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SOME COMMON FIXED POINT RESULTS IN HYPERBOLIC SPACES

KHANITIN SAMANMIT[1], BURASKORN NUNTADILOK[2], PITCHAYA KINGKAM[3,*]

[1]Department of Basic Science, Maejo University Phrae Campus, Phrae, Thailand

[2]Department of Mathematics, Faculty of Sciences, Maejo University, Chiangmai, Thailand

[3]Department of Mathematics, Faculty of Sciences, Lampang Rajabhat University, Lampang, Thailand

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Abstract. In this manuscript, we investigate common fixed point results for two infinite families of uniformly L-Lipschitzian total asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces which are essentially more general than asymptotically nonexpansive mappings in the intermediate sense. We prove strong convergence and Δ-convergence theorems for such mappings via Ishikawa type iterative schemes. Our results generalize, extend and improve several corresponding results in the existing literature.

Keywords: fixed point; Δ-convergence; total asymptotically nonexpansive mapping; Banach space; CAT(0) space; CAT(κ) space; hyperbolic space.

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1. Introduction

We assume that $(\mathfrak{M},\mathfrak{d})$ is a metric space and let $F(\mathscr{A})=\{x\in\mathfrak{M}:\mathscr{A}x=x\}$ be the set of all fixed points of the mappings $\mathscr{A}$. Let $\mathscr{K}$ be a nonempty subset of a metric space $(\mathfrak{M},\mathfrak{d})$. Throughout this paper, let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers.

Recall that a mapping $\mathscr{A}:\mathscr{K}\to\mathfrak{M}$ is said to be:

*Corresponding author

E-mail address: pitchaya@g.lpru.ac.th

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(i). nonexpansive if

$$\mathfrak{d}(\mathscr{A}x,\mathscr{A}y) \leq \mathfrak{d}(x,y),\ \forall x,y \in \mathscr{K}. \tag{1}$$

(ii). quasi-nonexpansive if $F(\mathscr{A}) \neq \emptyset$ and

$$\mathfrak{d}(\mathscr{A}x,p) \leq \mathfrak{d}(x,p),\ \forall p \in F(\mathscr{A}). \tag{2}$$

(iii). asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\mathfrak{d}(\mathscr{A}^n x,\mathscr{A}^n y) \leq k_n \mathfrak{d}(x,y),\ \forall\, x,y \in \mathscr{K} \text{ and } \forall\, n \in \mathbb{N}. \tag{3}$$

(iv). uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\mathfrak{d}(\mathscr{A}^n x,\mathscr{A}^n y) \leq L\mathfrak{d}(x,y),\ \forall x,y \in \mathscr{K}, \forall\, n \in \mathbb{N}. \tag{4}$$

Remark 1. *It is easy to see that if $F(\mathscr{A}) \neq \emptyset$, then nonexpansive mapping, quasi-nonexpansive mapping, all are asymptotically nonexpansive mappings, but the converse is not true in general.*

In 1993, Bruck et al. [3] introduced the following definition.

Definition 1. [3] *A mapping $\mathscr{A} : \mathscr{K} \to \mathscr{K}$ is said to be asymptotically nonexpansive in the intermediate sense, if $\mathscr{A}$ is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x,y\in\mathscr{K}} \{\mathfrak{d}(\mathscr{A}^n x,\mathscr{A}^n y) - \mathfrak{d}(x,y)\} \leq 0. \tag{5}$$

One can see that the class of asymptotically nonexpansive mappings in the intermediate sense is more general than the class of asymptotically nonexpansive mappings.

Definition 2. [1] *A mapping $\mathscr{A} : \mathscr{K} \to \mathscr{K}$ is said to be $(\{\mu_n\},\{\gamma_n\},\zeta)$-total asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\},\{\gamma_n\}$ with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \gamma_n$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that*

$$\mathfrak{d}(\mathscr{A}^n x,\mathscr{A}^n y) \leq \mathfrak{d}(x,y) + \gamma_n \zeta\big(\mathfrak{d}(x,y)\big) + \mu_n, \forall x,y \in \mathscr{K}, n \geq 1. \tag{6}$$

Remark 2. *Note that the notion of total asymptotically nonexpansive mappings is more general than that of asymptotically nonexpansive mappings in the intermediate sense (see [6]).*

From last two decades, CAT(0) spaces have attracted the attention of many authors because they have played a very important role in different aspects of geometry [10]. Kirk [18, 19] showed that a nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point.

In 2011, Khan et al.[13] proposed and analyzed a general algorithm for strong convergence results in CAT(0) spaces. Later, in 2012, Khan et al.[12] proposed implicit algorithm for two finite families of nonexpansive maps in the more general setting of hyperbolic spaces. Their results were refinement and generalization of several recent results in CAT(0) spaces and uniformly convex Banach spaces.

In 2012, Chang et al.[5] studied the demiclosedness principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Since then the convergence of several iteration procedures for this type of mappings has been rapidly developed, and many of articles have appeared (see, e.g., [2, 4, 15, 16, 25, 26]).

Let $\mathscr{K}$ be a nonempty closed convex subset of a CAT(0) space $(\mathfrak{M}, \mathfrak{d})$ and $\mathscr{A} : \mathscr{K} \to \mathscr{K}$ be a total asymptotically nonexpansive mapping. Given $x_1 \in \mathscr{K}$, let $\{x_n\} \subset \mathscr{K}$ be defined by

$$x_{n+1} = (1-\alpha_n)x_n \bigoplus \alpha_n \mathscr{A}^n\big((1-\beta_n)x_n \bigoplus \mathscr{A}^n x_n\big), n \in \mathbb{N}, \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$are sequences in $[0,1]$. In 2013, under some suitable assumptions, E. Karapinar et al.[17] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for the iteration (7).

Recently, Panyanak [23] obtained the demiclosedness principle, fixed point theorems, and convergence theorems for total asymptotically nonexpansive mappings on CAT(κ) space with $\kappa > 0$, which generalize the results of Chang et al.[5], Karapinar et al.[17], Tang et al.[25].

Later, Chang et al.[6] studied the strong convergence of a sequence generated by an infinite family of total asymptotically nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Their results are extensions and improvements of the corresponding results of Chang et al.[5], Hea et al.[11], Karapinar et al.[17], Panyanak [23], Tang et Tal.[25], and many others.

The purpose of this manuscript is to establish common fixed point theorems for two infinite families of uniformly L-Lipschitzian $(\{\mu_n\},\{\gamma_n\},\zeta)$-total asymptotically nonexpansive mappings in the setting of hyperbolic spaces. Our results are significantly viewed as generalization of many other related results in the literature.

2. Preliminaries

Let $(\mathfrak{M},\mathfrak{d})$ be a metric space. A geodesic path joining $x\in\mathfrak{M}$ to $y\in\mathfrak{M}$ is a mapping $\omega:[0,l]\to\mathfrak{M}$, where $[0,l]\subset\mathbb{R}$, such that $\omega(0)=x,\omega(l)=y$, and $\mathfrak{d}\big(\omega(t),\omega(t')\big)=|t-t'|$ for all $t,t'\in[0,l]$. Note that, ω is an isometry and $d(x,y)=l$. The image $\omega([0,l])$ of ω is called a geodesic segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z\in[x,y]$ if and only if there exists $\alpha\in[0,1]$ such that $\mathfrak{d}(x,z)=(1-\alpha)\mathfrak{d}(x,y)$, and $\mathfrak{d}(y,z)=\alpha\mathfrak{d}(x,y)$. For such z, we write $z=(1-\alpha)x\bigoplus\alpha y$.

A geodesic triangle $\Delta(x,y,z)$ in a geodesic space $(\mathfrak{M},\mathfrak{d})$ consists of three points x,y,z in $\mathfrak{M}$ (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in $(\mathfrak{M},\mathfrak{d})$ is a triangle $\bar{\Delta}(\bar{x},\bar{y},\bar{z})$ in M^2_κ such that

$$\mathfrak{d}(x,y)=\mathfrak{d}_{M^2_\kappa}(\bar{x},\bar{y}),\quad \mathfrak{d}(y,z)=\mathfrak{d}_{M^2_\kappa}(\bar{y},\bar{z}),\quad \mathfrak{d}(z,x)=\mathfrak{d}_{M^2_\kappa}(\bar{z},\bar{x}).$$

If $\kappa<0$, then such a comparison triangle always exists in M^2_κ. If $\kappa>0$, then such a triangle exists whenever $\mathfrak{d}(x,y)+\mathfrak{d}(y,z)+\mathfrak{d}(z,x)<2\mathscr{D}_\kappa$, where $\mathscr{D}_\kappa=\dfrac{\pi}{\sqrt{\kappa}}$. A point $\bar{p}\in[\bar{x},\bar{y}]$ is called a comparison point for $p\in[x,y]$ if $\mathfrak{d}(x,p)=\mathfrak{d}_{M^2_\kappa}(\bar{x},\bar{p})$.

Remark 3. *For more details on model spaces M^n_k, the reader is referred to* [10, 26].

A geodesic triangle $\Delta(x,y,z)$ in $\mathfrak{M}$ is said to satisfy the CAT(κ) inequality if for any $p,q\in\Delta(x,y,z)$ and for their comparison points $\bar{p},\bar{q}\in\bar{\Delta}(\bar{x},\bar{y},\bar{z})$, one has

$$\mathfrak{d}(p,q)\le\mathfrak{d}_{M^2_\kappa}(\bar{p},\bar{q}).$$

Definition 3. *A metric space $(\mathfrak{M},\mathfrak{d})$ is called a CAT(0) space if $\mathfrak{M}$ is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.*

Note that $\mathfrak{M}$ is called a CAT(κ) space if $\mathfrak{M}$ is $\mathscr{D}_\kappa$-geodesic and any geodesic triangle $\Delta(x,y,z)$ in $\mathfrak{M}$ with $\mathfrak{d}(x,y)+\mathfrak{d}(y,z)+\mathfrak{d}(z,x) < 2\mathscr{D}_\kappa$ satisfies the CAT(κ) inequality. For more details on CAT(κ) spaces, we refer readers to [6, 9].

Definition 4. [27] *A geodesic space* $(\mathfrak{M},\mathfrak{d})$ *is said to be R-convex with* $R \in (0,2]$ *if for any three points* $x,y,z \in \mathfrak{M}$*, we have*

$$\mathfrak{d}^2(x,(1-\alpha)y\bigoplus\alpha z) \leq (1-\alpha)\mathfrak{d}^2(x,y)+\alpha\mathfrak{d}^2(x,z)-\frac{R}{2}\alpha(1-\alpha)\mathfrak{d}^2(y,z). \tag{8}$$

If $(\mathfrak{M},\mathfrak{d})$ is a geodesic space, then the following statements are equivalent:

(i) $(\mathfrak{M},\mathfrak{d})$ is a CAT(0) space,

(ii) $(\mathfrak{M},\mathfrak{d})$ is R-convex with $R=2$, i.e., it satisfies the following inequality:

$$\mathfrak{d}^2\big(x,(1-\alpha)y\bigoplus\alpha z\big) \leq (1-\alpha)\mathfrak{d}^2(x,y)+\alpha\mathfrak{d}^2(x,z)-\alpha(1-\alpha)\mathfrak{d}^2(y,z) \tag{9}$$

for all $\alpha \in (0,1]$ and $x,y,z \in \mathfrak{M}$.

It is wellknown that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets:

We now recall another convex structure on a metric space, called a Hyperbolic Space.

Definition 5. [14, 21] *A hyperbolic space is a triple* $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ *where* $(\mathfrak{M},\mathfrak{d})$ *is a metric space and* $\mathscr{W}:\mathfrak{M}\times\mathfrak{M}\times[0,1]\to\mathfrak{M}$ *is such that*

(H1). $\mathfrak{d}\big(z,\mathscr{W}(x,y,\alpha)\big) \leq (1-\alpha)\mathfrak{d}(z,x)+\alpha\mathfrak{d}(z,y)$,

(H2). $\mathfrak{d}\big(\mathscr{W}(x,y,\alpha),\mathscr{W}(x,y,\beta)\big) = |\alpha-\beta|\mathfrak{d}(x,y)$,

(H3). $\mathscr{W}(x,y,\alpha) = \mathscr{W}(y,x,(1-\alpha))$,

(H4). $\mathfrak{d}\big(\mathscr{W}(x,z,\alpha),\mathscr{W}(y,w,\alpha)\big) \leq (1-\alpha)\mathfrak{d}(x,y)+\alpha\mathfrak{d}(z,w)$ *for all* $x,y,z,w \in \mathfrak{M}, \alpha,\beta \in [0,1]$.

We can see from (H1) that, for each $x,y \in \mathfrak{M}$ and $\alpha \in [0,1]$,

$$\mathfrak{d}\big(x,\mathscr{W}(x,y,\alpha)\big) \leq \alpha\mathfrak{d}(x,y), \quad \mathfrak{d}\big(y,\mathscr{W}(x,y,\alpha)\big) \leq (1-\alpha)\mathfrak{d}(x,y). \tag{10}$$

A subset $\mathscr{K}$ of a hyperbolic space $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ is convex if $\mathscr{W}(x,y,a) \in \mathscr{K}$ for all $x,y \in \mathscr{K}$ and $a \in [0,1]$.

An example of hyperbolic spaces is the family of Banach vector spaces or any normed vector spaces. A subset $\mathscr{K}$ of a hyperbolic space $\mathfrak{M}$ is said to be convex if $[x,y] \subset \mathscr{K}$, whenever $x,y \in \mathscr{K}$.

Lemma 1. [20]. *The hyperbolic space $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$ is called uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$such that, for all $x,y,z \in \mathfrak{M}$,*

$$\left.\begin{aligned} \mathfrak{d}(x,z) &\leq r \\ \mathfrak{d}(y,z) &\leq r \\ \mathfrak{d}(x,y) &\geq \varepsilon r \end{aligned}\right\} \quad \Rightarrow \quad \mathfrak{d}\big(\mathscr{W}(x,y,\frac{1}{2}),z\big) \leq (1-\delta)r. \tag{11}$$

A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta := \eta(r,\varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0,2]$, is called modulus of uniform convexity. A uniformly convex hyperbolic space is strictly convex (see [20]).

The following lemmas are due to Leustean [21, 22].

Lemma 2. [21] *Let $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$ be a uniformly convex hyperbolic space with modulus of uniform convexity η. For any $r > 0, \varepsilon \in (0,2], \lambda \in [0,1]$, and $x,y,z \in \mathfrak{M}$,*

$$\left.\begin{aligned} \mathfrak{d}(x,z) &\leq r \\ \mathfrak{d}(y,z) &\leq r \\ \mathfrak{d}(x,y) &\geq \varepsilon r \end{aligned}\right\} \quad \Rightarrow \quad \mathfrak{d}\big(\mathscr{W}(x,y,\lambda),z\big) \leq \big(1-2\lambda(1-\lambda)\eta(r,\varepsilon)\big)r.$$

Lemma 3. [22] *Let $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in $\mathfrak{M}$ has a unique asymptotic center with respect to any nonempty closed convex subset $\mathscr{K}$ of $\mathfrak{M}$.*

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space space $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$. For $x \in \mathfrak{M}$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \mathfrak{d}(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in \mathfrak{M}\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in \mathfrak{M} : r(x,\{x_n\}) = r(\{x_n\})\}.$$

The asymptotic radius $r(\{x_n\})$ with respect to $\mathscr{K} \subseteq \mathfrak{M}$ of $\{x_n\}$ is given by

$$r_{\mathscr{K}}(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in \mathscr{K}\}.$$

The asymptotic center $A_{\mathscr{K}}(\{x_n\})$with respect to $\mathscr{K} \subseteq \mathfrak{M}$ of $\{x_n\}$ is the set

$$A_{\mathscr{K}}(\{x_n\}) = \{x \in \mathscr{K} : r(x,\{x_n\}) = r(\{x_n\})\}.$$

The concept of Δ-convergence in a metric space was introduced by Lim [20] and its analogue in CAT(0) spaces has been investigated by Dhompongsa and Panyanak [7]. Later, Khan et al.[12] continued the investigation of Δ-convergence in the general setting of hyperbolic spaces.

Lemma 4. [12] *Let $\mathscr{K}$ be a nonempty,closed, and convex subset of a uniformly convex hyperbolic space $(\mathfrak{M},\mathfrak{d},\mathscr{W})$, and let $\{x_n\}$ be a bounded sequence in $\mathscr{K}$ such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in $\mathscr{K}$ such that $\lim_{n\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 5. [12] *Let $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathfrak{M}$ be a given point and $\{\alpha_n\}$ be a sequence in $[b,c]$ with $b,c \in (0,1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in $\mathfrak{M}$ such that*

$$\limsup_{n\to\infty} \mathfrak{d}(x_n,x) \leq r, \quad \limsup_{n\to\infty} \mathfrak{d}(y_n,x) \leq r \quad \textit{and}$$

$$\lim_{n\to\infty} \mathfrak{d}\big(\mathscr{W}(x_n,y_n,\alpha_n),x\big) = r \quad \textit{for some } r \geq 0.$$

Then

$$\lim_{n\to\infty} \mathfrak{d}(x_n,y_n) = 0.$$

The following lemmas are crucial.

Lemma 6. *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n,\, n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i). $\lim_{n\to\infty} a_n$ exists.

(ii). In particular, if $\{a_n\}_{n=1}^{\infty}$, has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 7. [24] *(1) For each positive integer $n \geq 1$, the unique solutions $i(n)$ and $k(n)$ with $k(n) \geq i(n)$ to the following positive integer equation*

$$n = i(n) + \frac{(k(n)-1)k(n)}{2} \tag{12}$$

are as follows:

$$\begin{aligned} i(n) &= n - \frac{(k(n)-1)k(n)}{2}, \\ k(n) &= [\frac{1}{2} + \sqrt{2n - \frac{7}{4}}], k(n) \geq i(n) \end{aligned}$$

and $k(n) \to \infty$ (as $n \to \infty$), where $[x]$ denotes the maximal integer that is not larger than x.

(2) For each $i \geq 1$, denote by

$$\begin{aligned} \Gamma_i &= \{n \in \mathbb{N} : n = i + \frac{(k(n)-1)k(n)}{2},\ k(n) \geq i\} \quad \text{and} \\ \Omega_i &= \{k(n) : n \in \Gamma_i, n = i + \frac{(k(n)-1)k(n)}{2},\ k(n) \geq i\}, \end{aligned}$$

then $k(n) + 1 = k(n+1), \forall n \in \Gamma_i$.

Let $\mathscr{A}_i : \mathscr{K} \to \mathscr{K}$, for each $i \geq 1$, be uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by(6), and for each positive integer $n \geq 1, i(n)$ and $k(n)$ are the unique solutions of the positive integer equation (12). Very recently, Chang et al.[6] proved a strong convergence theorem for $(\{\mu_n\}, \{\gamma_n\}, \zeta)$-total asymptotictally nonexpansive mappings in $CAT(\kappa)$ spaces via the following iterative scheme.

$$\begin{aligned} x_1 &\in \mathscr{K}, \\ x_{n+1} &= (1-\alpha_n)x_n \bigoplus \alpha_n \mathscr{A}_{i(n)}^{k(n)} y_n, \\ y_n &= (1-\beta_n)x_n \bigoplus \beta_n \mathscr{A}_{i(n)}^{k(n)} x_n, n \geq 1, \end{aligned} \tag{13}$$

where $\mathscr{K}$ is a nonempty closed and convex subset of a complete CAT(κ) space $\mathfrak{M}$ with $\kappa > 0$.

More precisely Chang et al.[6] obtained the following result.

Theorem 1. [6] *Let $(\mathfrak{M}, \mathfrak{d})$ be a complete uniformly convex CAT(κ) space with $\kappa > 0$ and $diam(\mathfrak{M}) \leq \frac{\frac{\pi}{2} - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$. Let $\mathscr{K}$ be a nonempty closed and convex subset of $\mathfrak{M}$ and, for each $i \geq 1$, let $\mathscr{A}_i : \mathscr{K} \to \mathscr{K}$ be uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by* (6) *such that* *(i)* $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty$,

(ii) *there exist constant* $\mathscr{M} > 0$ *such that* $\zeta^{(i)}(c) \leq \mathscr{M}\cdot c, \forall c \geq 0, i = 1,2,...$

(iii) *there exist constants* $a,b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\},\{\beta_n\} \subset [a,b]$.

If $\mathfrak{F} := \cap_{i=1}^{\infty} F(\mathscr{A}_i) \neq \emptyset$ *and there exist a mappings* $\mathscr{A}_{n_0} \in \{\mathscr{A}_i\}_{i=1}^{\infty}$ *and a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0, \forall r > 0$ *such that*

$$f(\mathfrak{d}(x_n,\mathfrak{F})) \leq \mathfrak{d}(x_n,\mathscr{A}_{n_0}x_n), \forall n \geq 1. \tag{14}$$

Then the sequence $\{x_n\}$ *defined by* (13) *converges strongly to some point* $p* \in \mathfrak{F}$.

In this manuscript, inspired and motivated by Chang et al.in [6] and Khan et al. [12], we establish common fixed point theorems for two infinite families of uniformly L-Lipschitzian $(\{\mu_n\},\{\gamma_n\},\zeta)$-total asymptotically nonexpansive mappings in the setting of hyperbolic spaces. Our results refine, extend and improve the works of Chang et al.in [6] as well as many other related results in the literature.

3. Main Results

Let $\mathscr{K}$ be a nonempty closed and convex subset of be a complete uniformly convex hyperbolic space $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ and $\mathscr{A}_i,\mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive mappings defined by (6). We will establish common fixed point theorems for two infinite families of such mappings in the setting of hyperbolic spaces via the following iterative scheme:

$$\begin{aligned} x_{n+1} &= \mathscr{W}(x_n,\mathscr{A}_{i(n)}^{k(n)}y_n,\alpha_n), \\ y_n &= \mathscr{W}(x_n,\mathscr{B}_{i(n)}^{k(n)}x_n,\beta_n), n \geq 1, \end{aligned} \tag{15}$$

and for each positive integer $n \geq 1, i(n)$ and $k(n)$ are the unique solutions of the positive integer equation (12).

We denote $\mathbb{F} = \cap_{i=1}^{\infty}(F(\mathscr{A}_i) \cap F(\mathscr{B}_i)), i \geq 1$. We will prove the following two technical lammas.

Lemma 8. *Let* $\mathscr{K}$ *be a nonempty closed and convex subset of be a complete uniformly convex hyperbolic space* $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ *with monotone modulus of uniform convexity* η. *And, for each*

$i \geq 1$, let $\mathscr{A}_i, \mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by(6) *such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)} < \infty,$

(ii) there exists $\mathscr{M} > 0$ such that $\zeta^{(i)}(c) \leq \mathscr{M}\cdot c, \forall c \geq 0, i = 1, 2, ...$

(iii) there exist $a, b \in (0,1)$ with $0 < b(1-a) \leq \frac{1}{2}$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.

If $\{x_n\}$ is the sequence defined by (15)*, then $\lim_{n\to\infty} d(x_n, p)$ and $\lim_{n\to\infty} d(x_n, \mathbb{F})$ exist.*

Proof. Let $p \in \mathbb{F}$. We know that for each $i \geq 1, \mathscr{A}_i, \mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ are $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings. Using condition (ii), for each $n \geq 1$ and any $x, y \in \mathscr{K}$, we have

$$\mathfrak{d}(\mathscr{A}_i^n x, \mathscr{A}_i^n y) \leq (1 + \gamma_n^{(i)}\mathscr{M})\mathfrak{d}(x,y) + \mu_n^{(i)} \tag{16}$$

and

$$\mathfrak{d}(\mathscr{B}_i^n x, \mathscr{B}_i^n y) \leq (1 + \gamma_n^{(i)}\mathscr{M})\mathfrak{d}(x,y) + \mu_n^{(i)}. \tag{17}$$

We will prove that $\lim_{n\to\infty}\mathfrak{d}(x_n, p)$ and $\lim_{n\to\infty}\mathfrak{d}(x_n, \mathbb{F})$ exist for each $p \in \mathbb{F}$.

In fact, since $p \in \mathbb{F}$ and $\mathscr{A}_i, \mathscr{B}_i, i \geq 1$ are $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings, it follows from (16), (17) and (W1) that

$$\begin{aligned}
\mathfrak{d}(y_n, p) &= \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{B}_{i(n)}^{k(n)}x_n, \beta_n), p\big) \\
&\leq (1-\beta_n)\mathfrak{d}(x_n,p) + \beta_n\mathfrak{d}(\mathscr{B}_{i(n)}^{k(n)}x_n, p) \\
&\leq (1-\beta_n)\mathfrak{d}(x_n,p) + \beta_n\big[\mathfrak{d}(x_n,p) + \gamma_{k(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{d}(x_n,p)\big) + \mu_{k(n)}^{i(n)}\big] \\
&\leq \mathfrak{d}(x_n,p) + \beta_n\gamma_{k(n)}^{i(n)}\zeta^{(i(n)}\big(\mathfrak{d}(x_n,p)\big) + \beta_n\mu_{k(n)}^{i(n)} \\
&\leq (1 + \beta_n\gamma_{k(n)}^{i(n)}\mathscr{M})\mathfrak{d}(x_n,p) + \beta_n\mu_{k(n)}^{i(n)} \\
&\leq (1 + \gamma_{k(n)}^{i(n)}\mathscr{M})\mathfrak{d}(x_n,p) + \mu_{k(n)}^{i(n)}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)}y_n, \alpha_n), p\big) \\
&\leq (1-\alpha_n)\mathfrak{d}(x_n,p) + \alpha_n\mathfrak{d}(\mathscr{A}_{i(n)}^{k(n)}y_n, p) \\
&\leq (1-\alpha_n)\mathfrak{d}(x_n,p) + \alpha_n\big[\mathfrak{d}(y_n,p) + \gamma_{k(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{d}(y_n,p)\big) + \mu_{k(n)}^{i(n)}\big]
\end{aligned} \tag{19}$$

$$\leq (1-\alpha_n)\mathfrak{d}(x_n,p)+\alpha_n\mathfrak{d}(y_n,p)+\alpha_n\gamma_{k(n)}^{i(n)}\zeta^{i(n)}\big(\mathfrak{d}(y_n,p)\big)+\alpha_n\mu_{k(n)}^{i(n)}$$
$$\leq (1-\alpha_n)\mathfrak{d}(x_n,p)+\alpha_n(1+\gamma_{k(n)}^{i(n)}\mathscr{M})\mathfrak{d}(y_n,p)+\alpha_n\mu_{k(n)}^{i(n)}.$$

Substitute (18) in (19), we get

$$\begin{aligned}
\mathfrak{d}(x_{n+1},p) &\leq (1-\alpha_n)\mathfrak{d}(x_n,p)+\alpha_n\big(1+\gamma_{k(n)}^{i(n)}\mathscr{M}\big)\big[(1+\gamma_{k(n)}^{i(n)}\mathscr{M})\mathfrak{d}(x_n,p)+\mu_{k(n)}^{i(n)}\big]\\
&\quad +\alpha_n\mu_{k(n)}^{i(n)}\\
&= \big[1-\alpha_n+\alpha_n\big(1+2\gamma_{k(n)}^{i(n)}\mathscr{M}+(\gamma_{k(n)}^{i(n)}\mathscr{M})^2\big)\big]\mathfrak{d}(x_n,p)\\
&\quad +\big[\alpha_n(1+\gamma_{k(n)}^{i(n)}M)+\alpha_n\big]\mu_{k(n)}^{i(n)} \qquad (20)\\
&= [1+2\alpha_n\gamma_{k(n)}^{i(n)}\mathscr{M}+\alpha_n(\gamma_{k(n)}^{i(n)}\mathscr{M})^2]\mathfrak{d}(x_n,p)+\big(2\alpha_n+\alpha_n\gamma_{k(n)}^{i(n)}M\big)\mu_{k(n)}^{i(n)}\\
&= \big[1+\alpha_n(2+\gamma_{k(n)}^{i(n)}\mathscr{M})\gamma_{k(n)}^{i(n)}\mathscr{M}\big]\mathfrak{d}(x_n,p)+\alpha_n\big(2+\gamma_{k(n)}^{i(n)}\mathscr{M}\big)\mu_{k(n)}^{i(n)}\\
&\leq (1+\sigma_n)\mathfrak{d}(x_n,p)+\xi_n, \forall n\geq 1 \text{ and } p\in\mathbb{F},
\end{aligned}$$

where $\sigma_n=b\big(2+\gamma_{k(n)}^{i(n)}\mathscr{M}\big)\gamma_{k(n)}^{i(n)}\mathscr{M}$, $\xi_n=b\big(2+\gamma_{k(n)}^{i(n)}\mathscr{M}\big)\mu_{k(n)}^{i(n)}$ (because $\{\alpha_n\}\subset[a,b]$).

So

$$\mathfrak{d}(x_{n+1},p)\leq(1+\sigma_n)\mathfrak{d}(x_n,p)+\xi_n, \forall n\geq 1. \tag{21}$$

By condition (i), we have

$$\sum_{n=1}^{\infty}\sigma_n<\infty \text{ and } \sum_{n=1}^{\infty}\xi_n<\infty. \tag{22}$$

From (21), (22) and by Lemma 6, we conclude that $\lim_{n\to\infty}\mathfrak{d}(x_n,p)$ and $\lim_{n\to\infty}\mathfrak{d}(x_n,\mathbb{F})$ exist for each $p\in\mathbb{F}$. □

Lemma 9. *Let $\mathscr{K}$ be a nonempty closed and convex subset of be a complete uniformly convex hyperbolic space $(\mathfrak{M},\mathfrak{d},\mathscr{W})$ with monotone modulus of uniform convexity η. And, for each $i\geq 1$, let $\mathscr{A}_i,\mathscr{B}_i:\mathscr{K}\to\mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive mappings defined by* (6) *such that*

(i) $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty,\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu_n^{(i)}<\infty$,

(ii) there exists $\mathscr{M}>0$ *such that* $\zeta^{(i)}(c)\leq\mathscr{M}\cdot c,\forall c\geq 0,i=1,2,...$,

(iii) there exist $a,b\in(0,1)$ *with* $0<b(1-a)\leq\frac{1}{2}$ *such that* $\{\alpha_n\},\{\beta_n\}\subset[a,b]$.

Suppose $\{x_n\}$ *is the sequence defined by* (15)*, then*

$$\lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} x_n) = 0, \quad \lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{B}_{i(n)}^{k(n)} x_n) = 0.$$

In particular, we have

$$\lim_{m(\in\Gamma_i)\to\infty} \mathfrak{d}(x_m, \mathscr{A}_i x_m) = 0 \quad \textit{and} \quad \lim_{m(\in\Gamma_i)\to\infty} \mathfrak{d}(x_m, \mathscr{B}_i x_m) = 0,$$

for $i \geq 1$.

Proof. Firstly, we will show that

$$\lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} x_n) = 0, \quad \lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{B}_{i(n)}^{k(n)} x_n) = 0.$$

In fact, it follows from Lemma 8 that for each given $p \in \mathbb{F}$, the $\lim_{n\to\infty} \mathfrak{d}(x_n, p)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \mathfrak{d}(x_n, p) = r \geq 0. \tag{23}$$

From (18) we have

$$\limsup_{n\to\infty} \mathfrak{d}(y_n, p) \leq \lim_{n\to\infty} \{(1+\gamma_{k(n)}^{i(n)} \mathscr{M}) \mathfrak{d}(x_n, p) + \mu_{k(n)}^{i(n)}\} = r. \tag{24}$$

Since

$$\begin{aligned} \mathfrak{d}(\mathscr{A}_{i(n)}^{k(n)} y_n, p) &\leq \mathfrak{d}(y_n, p) + \gamma_{k(n)}^{i(n)} \zeta^{i(n)}\big(\mathfrak{d}(y_n, p)\big) + \mu_{k(n)}^{i(n)} \\ &\leq (1+\gamma_{k(n)}^{i(n)} \mathscr{M}) \mathfrak{d}(y_n, p) + \mu_{k(n)}^{i(n)} \quad \forall n \geq 1, \end{aligned}$$

by (24) we get

$$\limsup_{n\to\infty} \mathfrak{d}(\mathscr{A}_{i(n)}^{k(n)} y_n, p) \leq r. \tag{25}$$

In addition, it follows from (20) that

$$\begin{aligned} \mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n, \alpha_n), p\big) \\ &\leq (1+\sigma_n) \mathfrak{d}(x_n, p) + \xi_n. \end{aligned} \tag{26}$$

This implies that

$$\lim_{n\to\infty} \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n, \alpha_n), p\big) = r. \tag{27}$$

From (23), (25), (27) and Lemma 5, we have

$$\lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n) = 0. \tag{28}$$

Consider

$$\begin{aligned} \mathfrak{d}(x_n, p) &= \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n) + \mathfrak{d}(\mathscr{A}_{i(n)}^{k(n)} y_n, p) \\ &\leq \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n) + \{\mathfrak{d}(y_n, p) + \gamma_{k(n)}^{i(n)} \zeta^{i(n)}\big(\mathfrak{d}(y_n, p)\big) + \mu_{k(n)}^{i(n)}\} \\ &\leq \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n) + (1 + \gamma_{k(n)}^{i(n)} \mathscr{M})\mathfrak{d}(y_n, p) + \mu_{k(n)}^{i(n)}. \end{aligned} \tag{29}$$

Taking the lim inf on both sides of the above inequality, we have

$$r \leq \liminf_{n\to\infty} \mathfrak{d}(y_n, p). \tag{30}$$

This inequality (30) and (24) yield

$$\lim_{n\to\infty} \mathfrak{d}(y_n, p) = r. \tag{31}$$

Using (18) and (31), we have

$$\begin{aligned} r = \lim_{n\to\infty} \mathfrak{d}(y_n, p) &= \lim_{n\to\infty} \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)} x_n, \beta_n), p\big) \\ &\leq \lim_{n\to\infty} (1 + \gamma_{k(n)}^{i(n)} \mathscr{M})\mathfrak{d}(x_n, p) + \mu_{k(n)}^{i(n)} \\ &= r. \end{aligned} \tag{32}$$

This implies

$$\lim_{n\to\infty} \mathfrak{d}\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)} x_n, \beta_n), p\big) = r. \tag{33}$$

Moreover, we can also have that

$$\limsup_{n\to\infty} \mathfrak{d}(\mathscr{A}_{i(n)}^{k(n)} x_n, p) \leq \limsup_{n\to\infty} [\mathfrak{d}(x_n, p) + \gamma_{k(n)}^{i(n)} \zeta^{i(n)}\big(\mathfrak{d}(x_n, p)\big) + \mu_{k(n)}^{i(n)}] \leq r. \tag{34}$$

Applying (34) together with (23), (33), and Lemma 5, we obtain

$$\lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{A}_{i(n)}^{k(n)} x_n) = 0. \tag{35}$$

Similarly, one can show that

$$\lim_{n\to\infty} \mathfrak{d}(x_n, \mathscr{B}_{i(n)}^{k(n)} x_n) = 0. \tag{36}$$

Next, we will prove that, for each $i \geq 1$, there exist a subsequence $\{x_{m(\in\Gamma_i)}\} \subset \{x_n\}$ such that

$$\lim_{m(\in\Gamma_i)\to\infty} \eth(x_m, \mathscr{A}_i x_m) = 0 \quad and \lim_{m(\in\Gamma_i)\to\infty} \eth(x_m, \mathscr{B}_i x_m) = 0,$$

where Γ_i is the set of positive integers defined in Lemma 7.

Applying (36), we have

$$\begin{aligned} \eth(x_n, y_n) &= \eth\big(x_n, \mathscr{W}(x_n, \mathscr{B}_{i(n)}^{k(n)} x_n, \beta_n)\big) \\ &\leq \beta_n \eth(x_n, \mathscr{B}_{i(n)}^{k(n)} x_n) \to 0 \ (\text{as}\, n \to \infty). \end{aligned} \tag{37}$$

Furthermore, from (28) we obtain that

$$\begin{aligned} \eth(x_{n+1}, x_n) &= \eth\big(\mathscr{W}(x_n, \mathscr{A}_{i(n)}^{k(n)} y_n, \alpha_n), x_n\big) \\ &\leq \alpha_n \eth(\mathscr{A}_{i(n)}^{k(n)} y_n, x_n) \to 0 \ (\text{as}\, n \to \infty). \end{aligned} \tag{38}$$

From (37) and (38), we get

$$\eth(x_{n+1}, y_n) \leq \eth(x_{n+1}, x_n) + \eth(x_n, y_n) \to 0 \ (\text{as}\, n \to \infty). \tag{39}$$

From (28), (35), (38), (39) and Lemma 7, for each given positive integer $i \geq 1$, there exist subsequences $\{x_m\}_{m\in\Gamma_i}, \{y_m\}_{m\in\Gamma_i}$, and $\{k(m)\}_{m\in\Gamma_i} \subset \Omega_i := \{k(m) : m \in \Gamma_i, m = i + \dfrac{(k(m)-1)k(m)}{2}, k(m) \geq i\}$ such that

$$\begin{aligned} \eth(x_m, \mathscr{A}_i x_m) &\leq \eth(x_m, \mathscr{A}_i^{k(m)} x_m) + \eth(\mathscr{A}_i^{k(m)} x_m, \mathscr{A}_i^{k(m)} y_{m-1}) + \eth(\mathscr{A}_i^{k(m)} y_{m-1}, \mathscr{A}_i x_m) \\ &\leq \eth(x_m, \mathscr{A}_i^{k(m)} x_m) + \{\eth(x_m, y_{m-1}) + \gamma_{k(m)}^{(i)} \zeta^i(\eth(x_m, y_{m-1})) + \mu_{k(m)}^{(i)}\} \\ &\quad + L_i \eth(\mathscr{A}_i^{k(m)-1} y_{m-1}, x_m) \\ &\leq \eth(x_m, \mathscr{A}_i^{k(m)} x_m) + \{(1 + \gamma_{k(m)}^{(i)} \mathscr{M}) \eth(x_m, y_{m-1}) + \mu_{k(m)}^{(i)}\} \\ &\quad + L_i \eth(\mathscr{A}_i^{k(m)-1} y_{m-1}, x_{m-1}) + L_i \eth(x_{m-1}, x_m). \end{aligned} \tag{40}$$

Taking the limit as $m \to \infty$ on the above inequality, we obtain

$$\lim_{m(\in\Gamma_i)\to\infty} \eth(x_m, \mathscr{A}_i x_m) = 0.$$

Similarly, one can show that

$$\lim_{m(\in\Gamma_i)\to\infty} \eth(x_m, \mathscr{B}_i x_m) = 0.$$

□

Theorem 2. *Let $\mathscr{K}$ be a nonempty closed and convex subset of be a complete uniformly convex hyperbolic space $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$ with monotone modulus of uniform convexity η. And, for each $i \geq 1$, let $\mathscr{A}_i, \mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by(6) such that*

(i) $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$,

(ii) there exists a constant $\mathscr{M} > 0$ such that $\zeta^{(i)}(c) \leq \mathscr{M} \cdot c, \forall c \geq 0, i = 1, 2, ...,$

(iii) there exist $a, b \in (0, 1)$ with $0 < b(1-a) \leq \frac{1}{2}$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$.

If $\mathbb{F} := \cap_{i=1}^{\infty} \big(F(\mathscr{A}_i) \cap F(\mathscr{B}_i)\big) \neq \emptyset$ and there exist mappings $\mathscr{A}_{n_0} \in \{\mathscr{A}_i\}_{i=1}^{\infty}, \mathscr{B}_{n_0} \in \{\mathscr{B}_i\}_{i=1}^{\infty}$ and a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0, \forall r > 0$ such that

$$f(\mathfrak{d}(x_n, \mathbb{F})) \leq \mathfrak{d}(x_n, \mathscr{A}_{n_0} x_n), \forall n \geq 1, \tag{41}$$

and

$$f(\mathfrak{d}(x_n, \mathbb{F})) \leq \mathfrak{d}(x_n, \mathscr{B}_{n_0} x_n), \forall n \geq 1. \tag{42}$$

Then the sequence $\{x_n\}$ defined by (15) *converges strongly to a common fixed point $p \in \mathbb{F}$.*

Proof. In fact, it follows from Lemma 9 that for given mapping $\mathscr{A}_{n_0}, \mathscr{B}_{n_0}$, there exists a subsequence $\{x_m\}_{m \in \Gamma_{n_0}}$ of $\{x_n\}$ such that

$$\lim_{m(\in \Gamma_{n_0}) \to \infty} \mathfrak{d}(x_m, \mathscr{A}_{n_0} x_m) = 0 \quad \text{and} \quad \lim_{m(\in \Gamma_{n_0}) \to \infty} \mathfrak{d}(x_m, \mathscr{B}_{n_0} x_m) = 0. \tag{43}$$

By (41), (42), we have

$$f(\mathfrak{d}(x_m, \mathbb{F})) \leq \mathfrak{d}(x_m, \mathscr{A}_{n_0} x_m), \ \forall m \geq 1, \tag{44}$$

and

$$f(\mathfrak{d}(x_m, \mathbb{F})) \leq \mathfrak{d}(x_m, \mathscr{B}_{n_0} x_m), \ \forall m \geq 1. \tag{45}$$

Taking the limit as $m \to \infty$ on the above inequalities, we have $\lim_{m \to \infty} f(\mathfrak{d}(x_m, \mathbb{F})) = 0$. This implies that $\lim_{m(\in \Gamma_{n_0}) \to \infty} \mathfrak{d}(x_m, \mathbb{F})) = 0$.

Next we show that $\{x_{m \in \Gamma_{n_0}}\}$ is a Cauchy sequence in $\mathscr{K}$. In fact, from (20), for any $p \in \mathbb{F}$, we have

$$\mathfrak{d}(x_{m+1}, p) \leq (1 + \sigma_m) \mathfrak{d}(x_m, p) + \xi_m, \forall m \geq 1, \forall m \in \Gamma_{n_0}, \tag{46}$$

where $\sum_{m=1}^{\infty}\sigma_m<\infty$ and $\sum_{m=1}^{\infty}\xi_m<\infty$. Hence, for any positive integers $j,n\in\Gamma_{n_0}, n>j$ and $n=m+j$ for some positive integer m, and since $1+x\le e^x$ for each $x\ge 0$, we have

$$\begin{aligned}
\mathfrak{d}(x_n,x_j) &= \mathfrak{d}(x_{m+j},x_j)\\
&\le \mathfrak{d}(x_{m+j},p)+\mathfrak{d}(p,x_j)\\
&\le (1+\sigma_{m+j-1})\mathfrak{d}(x_{m+j-1},p)+\xi_{m+j-1}+\mathfrak{d}(x_j,p)\\
&\le e^{(\sigma_{m+j-1})}\mathfrak{d}(x_{m+j-1},p)+\xi_{m+j-1}+\mathfrak{d}(x_j,p)\\
&\le e^{(\sigma_{m+j-1}+\sigma_{m+j-2})}\mathfrak{d}(x_{m+j-2},p)+e^{(\sigma_{m+j-1})}\xi_{m+j-2}\\
&\quad +\xi_{m+j-1}+\mathfrak{d}(x_j,p)\\
&\le e^{(\sigma_{m+j-1}+\sigma_{m+j-2}+\sigma_{m+j-3})}\mathfrak{d}(x_{m+j-3},p)+e^{(\sigma_{m+j-2}+\sigma_{m+j-3})}\xi_{m+j-3}\\
&\quad +e^{(\sigma_{m+j-1})}\xi_{m+j-2}+\xi_{m+j-1}+\mathfrak{d}(x_j,p)\\
&\;\;\vdots\\
&\le e^{(\sum_{i=j}^{m+j-1}\sigma_i)}\mathfrak{d}(x_j,p)+e^{(\sum_{i=j+1}^{m+j-1}\sigma_i)}\xi_j+e^{(\sum_{i=j+2}^{m+j-2}\sigma_i)}\xi_{j+1}+\cdots+\\
&\quad e^{(\sum_{i=j+m-2}^{m+j-3}\sigma_i)}\xi_{m+j-3}+e^{(\sigma_{m+j-1})}\xi_{m+j-2}+\xi_{m+j-1}+\mathfrak{d}(x_j,p)\\
&\le (1+M)\mathfrak{d}(x_j,p)+M\sum_{i=j}^{m+j-1}\xi_i\\
&= (1+M)\mathfrak{d}(x_j,p)+M\sum_{i=j}^{n-1}\xi_i \text{ for each } p\in\mathbb{F}.
\end{aligned}$$

Therefore we have

$$\mathfrak{d}(x_n,x_j)=\mathfrak{d}(x_{j+m},x_j)\le(1+M)\mathfrak{d}(x_j,\mathbb{F})+M\sum_{i=j}^{n-1}\xi_i,$$

where $M=e^{(\sum_{i=1}^{\infty}\sigma_i)}<\infty$. Therefore, we have

$$\mathfrak{d}(x_n,x_j)\le(1+M)\mathfrak{d}(x_j,\mathbb{F})+M\sum_{i=j}^{n-1}\xi_i\to 0,(\text{as } n,j\in\Gamma_{n_0}\to\infty).$$

This shows that the subsequence $\{x_m\}_{m\in\Gamma_{n_0}}$ is a Cauchy sequence in $\mathscr{K}$. Since $\mathscr{K}$ is a closed subset of a complete uniformly convex hyperbolic space $\mathfrak{M}$, without loss of generality, we can assume that the subsequence $\{x_m\}$ converges strongly to some common fixed point $p\in\mathscr{K}$. It is easy to see that $\mathbb{F}$ is a closed subset in $\mathscr{K}$. Since $\lim_{m\to\infty}\mathfrak{d}(x_m,\mathbb{F})=0, p\in\mathbb{F}$. By Lemma 6, it yields that the whole sequence $\{x_n\}$ converges strongly to a common fixed point $p \in \mathbb{F}$. This completes our proof. □

We now obtain the following corollary.

Corollary 1. *Let $\mathscr{K}$ be a nonempty closed and convex subset of a complete CAT(0) space $(\mathfrak{M}, d)$, and for each $i \geq 1$, $\mathscr{A}_i, \mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \mu_n(i)\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by*(6) *such that*

(i) $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty,$

(ii) there exist constant $\mathscr{M} > 0$ *such that* $\zeta^{(i)}(c) \leq \mathscr{M} \cdot c, \forall c \geq 0, i = 1, 2, ...,$

(iii) there exist constants $a, b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.

If $\mathbb{F} := \cap_{i=1}^{\infty} \big(F(\mathscr{A}_i) \cap F(\mathscr{B}_i)\big) \neq \emptyset$ *and there exist mappings* $\mathscr{A}_{n_0} \in \{\mathscr{A}_i\}_{i=1}^{\infty}, \mathscr{B}_{n_0} \in \{\mathscr{B}_i\}_{i=1}^{\infty}$ *and a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0, \forall r > 0$ *such that*

(47) $$f(\mathfrak{d}(x_n, \mathbb{F})) \leq \mathfrak{d}(x_n, \mathscr{A}_{n_0} x_n), \forall n \geq 1,$$

and

(48) $$f(\mathfrak{d}(x_n, \mathbb{F})) \leq \mathfrak{d}(x_n, \mathscr{B}_{n_0} x_n), \forall n \geq 1.$$

Then the sequence $\{x_n\}$ *defined by* (15) *converges strongly to some common fixed point* $p \in \mathbb{F}$.

Next we will prove a Δ-convergence theorem.

Theorem 3. *Let $\mathscr{K}$ be a nonempty closed and convex subset of be a complete uniformly convex hyperbolic space $(\mathfrak{M}, \mathfrak{d}, \mathscr{W})$ with monotone modulus of uniform convexity η. And, for each $i \geq 1$, let $\mathscr{A}_i, \mathscr{B}_i : \mathscr{K} \to \mathscr{K}$ be two uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings defined by*(6) *such that*

(i) $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty, \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty,$

(ii) there exist constant $\mathscr{M} > 0$ *such that* $\zeta^{(i)}(c) \leq \mathscr{M} \cdot c, \forall c \geq 0, i = 1, 2, ...,$

(iii) there exist constants $a, b \in (0,1)$ *with* $0 < b(1-a) \leq \frac{1}{2}$ *such that* $\{\alpha_n\}, \{\beta_n\} \subset [a,b]$.

Then the sequence $\{x_n\}$ *defined by* (15) Δ*-converges strongly to a commom fixed point* $p \in \mathbb{F}$.

Proof. It follows from Lemma 8 that $\{x_n\}$ is bounded. Therefore by Lemma 4, $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. And, by Lemma 9, we have $\lim_{n\to\infty} \eth(u_n, \mathscr{A}_i u_n) = 0 = \lim_{n\to\infty} \eth(u_n, \mathscr{B}_i u_n)$ for each $i \geq 1$.

We will prove that u is the common fixed point of $\{\mathscr{A}_i : i \geq 1\}$ and $\{\mathscr{B}_i : i \geq 1\}$. For each given positive integer $i \geq 1$, and $\{k(m)\}_{m\in\Gamma_i} \subset \Omega_i := \{k(m) : m \in \Gamma_i, m = i + \frac{(k(m)-1)k(m)}{2}, k(m) \geq i\}$, we define a sequence $\{z_m\}$ in $\mathscr{K}$ by $z_m = \mathscr{A}_i^{k(m)} u$. Observe that

$$\begin{aligned}
\eth(z_m, u_n) &\leq \eth(\mathscr{A}_i^{k(m)} u, \mathscr{A}_i^{k(m)} u_n) + \eth(\mathscr{A}_i^{k(m)} u_n, \mathscr{A}_i^{k(m)-1} u_n) + \cdots + \eth(\mathscr{A}_i u_n, u_n) \\
&\leq \eth(\mathscr{A}_i^{k(m)} u, \mathscr{A}_i^{k(m)} u_n) + L_i \eth(\mathscr{A}_i u_n, u_n) + \cdots + \eth(\mathscr{A}_i u_n, u_n) \\
&\leq \{\eth(u, u_n) + \gamma_{k(m)}^{(i)} \zeta^i(\eth(u, u_n)) + \mu_{k(m)}^{(i)}\} + M\eth(\mathscr{A}_i u_n, u_n) \\
&\leq \{(1 + \gamma_{k(m)}^{(i)} \mathscr{M})\eth(u, u_n) + \mu_{k(m)}^{(i)}\} + M\eth(\mathscr{A}_i u_n, u_n),
\end{aligned} \tag{49}$$

for some constant M. Therefore, we obtain

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} \eth(z_m, u_n) \leq \limsup_{n\to\infty} \eth(u, u_n) = r(u, \{u_n\}).$$

Hence $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 4 that $\mathscr{A}_i u = u$. Hence u is the common fixed point of $\{\mathscr{A}_i : i \geq 1\}$. Similarly, we can show that u is the common fixed point of $\{\mathscr{B}_i : i \geq 1\}$. Therefore u is the common fixed point of $\{\mathscr{A}_i : i \geq 1\}$ and $\{\mathscr{B}_i : i \geq 1\}$. Note that $\lim_{n\to\infty} \eth(x_n, u)$ exists by Lemma 8. Suppose $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} \eth(u_n, u) &< \limsup_{n\to\infty} \eth(u_n, x) \\
&\leq \limsup_{n\to\infty} \eth(x_n, x) \\
&\leq \limsup_{n\to\infty} \eth(x_n, u) \\
&\leq \limsup_{n\to\infty} \eth(u_n, u).
\end{aligned} \tag{50}$$

This is a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{\mathscr{A}_i : i \geq 1\}$ and $\{\mathscr{B}_i : i \geq 1\}$. □

4. Conclusions

In this manuscript, we establish common fixed point results for two infinite families of uniformly L_i-Lipschitzian and $(\{\gamma_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings in the setting of hyperbolic space, a more general space. Moreover, we prove a Δ-convergence result for such mappings. Our results significantly refine, generalize, extend and improve the results obtained by Chang et al.[6], as well as some other related existing results in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests.

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