xi)) &\leq S(\chi\varpi,\chi\varpi,\chi\varpi_{\ell+1}) + S(\chi\varpi,\chi\varpi,\chi\varpi_{\ell+1}) \\
&\quad + S(\chi Q(\varpi,\xi),\chi Q(\varpi,\xi),\chi\varpi_{\ell+1}) \\
&= 2S(\chi\varpi,\chi\varpi,\chi\varpi_{\ell+1}) + S(\chi Q(\varpi,\xi),\chi Q(\varpi,\xi),\chi Q(\varpi_\ell,\xi_\ell)) \\
&\leq 2S(\chi\varpi,\chi\varpi,\chi\varpi_{\ell+1}) + \theta(\max\{S(\chi\varpi,\chi\varpi,\chi\varpi_\ell), S(\chi\xi,\chi\xi,\chi\xi_\ell)\}).
\end{aligned}$$

Taking limit $\ell \to \infty$ and using (21),

Hence

$$S(\chi\varpi,\chi\varpi,\chi Q(\varpi,\xi)) = 0.$$

So, $\chi\varpi = \chi Q(\varpi,\xi)$ and Since χ is injective so $\varpi = Q(\varpi,\xi)$.

Similarly, we can show that $\xi = Q(\xi,\varpi)$.

Thus, we have shown Q has a CFP in ∇. □

Example 3.4. *Consider a $br_\perp$ on a non-void set ∇ and $(\nabla,S,\perp)$ be an orthogonal S-metric space. Let $\nabla = \{0,1,2,3,...\}$ and $S : \nabla\times\nabla\times\nabla \to \mathbb{R}^+$ be defined as follows:*

$$S(\varpi,\xi,\rho) = \begin{cases}
\varpi+\xi+\rho,\ if\ \varpi,\xi,\rho\ are\ all\ distinct\ and\ different\ from\ zero, \\
\varpi+\rho,\ if\ \varpi=\xi\neq\rho\ and\ all\ are\ different\ from\ zero, \\
\xi+\rho+1,\ if\ \varpi=0,\xi\neq\rho\ and\ \xi,\rho\ are\ different\ from\ zero, \\
\xi+2,\ if\ \varpi=0,\xi=\rho\neq 0, \\
\rho+1,\ if\ \varpi=0,\xi=0,\rho\neq 0, \\
0,\ if\ \varpi=\xi=\rho
\end{cases}$$

then $(\nabla,S,\perp)$ is an complete orthogonal S-metric space.

Let a partial order $'\leq'$ on ∇ be defined as follows: For $\varpi,\xi\in\nabla$, with $\varpi\perp\xi$, $\varpi\leq\xi$ holds if

$\varpi > \xi$ and 3 divides $(\varpi - \xi)$ and $3 \leq 1$ and $0 \leq 1$ hold.

Let $Q : \nabla \times \nabla \to \nabla$ be defined as follows:

$$Q(\varpi, \xi) = \begin{cases} 1, & if \quad \varpi < \xi \\ 0, & if \quad otherwise. \end{cases}$$

Let $w \leq \mathfrak{u} \leq \varpi < \xi \leq \mathfrak{v} \leq \rho$ hold then equivalently, we have $w \geq \mathfrak{u} \geq \varpi > \xi \geq \mathfrak{v} \geq \rho$.

Then

$$Q(\varpi, \xi) = Q(\mathfrak{u}, \mathfrak{v}) = Q(w, \rho) = 1.$$

Thus $S(1,1,1) = 0$ and Theorem (3.3) is satisfied. It may observed that the CFP is not unique. Hence $(0,0)$ and $(1,0)$ are two CFP of Q.

Theorem 3.5. *Let $(\nabla, \leq)$ be a partially ordered set and $(\nabla, S, \perp)$ is an complete orthogonal S-metric space. Suppose $\chi : \nabla \to \nabla$ be $\perp$-preserving, $(S, \perp)$-continuous, ICS mappings, and $Q : \nabla \times \nabla \to \nabla$ is s.t Q has the mixed monotone property. Assume that $\exists\ \theta \in \theta$ s.t*

$$S(\chi Q(\varpi, \xi), \chi Q(\varpi, \xi), \chi Q(\omega, \vartheta)) \leq \theta(\max\{S(\chi\varpi, \chi\varpi, \chi\omega), S(\chi\xi, \chi\xi, \chi\vartheta)\}) \tag{22}$$

for any $\varpi, \xi, \omega, \vartheta \in \nabla$ with $\varpi \perp \omega$ and $\xi \perp \vartheta$ for which $\varpi \leq \omega, \vartheta \leq \xi$. Suppose that $\forall\ (\varpi, \xi), (\omega, \vartheta) \in \nabla \times \nabla$, $\exists\ (\mathfrak{c}, \mathfrak{d}) \in \nabla \times \nabla$ such that $(Q(\mathfrak{c}, \mathfrak{d}), Q(\mathfrak{d}, \mathfrak{c}))$ is comparable to $(Q(\varpi, \xi), Q(\xi, \varpi))$ and $(Q(\omega, \vartheta), Q(\vartheta, \omega))$ then Q has a unique CFP (ϖ, ξ).

Proof. The orthogonality of a non-void set implies that $\exists\ \varpi_0 \in \nabla$, such that

$$(\forall \quad \xi \in \nabla, \varpi_0 \perp \xi) \quad or \quad (\forall \quad \xi \in \nabla, \xi \perp \varpi_0).$$

Since χ is $\perp$-preserving, $\varpi_0 \perp \chi\varpi_0$ or $\chi\varpi_0 \perp \varpi_0$.

From Theorem (3.3), we know that the set of CFP of Q is non-void.

Suppose, now (ϖ, ξ) and (ω, ϑ) are two CFP of Q,

$$Q(\varpi, \xi) = \varpi, Q(\xi, \varpi) = \xi \quad and \quad Q(\omega, \vartheta) = \omega, Q(\vartheta, \omega) = \vartheta.$$

we shall prove that (ϖ, ξ) and (ω, ϑ) are equal.

Suppose that $\forall\ (\varpi, \xi), (\omega, \vartheta) \in \nabla \times \nabla$, $\exists\ (\mathfrak{c}, \mathfrak{d}) \in \nabla \times \nabla$ such that $(Q(\mathfrak{c}, \mathfrak{d}), Q(\mathfrak{d}, \mathfrak{c}))$ is

comparable to $(Q(\varpi,\xi),Q(\xi,\varpi))$ and $(Q(\omega,\vartheta),Q(\vartheta,\omega))$.
Now construct $\perp$-sequences $\{\mathfrak{c}_\ell\}$ and $\{\mathfrak{d}_\ell\}$ such that $\mathfrak{c}_0=\mathfrak{c}$, $\mathfrak{d}_0=\mathfrak{d}$ and for any $\ell\geq 1$.

$$\mathfrak{c}_\ell = Q(\mathfrak{c}_{\ell-1},\mathfrak{d}_{\ell-1}), \quad \mathfrak{d}_\ell = Q(\mathfrak{d}_{\ell-1},\mathfrak{c}_{\ell-1}) \quad \forall \ell. \tag{23}$$

Let $\varpi_0=\varpi$, $\xi_0=\xi$ and $\omega_0=\omega$, $\vartheta_0=\vartheta$ and construct $\perp$-sequences $\{\varpi_\ell\}$, $\{\xi_\ell\}$, $\{\omega_\ell\}$ and $\{\vartheta_\ell\}$. Then

$$\varpi_\ell = Q(\varpi,\xi), \quad \xi_\ell = Q(\xi,\varpi), \quad \omega_\ell = Q(\omega,\vartheta), \quad \vartheta_\ell = Q(\vartheta,\omega), \quad \forall \ell\geq 1. \tag{24}$$

Since $(Q(\varpi,\xi),Q(\xi,\varpi))=(\varpi_1,\xi_1)=(\varpi,\xi)$ is comparable to $(Q(\mathfrak{c},\mathfrak{d}),Q(\mathfrak{d},\mathfrak{c}))=(\mathfrak{c}_1,\mathfrak{d}_1)$, then

$$(\varpi,\xi)\geq(\mathfrak{c}_1,\mathfrak{d}_1).$$

similarly,

$$(\varpi,\xi)\geq(\mathfrak{c}_\ell,\mathfrak{d}_\ell) \quad \forall \ell \tag{25}$$

From equation (25) and (2), we get

$$\begin{aligned} S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_{\ell+1}) &\leq S(\chi Q(\varpi,\xi),\chi Q(\varpi,\xi),\chi Q(\mathfrak{c}_\ell,\mathfrak{d}_\ell)) \\ &\leq \theta(\max\{S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_\ell),S(\chi\xi,\chi\xi,\chi\mathfrak{d}_\ell)\}) \end{aligned} \tag{26}$$

and

$$\begin{aligned} S(\chi\xi,\chi\xi,\chi\mathfrak{d}_{\ell+1}) &= S(\chi\mathfrak{d}_{\ell+1},\chi\mathfrak{d}_{\ell+1},\chi\xi) \\ &= S(\chi Q(\mathfrak{d}_\ell,\mathfrak{c}_\ell),\chi Q(\mathfrak{d}_\ell,\mathfrak{c}_\ell),\chi Q(\xi,\varpi)) \\ &\leq \theta(\max\{S(\chi\mathfrak{d}_\ell,\chi\mathfrak{d}_\ell,\chi\xi),S(\chi\mathfrak{c}_\ell,\chi\mathfrak{c}_\ell,\chi\varpi)\}) \end{aligned} \tag{27}$$

From equation (26) and (27),

$$\theta(\max\{S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_{\ell+1}),S(\chi\xi,\chi\xi,\chi\mathfrak{d}_{\ell+1})\}) \leq \theta(\max\{S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_\ell),S(\chi\xi,\chi\xi,\chi\mathfrak{d}_\ell)\})$$

So, $\forall\ \ell\geq 1$

$$\max\{S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_\ell),S(\chi\xi,\chi\xi,\chi\mathfrak{d}_\ell)\} \leq \theta^\ell(\max\{S(\chi\varpi,\chi\varpi,\chi\mathfrak{c}_0),S(\chi\xi,\chi\xi,\chi\mathfrak{d}_0)\}). \tag{28}$$

Since $\theta(\pi) < \pi$ and $\lim_{\wp \to \pi^+} \theta(\wp) < \pi$ implies

$$\lim_{\ell \to \infty} \theta^\ell(\pi) = 0 \quad \forall \quad \pi > 0.$$

From equation (28), we get

$$\lim_{\ell \to \infty} \max\{S(\chi\varpi, \chi\varpi, \chi\mathfrak{c}_\ell), S(\chi\xi, \chi\xi, \chi\mathfrak{d}_\ell)\} = 0$$

which implies

$$\lim_{\ell \to \infty} \{S(\chi\varpi, \chi\varpi, \chi\mathfrak{c}_\ell)\} = 0 \quad \textit{and} \quad \lim_{\ell \to \infty} \{S(\chi\xi, \chi\xi, \chi\mathfrak{d}_\ell)\} = 0. \tag{29}$$

Similarly

$$\lim_{\ell \to \infty} \{S(\chi\omega, \chi\omega, \chi\mathfrak{c}_\ell)\} = 0 \quad \textit{and} \quad \lim_{\ell \to \infty} \{S(\chi\vartheta, \chi\vartheta, \chi\mathfrak{d}_\ell)\} = 0. \tag{30}$$

From equation (29) and (30), we get
$(\chi\varpi, \chi\xi)$ and $(\chi\omega, \chi\vartheta)$ are equal.
Since χ is injective,

$$\varpi = \omega \quad \textit{and} \quad \xi = \vartheta.$$

□

4. Application

We examine a coupled system of nonlinear Langevin equations with fractional components, represented by the following form

$$\begin{cases} \mathscr{D}^{\mu_1}(\phi_1 \mathscr{D}^{\sigma_1} + \lambda_1)\wp_1(\pi) = \eta_1(\pi, \wp_1(\pi), \wp_2(\pi)), \pi \in \mathscr{F} = [1, \mathscr{A}], \\ \mathscr{D}^{\mu_2}(\phi_2 \mathscr{D}^{\sigma_2} + \lambda_2)\wp_2(\pi) = \eta_2(\pi, \wp_1(\pi), \wp_2(\pi)), \pi \in \mathscr{F} = [1, \mathscr{A}], \\ \wp_1(1) = \xi_1 \wp_1(\mathscr{A}), \wp_1^1(1) = 0 = \wp_1^1(\mathscr{A}), \xi_1 \neq 1, \\ \wp_2(1) = \xi_2 \wp_2(\mathscr{A}), \wp_2^1(1) = 0 = \wp_2^1(\mathscr{A}), \xi_2 \neq 2, \end{cases} \tag{31}$$

where $\mu_1, \mu_2 \in (0, 1]$, $\sigma_1, \sigma_2 \in (1, 2]$, $\phi_1, \phi_2 > 0$, and $\mathscr{D}^\vartheta$ is the Caputo-Hadamard type fractional derivative where $\vartheta \in \{\mu_1, \mu_2, \sigma_1, \sigma_2\}$.

Let $\nabla = \{\wp \in \mathscr{C}^{'}(\mathscr{F} = [1, \mathscr{A}]) : \wp(\pi) > 1, \pi \in \mathscr{F}\}$ be the banach space of functions whose first derivatives are continuous on $\mathscr{F}$. Suppose the mapping $S : \nabla \times \nabla \times \nabla \to \mathbb{R}^{+}$ is defined by

$$S(\wp_1, \wp_2, \wp_3) = \max_{\pi \in \mathscr{F}} |\wp_1(\pi) - \wp_3(\pi)| + \max_{\pi \in \mathscr{F}} |\wp_2(\pi) - \wp_3(\pi)|$$
$$\forall \wp_1, \wp_2, \wp_3 \in \nabla \quad and \quad \pi \in \mathscr{F}$$

Define the relation $\perp$ in ∇: $\wp_1 \perp \wp_2$ if $\wp_1(\pi)\wp_2(\pi) \geq \wp_2(\pi)$, $\forall \pi \in \mathscr{F}$, then $(\nabla, S, \perp)$ is an orthogonal complete S metric space. Let $\chi : \nabla \to \nabla$ is an ICS mapping and $Q : \nabla \times \nabla \to \nabla \times \nabla$, where $(\wp_1, \wp_2), (\omega_1, \omega_2) \in \nabla \times \nabla$, and $Q_1, Q_2 \in Q$ is defined by

$$\begin{aligned} Q_1(\wp_1, \wp_2)(\pi) = & \frac{1}{\phi_1} \mathscr{I}^{\mu_1+\sigma_1} \eta_1(\pi, \wp_1(\pi), \wp_2(\pi)) - \frac{\xi_1}{\phi_1(\xi_1 - 1)} \mathscr{I}^{\mu_1+\sigma_1} \eta_1(\mathscr{A}, \wp_1(\mathscr{A}), \wp_2(\mathscr{A})) \\ & + \frac{\mathscr{A} log \mathscr{A}}{\sigma_1 \phi_1} \left(\frac{\xi_1}{\xi_1 - 1} - \left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_1} \right) \mathscr{I}^{\mu_1+\sigma_1-1} \eta_1(\mathscr{A}, \wp_1(\mathscr{A}), \wp_2(\mathscr{A})) \\ & + \frac{\lambda_1 \mathscr{A} log \mathscr{A}}{\sigma_1 \phi_1} \left(\left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_1} - \frac{\xi_1}{\xi_1 - 1} \right) \mathscr{I}^{\sigma_1 - 1} \wp_1(\mathscr{A}) + \frac{\lambda_1 \xi_1}{\phi_1(\xi_1 - 1)} \mathscr{I}^{\sigma_1} \wp_1(\mathscr{A}) \\ & - \frac{\lambda_1}{\phi_1} \mathscr{I}^{\sigma_1} \wp_1(\pi), \quad \pi \in \mathscr{F}, \end{aligned}$$

$$\begin{aligned} Q_2(\wp_1, \wp_2)(\pi) = & \frac{1}{\phi_2} \mathscr{I}^{\mu_2+\sigma_2} \eta_2(\pi, \wp_1(\pi), \wp_2(\pi)) - \frac{\xi_2}{\phi_2(\xi_2 - 1)} \mathscr{I}^{\mu_2+\sigma_2} \eta_2(\mathscr{A}, \wp_1(\mathscr{A}), \wp_2(\mathscr{A})) \\ & + \frac{\mathscr{A} log \mathscr{A}}{\sigma_2 \phi_2} \left(\frac{\xi_2}{\xi_2 - 1} - \left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_2} \right) \mathscr{I}^{\mu_2+\sigma_2-1} \eta_2(\mathscr{A}, \wp_1(\mathscr{A}), \wp_2(\mathscr{A})) \\ & + \frac{\lambda_2 \mathscr{A} log \mathscr{A}}{\sigma_2 \phi_2} \left(\left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_2} - \frac{\xi_2}{\xi_2 - 1} \right) \mathscr{I}^{\sigma_2 - 1} \wp_2(\mathscr{A}) \\ & + \frac{\lambda_2 \xi_2}{\phi_2(\xi_2 - 1)} \mathscr{I}^{\sigma_2} \wp_2(\mathscr{A}) - \frac{\lambda_2}{\phi_2} \mathscr{I}^{\sigma_2} \wp_2(\pi), \quad \pi \in \mathscr{F}. \end{aligned}$$

Clearly, $(\wp_1, \wp_2)$ is a fixed point of Q iff $(\wp_1, \wp_2)$ is a solution of system (31)

Furthermore,

$$\begin{aligned} Q_1^{'}(\wp_1, \wp_2)(\pi) = & \frac{1}{\phi_1} \mathscr{I}^{\mu_1+\sigma_1-1} \eta_1(\pi, \wp_1(\pi), \wp_2(\pi)) \\ & - \frac{\mathscr{A}}{\phi_1 \pi} \left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_1 - 1} \mathscr{I}^{\mu_1+\sigma_1-1} \eta_1(\mathscr{A}, \wp_1(\mathscr{A}), \wp_2(\mathscr{A})) \\ & + \frac{\lambda_1 \mathscr{A}}{\phi_1 \pi} \left(\frac{log \pi}{log \mathscr{A}} \right)^{\sigma_1 - 1} \mathscr{I}^{\sigma_1 - 1} \wp_1(\mathscr{A}) - \frac{\lambda_1}{\phi_1} \mathscr{I}^{\sigma_1 - 1} \wp_1(\pi), \quad \pi \in \mathscr{F}, \end{aligned}$$

$$\begin{aligned} Q_2'(\wp_1,\wp_2)(\pi) =& \frac{1}{\phi_2}\mathscr{I}^{\mu_2+\sigma_2-1}\eta_2(\pi,\wp_1(\pi),\wp_2(\pi)) \\ & -\frac{\mathscr{A}}{\phi_2\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_2-1}\mathscr{I}^{\mu_2+\sigma_2-1}\eta_2(\mathscr{A},\wp_1(\mathscr{A}),\wp_2(\mathscr{A})) \\ & +\frac{\lambda_2\mathscr{A}}{\phi_2\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_2-1}\mathscr{I}^{\sigma_2-1}\wp_2(\mathscr{A})-\frac{\lambda_2}{\phi_2}\mathscr{I}^{\sigma_2-1}\wp_2(\pi), \quad \pi\in\mathscr{F}. \end{aligned}$$

Notations and assumptions:

$$\begin{aligned} \Lambda_{11} &= \frac{(log\mathscr{A})^{\mu_1+\sigma_1}}{\phi_1\Gamma(\mu_1+\sigma_1)}\left(\frac{1}{\mu_1+\sigma_1}+\frac{\mathscr{A}}{\sigma_1}\right)\left(\left|\frac{\xi_1}{\xi_1-1}\right|+1\right), \\ \Lambda_{12} &= \frac{(\mathscr{A}+1)|\lambda_1|(log\mathscr{A})^{\sigma_1}}{\phi_1\Gamma(\sigma_1+1)}\left(\left|\frac{\xi_1}{\xi_1-1}\right|+1\right), \\ \Lambda_{21} &= \frac{(log\mathscr{A})^{\mu_2+\sigma_2}}{\phi_2\Gamma(\mu_2+\sigma_2)}\left(\frac{1}{\mu_2+\sigma_2}+\frac{\mathscr{A}}{\sigma_2}\right)\left(\left|\frac{\xi_2}{\xi_2-1}\right|+1\right), \\ \Lambda_{22} &= \frac{(\mathscr{A}+1)|\lambda_2|(log\mathscr{A})^{\sigma_2}}{\phi_2\Gamma(\sigma_2+1)}\left(\left|\frac{\xi_2}{\xi_2-1}\right|+1\right), \\ \Omega_{11} &= \frac{(\mathscr{A}+1)(log\mathscr{A})^{\mu_1+\sigma_1-1}}{\phi_1\Gamma(\mu_1+\sigma_1)}, \\ \Omega_{12} &= \frac{(\mathscr{A}+1)|\lambda_1|(log\mathscr{A})^{\sigma_1-1}}{\phi_1\Gamma(\sigma_1)}, \\ \Omega_{21} &= \frac{(\mathscr{A}+1)(log\mathscr{A})^{\mu_2+\sigma_2-1}}{\phi_2\Gamma(\mu_2+\sigma_2)}, \\ \Omega_{22} &= \frac{(\mathscr{A}+1)|\lambda_2|(log\mathscr{A})^{\sigma_2-1}}{\phi_2\Gamma(\sigma_2)}. \end{aligned}$$

$$\theta=(\theta_1+\theta_2)(\Lambda_{11}+\Omega_{11})+(\theta_3+\theta_4)(\Lambda_{21}+\Omega_{21})+(\Lambda_{12}+\Omega_{12})+(\Lambda_{22}+\Omega_{22})>0,$$

$$where \quad \theta_1,\theta_2,\theta_3,\theta_4\in\theta$$

We now illustrate the subsequent theorem to validate the existence and uniqueness of a solution for the system (31).

Theorem 4.1. *Assume that the following axioms hold:*

(i) $\eta_1,\eta_2:[1,\mathscr{A}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous;*

(ii) Let Θ *be the set of all functions* $\theta:[0,1)\to[0,1)\ \exists\ \theta_1,\theta_2,\theta_3,\theta_4\in\theta$ *such that*

$$|\eta_1(\pi,\wp_1,\wp_2)-\eta_1(\pi,\omega_1,\omega_2)|\leq\theta_1|\wp_1-\omega_1|+\theta_2|\wp_2-\omega_2|,$$

and

$$|\eta_2(\pi,\wp_1,\wp_2)-\eta_2(\pi,\omega_1,\omega_2)|\leq \theta_3|\wp_1-\omega_1|+\theta_4|\wp_2-\omega_2|,$$
$$\forall(\pi,\wp_1,\wp_2),(\pi,\omega_1,\omega_2)\in[0,\mathscr{A}]\times\mathbb{R}\times\mathbb{R}.$$

(iii) $\max_{\pi\in[0,\mathscr{A}]}\eta_1(\pi,0,0)=\mathscr{M}_1, \quad \max_{\pi\in[0,\mathscr{A}]}\eta_2(\pi,0,0)=\mathscr{M}_2.$

(iv) $\exists$ *non-negative functions* $\Psi_1,\Psi_2\in\nabla$ *such that*

$$|\eta_1(\pi,\wp_1,\wp_2)|\leq\Psi_1(\pi), \quad |\eta_2(\pi,\wp_1,\wp_2)|\leq\Psi_2(\pi)$$
$$\forall \quad (\pi,\wp_1,\wp_2)\in[1,\mathscr{A}]\times\mathbb{R}\times\mathbb{R}$$

then (31) *has a unique solution* $(\wp_1,\wp_2)\in\nabla\times\nabla$.

Proof. Let $Q:\nabla\times\nabla\to\nabla\times\nabla$, $(\wp_1,\wp_2),(\omega_1,\omega_2)\in\nabla\times\nabla$, $Q_1,Q_2,Q_1^{'},Q_2^{'}\in Q$, $\pi\in\mathscr{F}$ and $\chi:\nabla\to\nabla$ is an ICS mapping.

Now verify that the hypothesis in Theorem (3.3) is satisfied. to do this, we show that

(a) There exists $\wp_1\in\nabla$ such that $\wp_1\perp\chi\wp_2\ \forall\wp_2\in\nabla$.
(b) χ is $\perp$-preserving.
(c) χ is $(S,\perp)$- contraction.
(d) χ is $(S,\perp)$-continuous.

Proof. (a) Put $\wp_0=\mathfrak{b}$(the constant function), we have $\mathfrak{b}\perp\chi\wp\ \forall\wp\in\nabla$.

(b) $\forall\ \wp_1,\wp_2\in\nabla$, $\wp_1\perp\wp_2$, if $\wp_1(\pi)\wp_2(\pi)\geq\wp_2(\pi)$ for every $\pi\in\mathscr{F}$.
Since $\chi:\nabla\to\nabla$ is an ICS mapping, $\chi\wp_1(\pi)\chi\wp_2(\pi)\geq\chi\wp_2(\pi)\ \forall\pi\in\mathscr{F}$. So $\chi\wp_1\perp\chi\wp_2$.
Hence χ is $\perp$-preserving
(c) Let $Q:\nabla\times\nabla\to\nabla\times\nabla$, $(\wp_1,\wp_2),(\omega_1,\omega_2)\in\nabla\times\nabla$, $Q_1,Q_2\in Q$, $\pi\in\mathscr{F}$,
consider

$$|Q_1(\wp_1,\wp_2)(\pi)-Q_1(\omega_1,\omega_2)(\pi)|\leq\frac{1}{\phi_1}\mathscr{I}^{\mu_1+\sigma_1}|\eta_1(\pi,\wp_1(\pi),\wp_2(\pi))-\eta_1(\pi,\omega_1(\pi),\omega_2(\pi))|$$
$$+\frac{1}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\mathscr{I}^{\mu_1+\sigma_1}$$
$$|\eta_1(\mathscr{A},\wp_1(\mathscr{A}),\wp_2(\mathscr{A}))-\eta_1(\mathscr{A},\omega_1(\mathscr{A}),\omega_2(\mathscr{A}))|$$

$$
\begin{aligned}
&+\frac{\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left|\frac{\xi_1}{\xi_1-1}-\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1}\right|\mathscr{I}^{\mu_1+\sigma_1-1}\\
&\quad|\eta_1(\mathscr{A},\wp_1(\mathscr{A}),\wp_2(\mathscr{A}))-\eta_1(\mathscr{A},\omega_1(\mathscr{A}),\omega_2(\mathscr{A}))|\\
&+\frac{|\lambda_1|\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left|\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1}-\frac{\xi_1}{\xi_1-1}\right|\\
&\quad\mathscr{I}^{\sigma_1-1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|\\
&+\frac{|\lambda_1|}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\mathscr{I}^{\sigma_1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|\\
&+\frac{|\lambda_1|}{\phi_1}\mathscr{I}^{\sigma_1}|\wp_1(\pi)-\omega_1(\pi)|\\
\leq&\frac{1}{\phi_1}\left[\theta_1\mathscr{I}^{\mu_1+\sigma_1}|\wp_1(\pi)-\omega_1(\pi)+\theta_2\mathscr{I}^{\mu_1+\sigma_1}|\wp_2(\pi)-\omega_2(\pi)\right]\\
&+\frac{1}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\Big[\theta_1\mathscr{I}^{\mu_1+\sigma_1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|\\
&+\theta_2\mathscr{I}^{\mu_1+\sigma_1}|\wp_2(\mathscr{A})-\omega_2(\mathscr{A})|\Big]\\
&+\frac{\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|-\left|\frac{log\pi}{log\mathscr{A}}\right|^{\sigma_1}\right]\\
&\quad[\theta_1\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|]\\
&+\theta_2\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_2(\mathscr{A})-\omega_2(\mathscr{A})|\\
&+\frac{|\lambda_1|\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|-\left|\frac{log\pi}{log\mathscr{A}}\right|^{\sigma_1}\right]\\
&\quad\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1}\\
&+\frac{|\lambda_1|}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\mathscr{A})^{\sigma_1}\\
&+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\pi)^{\sigma_1},
\end{aligned}
$$

$$
\begin{aligned}
|Q_1(\wp_1,\wp_2)(\pi)-Q_1(\omega_1,\omega_2)(\pi)|\leq&\frac{1}{\phi_1}\left[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1+1)}(log\pi)^{\mu_1+\sigma_1}+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1+1)}(log\pi)^{\mu_1+\sigma_1}\right]\\
&+\frac{1}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\Big[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}\\
&+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}\Big]\\
&+\frac{\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|-\left|\frac{log\pi}{log\mathscr{A}}\right|^{\sigma_1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}\right]\\
&+\frac{|\lambda_1|\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|-\left|\frac{log\pi}{log\mathscr{A}}\right|^{\sigma_1}\right]\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1}\\
&+\frac{|\lambda_1|}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\mathscr{A})^{\sigma_1}+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\pi)^{\sigma_1}\\
\le&\frac{1}{\phi_1}\left[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}\right]\\
&+\frac{1}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\left[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}\right.\\
&\left.+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1+1)}(log\mathscr{A})^{\mu_1+\sigma_1}\right]+\frac{\mathscr{A}\,log\mathscr{A}}{\sigma_1\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|+1\right]\\
&\left[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}\right]\\
&+\frac{|\lambda_1|\mathscr{A}}{\phi_1}\left[\left|\frac{\xi_1}{\xi_1-1}\right|+1\right]\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\mathscr{A})^{\sigma_1}\\
&+\frac{|\lambda_1|}{\phi_1}\left|\frac{\xi_1}{\xi_1-1}\right|\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\mathscr{A})^{\sigma_1}+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1+1)}(log\mathscr{A})^{\sigma_1}\\
=&\frac{(\theta_1+\theta_2)(log\mathscr{A})^{\mu_1+\sigma_1}}{\phi_1\Gamma(\mu_1+\sigma_1+1)}\left[\left|\frac{\xi_1}{\xi_1-1}\right|+1\right][\|\wp_1-\omega_1\|+\|\wp_2-\omega_2\|]\\
&+\frac{\mathscr{A}(\theta_1+\theta_2)(log\mathscr{A})^{\mu_1+\sigma_1}}{\sigma_1\phi_1\Gamma(\mu_1+\sigma_1)}\left[\left|\frac{\xi_1}{\xi_1-1}\right|+1\right]\\
&[\|\wp_1-\omega_1\|+\|\wp_2-\omega_2\|]\\
&+\frac{(\mathscr{A}+1)|\lambda_1|(log\mathscr{A})^{\sigma_1}}{\phi_1\Gamma(\sigma_1+1)}\left[\left|\frac{\xi_1}{\xi_1-1}\right|+1\right]\|\wp_1-\omega_1\|\\
=&\left[(\theta_1+\theta_2)\Lambda_{11}+\Lambda_{12}\right]\|\wp_1-\omega_1\|+(\theta_1+\theta_2)\Lambda_{11}\|\wp_2-\omega_2\|,
\end{aligned}
$$

$$
\|Q_1(\wp_1,\wp_2)(\pi)-Q_1(\omega_1,\omega_2)(\pi)\|\le\left[(\theta_1+\theta_2)\Lambda_{11}+\Lambda_{12}\right]\|\wp_1-\omega_1\|+(\theta_1+\theta_2)\Lambda_{11}\|\wp_2-\omega_2\|. \tag{32}
$$

Now, consider

$$
\begin{aligned}
|Q_1'(\wp_1,\wp_2)(\pi)-Q_1'(\omega_1,\omega_2)(\pi)|\le&\frac{1}{\phi_1}\mathscr{I}^{\mu_1+\sigma_1-1}|\eta_1(\pi,\wp_1(\pi),\wp_2(\pi))-\eta_1(\pi,\omega_1(\pi),\omega_2(\pi))|\\
&+\frac{\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1}\mathscr{I}^{\mu_1+\sigma_1-1}\\
&|\eta_1(\mathscr{A},\wp_1(\mathscr{A}),\wp_2(\mathscr{A}))-\eta_1(\mathscr{A},\omega_1(\mathscr{A}),\omega_2(\mathscr{A}))|\\
&+\frac{|\lambda_1|\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1}\mathscr{I}^{\sigma_1-1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\lambda_1|}{\phi_1}\mathscr{I}^{\sigma_1-1}|\wp_1(\pi)-\omega_1(\pi)| \\
\leq&\frac{1}{\phi_1}\Bigg[\theta_1\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_1(\pi)-\omega_1(\pi) \\
&+\theta_2\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_2(\pi)-\omega_2(\pi)\Bigg] \\
&+\frac{\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1}[\theta_1\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_1(\mathscr{A})-\omega_1(\mathscr{A})|] \\
&+\theta_2\mathscr{I}^{\mu_1+\sigma_1-1}|\wp_2(\mathscr{A})-\omega_2(\mathscr{A})|+\frac{|\lambda_1|\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1} \\
&\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1}+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1} \\
\leq&\frac{1}{\phi_1}\Bigg[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\pi)^{\mu_1+\sigma_1-1} \\
&+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\pi)^{\mu_1+\sigma_1-1}\Bigg] \\
&+\frac{\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1}\Bigg[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1} \\
&+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}\Bigg]+\frac{|\lambda_1|\mathscr{A}}{\phi_1\pi}\left(\frac{log\pi}{log\mathscr{A}}\right)^{\sigma_1-1} \\
&\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1}+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1},
\end{aligned}
$$

$$
\begin{aligned}
|Q_1'(\wp_1,\wp_2)(\pi)-Q_1'(\omega_1,\omega_2)(\pi)|\leq&\frac{1}{\phi_1}\Bigg[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1} \\
&+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}\Bigg] \\
&+\frac{\mathscr{A}}{\phi_1}\Bigg[\frac{\theta_1\|\wp_1-\omega_1\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1} \\
&+\frac{\theta_2\|\wp_2-\omega_2\|}{\Gamma(\mu_1+\sigma_1)}(log\mathscr{A})^{\mu_1+\sigma_1-1}\Bigg] \\
&+\frac{|\lambda_1|\mathscr{A}}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1} \\
&+\frac{|\lambda_1|}{\phi_1}\frac{\|\wp_1-\omega_1\|}{\Gamma(\sigma_1)}(log\mathscr{A})^{\sigma_1-1} \\
=&\frac{(\mathscr{A}+1)(\theta_1+\theta_2)(log\mathscr{A})^{\mu_1+\sigma_1-1}}{\phi_1\Gamma(\mu_1+\sigma_1)}[\|\wp_1-\omega_1\|+\|\wp_2-\omega_2\|]
\end{aligned}
$$

$$+\frac{(\mathscr{A}+1)|\lambda_1|(log\mathscr{A})^{\sigma_1-1}}{\phi_1\Gamma(\sigma_1)}\|\wp_1-\omega_1\|$$
$$=\big[(\theta_1+\theta_2)\Omega_{11}+\Omega_{12}\big]\|\wp_1-\omega_1\|+(\theta_1+\theta_2)\Omega_{11}\|\wp_2-\omega_2\|,$$

(33)
$$\|Q_1^{'}(\wp_1,\wp_2)-Q_1^{'}(\omega_1,\omega_2)\|\leq\big[(\theta_1+\theta_2)\Omega_{11}+\Omega_{12}\big]\|\wp_1-\omega_1\|+(\theta_1+\theta_2)\Omega_{11}\|\wp_2-\omega_2\|.$$

From equation (32) and (33)

$$\|Q_1(\wp_1,\wp_2)-Q_1(\omega_1,\omega_2)\|\leq\big[(\theta_1+\theta_2)(\Lambda_{11}+\Omega_{11})+(\Lambda_{12}+\Omega_{12})\big]\|\wp_1-\omega_1\|$$
$$+\big[(\theta_1+\theta_2)(\Lambda_{11}+\Omega_{11})\big]\|\wp_2-\omega_2\|. \tag{34}$$

Similarly,

$$\|Q_2(\wp_1,\wp_2)-Q_2(\omega_1,\omega_2)\|\leq\big[(\theta_3+\theta_4)(\Lambda_{21}+\Omega_{21})+(\Lambda_{22}+\Omega_{22})\big]\|\wp_1-\omega_1\|$$
$$+\big[(\theta_3+\theta_4)(\Lambda_{21}+\Omega_{21})\big]\|\wp_2-\omega_2\|. \tag{35}$$

From equation (34) and (35)

$$\|Q(\wp_1,\wp_2)-Q(\omega_1,\omega_2)\|\leq\theta\|(\wp_1,\wp_2)-(\omega_1,\omega_2)\|,\quad \theta\in(0,1].$$

$\chi:\nabla\to\nabla$ is an ICS mapping,

$$\|\chi Q(\wp_1,\wp_2)-\chi Q(\omega_1,\omega_2)\|\leq\theta\|(\chi\wp_1,\chi\wp_2)-(\chi\omega_1,\chi\omega_2)\|$$

Now,

$$\begin{aligned}S(\chi Q(\wp_1,\wp_2),\chi Q(\wp_1,\wp_2),\chi Q(\omega_1,\omega_2)) &= 2\max_{\pi\in\mathscr{F}}\big|\chi Q(\wp_1,\wp_2)(\pi)-\chi Q(\omega_1,\omega_2)(\pi)\big|\\ &\leq 2\max\|\chi Q(\wp_1,\wp_2)-\chi Q(\omega_1,\omega_2)\|\\ &\leq 2\theta\max\|(\chi\wp_1,\chi\wp_2)-(\chi\omega_1,\chi\omega_2)\|\\ &\leq\theta(\max\{S(\chi\wp_1,\chi\wp_1,\chi\omega_1),S(\chi\wp_2,\chi\wp_2,\chi\omega_2)\}).\end{aligned}$$

Therefore, χ is $(S,\perp)$- contraction with $\theta\in(0,1]$.

(d) Let $\{\wp_\ell\}$ be an $(S,\perp)$-sequence in ∇ such that $\{\wp_\ell\}$ converges to some $\wp\in\nabla$. Since χ is

$\perp$-preserving, $\{\chi\wp_\ell\}$ is an $(S,\perp)$-sequence.

For each $\ell \in \mathbb{N}$,

$$|\chi\wp_\ell - \chi\wp| \leq \theta|\wp_\ell - \wp|, \quad 0 < \theta < 1.$$

As $\ell \to \infty$, χ is $\perp$-continuous.

□

Hence, there exists a unique solution to coupled system of nonlinear fractional Langevin equations. □

5. Conclusions

In this manuscript, We introduced the CFP theorem for orthogonal S-metric space. Our findings expand on and generalize previous research. Furthermore, we apply our main findings to a coupled system of nonlinear fractional Langevin equations.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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