Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:45
https://doi.org/10.28919/afpt/8708
ISSN: 1927-6303

FINITE-TIME BLOW-UP AND SOLVABILITY OF A WEAK SOLUTION FOR A SUPERLINEAR REACTION-DIFFUSION PROBLEM WITH INTEGRAL CONDITIONS OF THE SECOND TYPE

IQBAL M. BATIHA[1,2,*], IQBAL H. JEBRIL[1], ZAINOUBA CHEBANA[3], TAKI-EDDINE OUSSAEIF[3], SOFIANE DEHILIS[3], SHAWKAT ALKHAZALEH[4]

[1]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan

[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE

[3]Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria

[4]Department of Mathematics, Jadara University, Irbid, Jordan

Abstract. The focus of this work is on a class of reaction-diffusion equations: a superlinear nonlocal issue with Neumann condition modeled by the integral condition of second type. By using the Fadeo-Galarkin method to get over the complications caused by the integral condition's existence, we are able to demonstrate the existence of the weak solution. Next, we demonstrate the uniqueness of the problem's weak solution by using an a priori estimate. In conclusion, we examine the blow-up solution for completeness in its finite-time case.

Keywords: parabolic equation; nonlinear equations; integral condition; existence and uniqueness; Fadeo-Galarkin method.

2020 AMS Subject Classification: 35K58, 35K61, 35B45, 35A01, 35A02.

1. Introduction

A significant class of parabolic equations known as nonlinear diffusion equations originated from a wide range of diffusion phenomena that are frequently found in nature [1, 2, 3, 4, 5, 6,

*Corresponding author
E-mail address: i.batiha@zuj.edu.jo
Received June 14, 2024

7, 8]. Numerous mathematicians and scientists in the nonlinear sciences have expressed great interest in the nonlinear evolution equations due to their complexity and the difficulties in studying them theoretically [9]. Partial differential equations with nonlocal conditions can be used to simulate a wide range of natural phenomena. On the other hand, integral conditions—which are gaining popularity—are a superior way to characterize a lot of events. Nonlocal and integral conditions for partial differential equations are used to formulate many contemporary physics and technology problems (see, for example, [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]). The the first type of these conditions can be given by

$$\int_{\Omega} k(x,t)u(x,t)dx = E(t),$$

where $t \in (0,T)$, $\Omega \subset \mathbb{R}^n$ and k is a given function. The second type of such conditions, where the Dirichlet or Neumann condition modeling by integral condition, is given by

$$u(x,t)|_{\partial\Omega} = \int k(x,t)u(x,t)dx.$$

In fact, the above condition can be used when it is impossible to directly measure the sought quantity on the border, its total value, or its average is known. Numerous strong and diverse techniques in nonlinear analysis, such as the fixed-point theorem, semi-group method, Galerkin [20], and monotone operator method [21, 22, 23, 24, 25], have been used to solve problems involving nonlinear evolution equations with various boundary conditions types (classical and non-classical condition).

Motivated by this, we use the Faedo-Galerkin method to demonstrate the existence and uniqueness of the weak solution for the linear problem in a superlinear parabolic equation with a classical Dirichlet condition and an integral condition of the second type, which is more general than any classical integral condition. The existence and uniqueness of the weak solution to the semilinear problem are then demonstrated by employing an iterative procedure based on the outcomes for the linear problem. Additionally, we theoretically examine the blow-up solution, concentrating on characterizing the main problem's finite time blow-up solution as it is affected by integral conditions.

2. Problem's Formulation

Let $\Omega = (0,l)$ be a bounded open of $\mathbb{R}$ and $Q = \Omega \times (0,T)$. We consider the following problem:

$$(P_1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + u^p - bu = f(x,t) & \forall (x,t) \in \overline{Q} \\ u(x,0) = \varphi(x) & \forall x \in (0,l) \\ \dfrac{\partial u}{\partial x}(0,t) = \int_0^l k_1(x,t)u(x,t)dx & \forall t \in [0,T] \\ \dfrac{\partial u}{\partial x}(l,t) = \int_0^l k_2(x,t)u(x,t)dx & \forall t \in [0,T] \end{cases},$$

where a, b and p are positive odd integers and $p \geq 1$. The purpose of this work is to show that the function $u = u(x,t)$ represents a solution of the problem (P_1) under certain assumptions (H) for which $x \in \Omega$ and $t \in [0,T]$. In order to properly pose the problem and to have the tools to solve it, we need to introduce some concepts and some functional basis that we will use later. Now, we define the space V by

$$V = \left\{ u \in H^1(\Omega) \cap L^{p+1}(\Omega) \right\},$$

where the space V is provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^{p+1}(\Omega)}$ and with the scalar product of $H^1(\Omega)$. We are now able to formulate the problem precisely (P_1) for studying it deeply. To this end, we need the following hypothesis:

$$(H): \begin{cases} f \in L^2\left(0,T;\, L^2(\Omega)\right) & (H.1) \\ \varphi \in H^1(\Omega) \cap L^{p+1}(\Omega) & (H.2) \\ k_i \in L^\infty((0,T)\times\Omega) \ \ \forall i \in \{1,2\} \end{cases}.$$

Definition 1. *The weak solution of the problem (P_1) is a function that verifies*

(1) $u \in L^2\left(0,T;\, H^1(\Omega)\right) \cap L^\infty\left(0,T;\, H^1(\Omega)\right)$.

(2) u admits a strong derivative $\dfrac{\partial u}{\partial t} \in L^2\left(0,T;\, H^1(\Omega)\right)$.

(3) $u(0) = \varphi$.

(4) Identity

$$(u_t, v) + a(u_x, v_x) + (u^p, v) - b(u,v) = (f,v) + av(l,t)\int_0^l k(x,t)u(x,t)dx,$$

for all $v \in V$ *and for all* $t \in [0,T]$.

3. Solution's Existence of the Semilinear Problem (P_1)

3.1. Variational Formulation. By multiplying the equation

$$\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}+u^p-bu=f(x,t), \tag{1}$$

by an element $v\in V$ and then integrating the result over Ω, we obtain

$$\int_\Omega \frac{\partial u}{\partial t}\cdot v dx-a\int_\Omega \frac{\partial^2 u}{\partial x^2}\cdot v dx+\int_\Omega u^p\cdot v dx-b\int_\Omega u\cdot v dx=\int_\Omega f\cdot v dx.$$

Then, we have

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial t}\cdot v dx+a\int_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}dx+\int_\Omega u^p\cdot v dx-b\int_\Omega u\cdot v dx-av(l)u_x(l,t)+av(0)u_x(0,t)\\ =\int_\Omega f\cdot v dx.\end{aligned} \tag{2}$$

Using the boundary conditions and Green's formula (2) yields

$$(u_t,v)+a(u_x,v_x)+(u^p,v)-b(u,v)-av(l)u_x(l,t)+av(0)u_x(0,t)=(f,v),\ \forall v\in V, \tag{3}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

3.2. Solution's Existence of the problem (P_1). The demonstration of the existence of the solution of the problem (P_1) is based on the method of Faedo-Galerkin which consists in carrying out the subsequent content. The space V is separable, and then there exists a sequence $w_1,w_2,\dots,w_m$ having the following properties:

$$\begin{cases} w_i\in V, & \forall i,\\ \forall m, w_1,w_2,\dots,w_m & \text{are linearly independent},\\ V_m=\langle\{w_1,w_2,\dots,w_m\}\rangle & \text{is dense in } V.\end{cases} \cdot \tag{4}$$

In particular, we can say that

$$\forall\varphi\in V \implies \exists(\alpha_{km})_m\in IN^*,\ \varphi_m=\sum_{k=1}^m\alpha_{km}w_k\longrightarrow\varphi \text{ when } m\longrightarrow+\infty. \tag{5}$$

Faedo Galerkin's approximation consists in searching for any integer $m\geq 1$ for which the following inequality is satisfied:

$$t\mapsto u_m(x,t)=\sum_{i=1}^m g_{im}(t)w_i(x).$$

The approximate solution satisfies the following identities:

$$(P_2) \quad \begin{cases} ((u_m(t))_t, w_k) + a(\Delta u_m(t), w_k) + \left(u_m^p(t) - bu_m(t), w_k\right) = (f(t), w_k) \quad \forall k = \overline{1,m} \\ (u_m(0), w_k) = \alpha_{km} \end{cases},$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. So, we have

$$\begin{aligned} ((u_m(t))_t, w_k) &= \left(\left(\sum_{i=1}^m g_{im}(t) w_i\right)_t, w_k\right) \\ &= \left(\sum_{i=1}^m \frac{\partial g_{im}}{\partial t}(t) w_i(x), w_k\right) \\ &= \sum_{i=1}^m (w_i, w_k) \frac{\partial g_{im}}{\partial t}(t), \end{aligned} \tag{6}$$

and

(7)

$$\begin{aligned} a(\Delta u_m(t), w_k) &= a\left(\Delta\left(\sum_{i=1}^m g_{im}(t) w_i\right), w_k\right) \\ &= a\sum_{i=1}^m g_{im}(t)\left[\frac{\partial w_i}{\partial x}(l) w_k(l) - \int_\Omega \frac{\partial w_i}{\partial x}\frac{\partial w_k}{\partial x} dx\right] \\ &= -a\sum_{i=1}^m g_{im}(t)\int_\Omega \frac{\partial w_i(x)}{\partial x}\frac{\partial w_k(x)}{\partial x} dx + a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(l) w_k(l) - a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(0) w_k(0) \\ &= -\sum_{i=1}^m a((w_i)_x, (w_k)_x) g_{im}(t) + a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(l) w_k(l) - a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(0) w_k(0). \end{aligned}$$

Additionally, we have

$$\begin{aligned} u_m(0) &= \sum_{i=1}^m g_{im}(0) w_i(x) \\ &= \varphi_m \\ &= \sum_{k=1}^m \alpha_{km} w_k(x). \end{aligned}$$

Now, the existence of such an α_{km} follows from $u_0 \in V \cap L^{P+1}(\Omega)$ and the fact that $\{w_k,\ k \in \mathbb{N}\}$ is the base in $V \cap L^{P+1}(\Omega)$. Thus, (P_1) is reduced to the initial value problem for a system of first-order differential equations with respect to g_{im}, i.e.,

$$(P_3)\quad \begin{cases} \sum_{i=1}^{m}(w_i,w_k)\dfrac{\partial g_{im}}{\partial t}(t)+a\sum_{i=1}^{m}((w_i)_x,(w_k)_x)\,g_{im}(t) \\ -a\sum_{i=1}^{m}g_{im}(t)\frac{\partial w_i}{\partial x}(l)w_k(l)+a\sum_{i=1}^{m}g_{im}(t)\frac{\partial w_i}{\partial x}(0)w_k(0)+\left(u_m^p-bu_m,w_k\right)=(f(t),w_k) \\ g_{km}(0)=\alpha_{km}\qquad \forall k=\overline{1,m}. \end{cases}.$$

Consequently, we consider the vectors

$$g_m=(g_{1m}(t),\dots,g_{mm}(t)),\ f_m=((f,w_1),\dots,(f,w_m))$$

with the matrices

$$B_m=\left((w_i,w_j)\right)_{\substack{1\le i\le m\\1\le j\le m}},\ A_m=\left(\left(\frac{\partial w_i}{\partial x},\frac{\partial w_j}{\partial x}\right)\right)_{\substack{1\le i\le m\\1\le j\le m}}$$

$$C_m=\left(\frac{\partial w_i}{\partial x}(l)\cdot w_j(l)\right)_{\substack{1\le i\le m\\1\le j\le m}},\ D_m=\left(\frac{\partial w_i}{\partial x}(0)\cdot w_j(0)\right)_{\substack{1\le i\le m\\1\le j\le m}},$$

and

$$G(g)=\left(\left(\left(\sum_{i=1}^{m}g_{im}(t)\,w_i\right)^p,w_j\right)\right)_{\substack{1\le i\le m\\1\le j\le m}}.$$

Now, we can write the problem (P_4) in the matrix form as follows:

$$(8)\qquad \begin{cases} B_m\frac{\partial g_m}{\partial t}(t)+aA_mg_m-bB_mg_m+G(g)=f_m+aC_mg_m-aD_mg_m \\ g_m(0)=(\alpha_{im})_{1\le i\le m} \end{cases}.$$

By using the Carathéodory's existence theorem, which can be used for ordinary differential equations, we can conclude that there exists t_m that depends only on $|\alpha_{im}|$ such that the problem (8) admits a unique local solution $g_m(t)\in C[0,t_m]$ in the interval $[0,t_m]$, for which $g_m'(t)\in L^2[0,T]$.

In the following content, we aim to study the a priori estimates for the approximate solution $u_m(x,t)$ obtained in the previous discussion.

Theorem 1. *For all $m\in\mathbb{N}^*$ and $\dfrac{p}{2}\ge b$, we suppose that $\varphi\in H^1(\Omega)\cap L^{p+1}(\Omega)$ and $f\in L^2\left(0,T,\ L^2(\Omega)\right)$. Then, the problem (P_1) admits a solution u such that*

$$u\in L^2(0,T;\ V)\cap L^\infty\left(0,T;\ H^1(\Omega)\right),$$

and

$$u' \in L^2(0,T;\, L^2(\Omega)).$$

Proof. To prove this result, we multiply both sides of equation (P_2) by $g_{im}(t)$, and then summing the result with respect to k to obtain

$$\sum_{k=1}^{m}((u_m)_t\,,w_k)\,g_{km}(t) - a\sum_{k=1}^{m}(\Delta u_m, w_k)\,g_{km}(t) + \sum_{k=1}^{m}(u_m^p - bu_m, w_k)\,g_{km}(t) = \sum_{k=1}^{m}(f, w_k)\,g_{km}(t).$$

Then, we find

$$((u_m)_t\,,u_m) - a\,(\Delta u_m, u_m) + (u_m^p - bu_m, u_m) = (f, u_m)\,,$$

which implies

$$\begin{aligned}((u_m)_t\,,u_m) + a\left(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial x}\right) + (u_m^p - bu_m, u_m)& \\ = (f,u_m) + a\frac{\partial u_m}{\partial x}(l,t)u_m(l,t) - a\frac{\partial u_m}{\partial x}(0,t)u_m(0,t).&\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|u_m\|^2_{L^2(\Omega)} + a\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} + \|u_m\|^{p+1}_{L^{p+1}(\Omega)}& \\ = (f,u_m) + b\|u_m\|^2_{L^2(\Omega)} + a\frac{\partial u_m}{\partial x}(l,t)u_m(l,t) - a\frac{\partial u_m}{\partial x}(0,t)u_m(0,t).&\end{aligned}$$

As a consequence, integrating the above equality from 0 to t gives

$$\begin{aligned}&\frac{1}{2}\int_0^t \frac{\partial}{\partial t}\|u_m\|^2_{L^2(\Omega)} + a\int_0^t \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} + \int_0^t \|u_m\|^{p+1}_{L^{p+1}(\Omega)} \\ &= \int_0^t (f,u_m) + b\int_0^t \|u_m\|^2_{L^2(\Omega)} + a\int_0^t \frac{\partial u_m}{\partial x}(l,\tau)u_m(l,\tau)d\tau - a\int_0^t \frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau.\end{aligned}$$

It consequently comes

$$\begin{aligned}\frac{1}{2}\|u_m\|^2_{L^2(\Omega)} + a\int_0^t \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} d\tau + \int_0^t \|u_m\|^{p+1}_{L^{p+1}(\Omega)} d\tau = \int_0^t (f,u_m)d\tau + b\int_0^t \|u_m\|^2_{L^2(\Omega)} d\tau& \\ + \frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)} + a\int_0^t \frac{\partial u_m}{\partial x}(l,\tau)\cdot u_m(l,\tau)d\tau - a\int_0^t \frac{\partial u_m}{\partial x}(0,\tau)u_m(0,\tau)d\tau&\end{aligned}$$

By using the Cauchy inequality, we obtain

$$\int\limits_0^t \left(\frac{\partial u_m}{\partial x}(l,\tau)\cdot u_m(l,\tau)\right)d\tau < \frac{1}{2}\int_0^t u_m^2(l,\tau)\,d\tau + \frac{1}{2}\int_0^t u_{mx}^2(l,\tau)\,d\tau.$$

Now, to obtain the desired estimate, we need the inequality

$$u^2(l,t) \leqslant 2\int_x^l u_x^2 dx + 2u^2,$$

which easily followed from the equality

$$u(l,t) = \int_x^l u_x(x,t)\,dx + u(x,t).$$

Also, by the second type of the integral condition, we can have

$$\begin{aligned}\int_0^t u_x(l,\tau)\,u(l,\tau)\,d\tau &\leqslant \frac{1}{8}\int_0^t u^2(l,\tau)\,d\tau + 2\int_0^t u_x^2(l,\tau)\,d\tau \\ &\leqslant \frac{1}{8}\int_0^t \left[2\int_x^l u_x^2 dx + 2u^2\right]d\tau + 2\int_0^t \left[\int_0^l k(x,t)\,u(x,t)\,dx\right]^2 d\tau.\end{aligned}$$

So, by using Holder's inequality, we get

$$\int_0^t u_x(l,t)\,u(l,t)\,dt \leqslant \frac{1}{4}\int_Q u_x^2 dxdt + \frac{1}{4}\int_0^t u^2 dt + 2K\int_Q u^2 dxdt,$$

where the constant $K = \max\limits_{\tau\in[0,T]} \int_\Omega k_i^2(x,t)\,dxdt$, for $i = 1,2$. Thus, by using the same method used previously, we get

$$\int_0^t u_x(0,t)\,u(0,t)\,dt \leqslant \frac{1}{4}\int_Q u_x^2 dxdt + \frac{1}{4}\int_0^t u^2 dt + 2K\int_Q u^2 dxdt.$$

So, we find

$$\begin{aligned}&\frac{1}{2}\|u_m\|_{L^2(\Omega)}^2 + a\int\limits_0^t \left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + \int\limits_0^t \|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\ &\qquad\le \frac{1}{2}\int\limits_0^t \|f\|_{L^2(\Omega)}^2 + \frac{1}{2}\int\limits_0^t \|u_m\|_{L^2(\Omega)}^2 + b\int\limits_0^t \|u_m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\varphi_m\|_{L^2(\Omega)}^2 \\ &\qquad + \frac{a}{2}\left[\int\limits_0^t \left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + \frac{1}{2l}\int\limits_0^t \|u_m\|_{L^2(\Omega)}^2 + 4K\int\limits_0^t \|u_m\|_{L^2(\Omega)}^2\right].\end{aligned}$$

Also, we obtain

$$\|u_m\|^2_{L^2(\Omega)} + a\int_0^t \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} + 2\int_0^t \|u_m\|^{p+1}_{L^{p+1}(\Omega)}$$
$$\leq \int_0^t \|f\|^2_{L^2(\Omega)} + \left(2b+1+\frac{a}{2l}+4aK\right)\int_0^t \|u_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)}$$

Using Gronwall's Lemma yields

$$\|u_m\|^2_{L^\infty(0,T,\,L^2(\Omega))} + \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^{p+1}_{L^{p+1}(0,T,\,L^{p+1}(\Omega))}$$
$$\leq \frac{\exp\left(\left(2b+1+\frac{a}{2l}+4aK\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)} + \|\varphi_m\|^2_{L^2(\Omega)}\right).$$

Now, by considering

$$C_T = \frac{\exp\left(\left(2b+1+\frac{a}{2l}+4aK\right)T\right)}{\min\{1,a\}}\left(\|f\|^2_{L^2(Q)} + \|\varphi_m\|^2_{L^2(\Omega)}\right), \tag{9}$$

we get

$$\|u_m\|^2_{L^\infty(0,T,\,L^2(\Omega))} + \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \|u_m\|^{p+1}_{L^{p+1}(0,T,\,L^{p+1}(\Omega))} \leq C_T, \tag{10}$$

where C_T is a positive constant depending only on $\int_0^T \|f(\tau)\|^2_{L^2(\Omega)}$, $\|\varphi_m\|^2_{L^2(\Omega)}$ and T. It follows from (10) that

$$\|u_m(t)\|^2_{L^2(\Omega)} \leq C_T.$$

This implies that the solution to the initial value problem for the system of ODE (8) can be extended to $[0,T]$. Consequently, we have the following uniform a priori estimates:

$$\begin{cases} u_m \text{ uniformly bounded in } L^\infty(0,T;\, L^2(\Omega)\ ) \\ u_m \text{ uniformly bounded in } L^2(0,T;\, H^1(\Omega)) \\ u_m \text{ uniformly bounded in } L^{p+1}\left(0,T;\, L^{p+1}(\Omega)\right) \end{cases}.$$

Now we would like to get more a priori estimates. In doing so, we use the same formulation variational by $g^{'}_{km}(t)$, and then sum the result over k to get

$$\sum_{k=1}^m ((u_m)_t, w_k)\, g^{'}_{km}(t) + \sum_{k=1}^m a(\Delta u_m, w_k)\, g'_{km}(t) + \sum_{k=1}^m (u_m^p - bu_m, w_k)\, g^{'}_{km}(t) = \sum_{k=1}^m (f, w_k)\, g^{'}_{km}(t). \tag{11}$$

So, it comes

$$\left((u_m)_t\,,(u_m)_t\right) - a\left(\frac{\partial^2 u_m}{\partial x^2},(u_m)_t\right) + \left(u_m^p - bu_m,(u_m)_t\right) = \left(f,(u_m)_t\right).$$

By integration the above equality over $(0,t)$, we obtain

$$\int_0^t \left((u_m)_t\,,(u_m)_t\right) - a\int_0^t \left(\frac{\partial^2 u_m}{\partial x^2},(u_m)_t\right) + \int_0^t \left(u_m^p - bu_m,(u_m)_t\right) = \int_0^t \left(f,(u_m)_t\right).$$

Then, we have

$$\begin{aligned}
&\|(u_m)_t\|_{L^2(Q)}^2 + \frac{a}{2}\|(u_m)_x\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\
&\qquad \le \int_0^t \left(f + bu_m,(u_m)_t\right) + \frac{a}{2}\|(\varphi_m)_x\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} \\
&\qquad + a\,(u_m)_x(l,t)\cdot(u_m)_t(l,t)\,dx + a\,(u_m)_x(0,t)\cdot(u_m)_t(0,t)\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|(u_m)_t\|_{L^2(Q)}^2 + \frac{a}{2}\|(u_m)_x\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\
&\qquad \le \frac{1}{2}\|(u_m)_t\|_{L^2(Q)}^2 + \frac{1}{2}\|f\|_{L^2(Q)}^2 + \frac{b}{2}\|u_m\|_{L^2(\Omega)}^2 + \frac{a}{2}\|(\varphi_m)_x\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{1}{p+1}\|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} - \frac{b}{2}\|\varphi_m\|_{L^2(\Omega)}^2 + a\frac{\varepsilon}{2}K^2\|u_m\|_{L^\infty\left(0,T,\,L^2(\Omega)\right)}^2 \\
&\qquad + a\frac{l}{\varepsilon}\|(u_m)_x\|_{L^2(\Omega)}^2 + a\frac{l}{\varepsilon}\|(\varphi_m)_x\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
&\frac{1}{2}\|(u_m)_t\|_{L^2(Q)}^2 + \left(\frac{a}{2} - a\frac{l}{\varepsilon}\right)\|(u_m)_x\|_{L^2(\Omega)}^2 + \frac{1}{p+1}\|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\
&\qquad \le \frac{1}{2}\|f\|_{L^2(Q)}^2 + \left(\frac{a}{2} + a\frac{l}{\varepsilon}\right)\|(\varphi_m)_x\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{1}{p+1}\|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} - \frac{b}{2}\|\varphi_m\|_{L^2(\Omega)}^2 + \left(a\frac{\varepsilon}{2}K^2 + \frac{b}{2}\right)C_T.
\end{aligned}$$

As a result, we get

$$\begin{aligned}
&\min\left\{\frac{1}{2},\frac{a}{T}\left(\frac{1}{2} - \frac{l}{\varepsilon}\right),\frac{l}{(p+1)\,T}\right\}\left[\|(u_m)_t\|_{L^2(Q)}^2 + \|(u_m)_x\|_{L^2(Q)}^2 + \int_0^t \|u_m\|_{L^{p+1}(\Omega)}^{p+1}\right] \\
&\quad \le \max\left\{\frac{1}{2},\left(\frac{a}{2} + a\frac{l}{\varepsilon}\right),\frac{1}{p+1},-\frac{b}{2}\right\}\left[\begin{array}{c} \|f\|_{L^2(Q)}^2 + \left(a\frac{\varepsilon}{2}K^2 + \frac{b}{2}\right)C_T \\ + \|(\varphi_m)_x\|_{L^2(\Omega)}^2 + \|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} + \|\varphi_m\|_{L^2(\Omega)}^2 \end{array}\right].
\end{aligned}$$

Now, by putting

$$C=\frac{\max\left\{\frac{1}{2},\left(\frac{a}{2}+a\frac{l}{\varepsilon}\right),\frac{1}{p+1},-\frac{b}{2}\right\}}{\min\left\{\frac{1}{2},\frac{a}{T}\left(\frac{1}{2}-\frac{l}{\varepsilon}\right),\frac{l}{(p+1)T}\right\}},$$

we obtain

$$\begin{aligned}&\|(u_m)_t\|^2_{L^2(Q)}+\|(u_m)_x\|^2_{L^2(Q)}+\int_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}\\&\qquad\leq C\left[\|f\|^2_{L^2(Q)}+\frac{bl}{2}C_T+\|(\varphi_m)_x\|^2_{L^2(\Omega)}+\|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)}+\|\varphi_m\|^2_{L^2(\Omega)}\right]\end{aligned}$$

This implies

$$\|(u_m)_t\|^2_{L^2(Q)}\leq C. \tag{12}$$

Then, we get the following further priori estimates:

$$\left\{\begin{array}{c}u_m \text{ uniformly bounded in } L^{p+1}(0,T;\,L^{P+1}(\Omega))\\ u_m \text{ uniformly bounded in } L^2(0,T;\,H^1(\Omega))\\ (u_m)_t \text{ uniformly bounded in } L^2\left(0,T;\,L^2(\Omega)\right)\end{array}\right. . \tag{13}$$

Thus, by Lemma 1.2, there is a subsequence of u_m, still denoted by u_m such that

$$\left\{\begin{array}{c}u_m\longrightarrow u \text{ weakly in } L^{p+1}(0,T;\,L^{P+1}(\Omega))\\ u_m\longrightarrow u \text{ weakly in } L^2(0,T;\,H^1(\Omega))\\ u_m\longrightarrow u \text{ weakly in } L^2\left(0,T;\,L^2(\Omega)\right)\end{array}\right. . \tag{14}$$

We deduce from Lemma 1.2 that there are subsequences denoted by (u_{m_k}) and $\left(\frac{\partial u_{m_k}}{\partial t}\right)$ of (u_m) and $(u_m)_t$, respectively, such that

$$u_{m_k}\rightharpoonup u \qquad \text{in } L^2(0,T;\,H^1(\Omega))\ , \tag{15}$$

and

$$\frac{\partial u_{m_k}}{\partial t}\rightharpoonup w \qquad \text{in } L^2\left(0,T;\,L^2(\Omega)\right)\ . \tag{16}$$

We know, according to Relikh-Kondrachoff's theorem, that the injection of $H^1(Q)$ into $L^2(Q)$ is compact, and like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence that converges strongly in $L^2(Q)$. So, we have

$$u_{m_k}\longrightarrow u \qquad \text{in } L^2(Q)\ . \tag{17}$$

On the other hand, from lemma (1.3) there is a subsequence of $(u_{m_k})_k$ is still denoted by u_{m_k} converges almost everywhere to u, such that

$$u_{m_k} \longrightarrow u \qquad \text{almost everywhere } Q \ . \tag{18}$$

By Lemma 1.3, there is a subsequence of u_m, still denoted by u_m, such that u_m converges almost everywhere to u in $Q_T = \Omega \times [0,T]$. It turns out that

$$(u_m)^p \ \text{almost everywhere converges to } u^p \text{in } Q_T.$$

On the other hand, (13) implies that $(u_m)^p$ is uniformly bounded in $L^{\frac{p+1}{p}}(Q_T)$. Therefore, we infer from Lemma 1.4 that

$$u_m^p \rightharpoonup u^p \qquad \text{weakly in } L^{\frac{p+1}{p}}\left(0,T\ ,L^{\frac{p+1}{p}}(\Omega)\right).$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$. For this purpose, it suffices to prove

$$u(t) = \varphi + \int_0^t w(\tau) d\tau \tag{19}$$

as

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T\ ; L^2(\Omega)) \ .$$

Actually, the proof of (19) is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0,T\ ; L^2(\Omega)) \ ,$$

which means

$$\lim\,(u_{m_k} - \varphi - \chi, v)_{L^2(0,T\ ; L^2(\Omega))} = 0,\ \forall v \in L^2(0,T\ ; L^2(\Omega)),$$

as

$$\chi(t) = \int_0^t w(\tau) d\tau.$$

Now, by using the equality

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau, \text{ for all } t \in [0,T],$$

we obtain, from $u_{m_k} \in L^2(0,T;V_{m_k})$ and $(u_{m_k})_t \in L^2(0,T;V_{m_k})$, that

$$\begin{aligned}
&\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))} \\
&= \left(u_{m_k} - \varphi_{m_k} - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))} \\
&= \left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau)\right) d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))},
\end{aligned}$$

for all $t \in [0,T]$. By virtue of **(ii)** of Lemma 1.6, it comes

$$\begin{aligned}
\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))} &= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau \\
&\quad + (\varphi_{m_k} - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))},
\end{aligned}$$

for all $t \in [0,T]$. On the one hand, we have

$$\lim_{k\longrightarrow\infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau = 0, \text{ for } t \in [0,T]. \tag{20}$$

Also, we have

$$\lim_{k\longrightarrow\infty} (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0. \tag{21}$$

So, we get

$$\lim_{k\longrightarrow\infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0, \ \forall v \in L^2\left(0,T;L^2(\Omega)\right).$$

Finally, from (20) and (21), we get

$$\lim_{k\longrightarrow\infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0, \ \forall v \in L^2\left(0,T;L^2(\Omega)\right).$$

Now, by passing the limit into (P_2) and by considering that each term on the left side of (P_2) is weakly convergent in $L^{\frac{p+1}{p}}(\Omega)$, we obtain that the following assertion holds in $L^{\frac{p+1}{p}}(\Omega)$:

$$((u_m(t))_t, w_k) + a(u_m(t), w_k) + (u_m^p(t) - bu_m(t), w_k) = (f(t), w_k), \ \forall k = \overline{1,m}. \tag{22}$$

Since $\{w_j, \ \ j \in \mathbb{N}\}$ is a base in $L^{\frac{p+1}{p}}(\Omega)$, we infer from (22) that the following assertion

$$u' - a\Delta u + u^p - bu = f \tag{23}$$

holds also in $L^{\frac{p+1}{p}}\left(0,T\ ,L^{\frac{p+1}{p}}(\Omega)\right)$. Since all u', Δu, and f belong to $L^2\left(0,T;L^2(\Omega)\right)$, u^p also belongs to $L^2\left(0,T;L^2(\Omega)\right)$, then (23) also holds in $L^2\left(0,T;L^2(\Omega)\right)$. Thus, we have the desired result. □

4. The Uniqueness of the Solution

Herein, we will study the uniqueness of the solution only in the case where p is odd. For this purpose, we present the subsequent theorem.

Theorem 2. *Suppose that $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and $f \in L^2\left(0,T;L^2(\Omega)\right)$. Then problem ($P_1$) admits a unique solution u such that*

$$u \in L^2\left(0,T,\ H^1(\Omega)\right) \cap\ L^{p+1}(0,T;\ L^{P+1}(\Omega))$$

and

$$u' \in L^2(0,T,\ L^2(\Omega)).$$

Proof. Suppose first that p is odd. Now, by multiplying the equation of the problem (P_1) by the following multiplier Mu for which

$$Mu = u,$$

and by integrating the result over the domain $\Omega = (0,l)$, we obtain

$$\begin{aligned}\int_\Omega [u_t - a\Delta u + u^p - bu].Mudx &= \int_\Omega [u_t(x,t) - a\Delta u(x,t) + u^p - bu]\cdot udx \\ &= \int_\Omega u_t(x,t)udx - a\int_\Omega \Delta u\cdot udx + \int_\Omega u^p udx - b\int_\Omega u^2dx \\ &= \int_\Omega f(x,t)udx,\end{aligned}$$

where u_x and u_t denotes the partial derivative of u with respect to x and t, respectively. This yields

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} + a\int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dx + \int_\Omega u^{p+1}dx - b\int_\Omega u^2dx = \int_\Omega fudx.$$

Then, by integrating the above equality over $(0,\tau)$, where $\tau\in(0,T)$, we get

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)}+a\int\limits_{Q_\tau}\left(\frac{\partial u}{\partial x}\right)^2 dxdt+\int\limits_{Q_\tau}u^{p+1}dxdt$$
$$\leq\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\left(\frac{\varepsilon}{2}+b\right)\|u\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}$$
$$+a\varepsilon\int\limits_0^t\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)}+\frac{a\varepsilon}{l}\int\limits_0^t\|u\|^2_{L^2(\Omega)}+\frac{aK}{2\varepsilon}\int\limits_0^t\|u\|^2_{L^2(\Omega)}.$$

Therefore, we have

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)}+(a-a\varepsilon)\int\limits_0^t\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)}d\tau+\left(1-\left(\frac{1}{2}+\frac{\varepsilon}{l}+\frac{K}{2\varepsilon}+b\right)\frac{2}{p+1}\right)\int\limits_0^t\|u\|^{p+1}_{L^{p+1}(\Omega)}d\tau$$
$$\leq\frac{1}{2}\int\limits_0^T\|f(\tau)\|^2_{L^2(\Omega)}d\tau+\frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}+\left(\frac{1}{2}+\frac{\varepsilon}{l}+\frac{K}{2\varepsilon}+b\right)\frac{(p+1)\,mes(\Omega)T}{p-1}.$$

Consequently, by assuming that

$$C_T=\frac{\frac{1}{2}\int\limits_0^T\|f(\tau)\|^2_{L^2(\Omega)}d\tau+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\left(\frac{1}{2}+\frac{\varepsilon}{l}+\frac{K}{2\varepsilon}+b\right)\frac{(p+1)mes(\Omega)T}{p-1}}{\min\left(\frac{1}{2},(a-a\varepsilon),\left(1-\left(\frac{1}{2}+\frac{\varepsilon}{l}+\frac{K}{2\varepsilon}+b\right)\frac{2}{p+1}\right)\right)},\tag{24}$$

we obtain

$$\|u_m(t)\|^2_{L^2(\Omega)}+\int\limits_0^t\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}d\tau+\int\limits_0^t\|u_m\|^{p+1}_{L^{p+1}(\Omega)}d\tau\leq C_T.\tag{25}$$

Now, we put

$$\|u\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)}+\|u\|^2_{L^2\left(0,T;L^2(\Omega)\right)}+\|u\|^{p+1}_{L^{p+1}\left(0,T\,;L^{p+1}(\Omega)\right)}\equiv\|u\|_B.$$

Also, we let u_1 and u_2 be two solutions to the problem (P_1), i.e.,

$$\begin{cases}Lu_1=\mathscr{F}\\ Lu_2=\mathscr{F}\end{cases}\Longrightarrow Lu_1-Lu_2=0,$$

where L is the differential operator of the main semilinear problem. This, immediately, implies

$$L(u_1-u_2)=0.$$

Therefore, we get

$$\|u_1-u_2\|_B\leq c\,\|0\|_F=0,$$

which directly gives $u_1 = u_2$. □

5. Finite-Time Blow-up Solution

To investigate the finite-time blow-up solution to the desired problem, we assume that

$$\Pi(t) = \int_\Omega u^2 dx.$$

Now, by integrating the above equation over Ω with null source, we get

$$u_t - a\Delta u - bu = u^p$$

$$\int_\Omega u_t u dx - a\int_\Omega \Delta u.u dx - \int_\Omega bu^2 dx = \int_\Omega u^{p+1} dx$$

By using Green's formula and the Pointcarré inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - \int_\Omega bu^2 dx \le a\left[u_x(l,t)u(l,t) - u_x(0,t)u(0,t)\right] + \int_\Omega u^{p+1} dx$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - \int_\Omega bu^2 dx \le a\max_{x,t\in Q}(k_{1,2})\left(\int_\Omega u dx\right)\left[\int_\Omega u_x\right] + \int_\Omega u^{p+1} dx$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - b\int_\Omega u^2 dx \le a\max_{x,t\in Q}(k_{1,2})\left[\frac{1}{2}\left(\int_\Omega u dx\right)^2 + \frac{1}{2}\left(\int_\Omega u_x\right)^2\right] + \int_\Omega u^{p+1} dx.$$

This implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega u_x^2 - b\int_\Omega u^2 dx \le a\max_{x,t\in Q}(k_{1,2})\left[\frac{l}{2}\int_\Omega u^2 + \frac{l}{2}\int_\Omega u_x^2\right] + \int_\Omega u^{p+1} dx.$$

As a result, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \left(a - \frac{al}{2}\max_{x,t\in Q}(k_{1,2})\right)\int_\Omega u_x^2 - \left(b + \frac{al\max_{x,t\in Q}(k_{1,2})}{2}\right)\int_\Omega u^2 dx &\le \int_\Omega u^{p+1} dx \\ \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx - \left(b + \frac{al\max_{x,t\in Q}(k_{1,2})}{2}\right)\int_\Omega u^2 dx &\le \int_\Omega u^{p+1} dx \end{aligned}$$

Consequently, we apply the Jensen inequality to obtain

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \left(b + \frac{al\max_{x,t\in Q}(k_{1,2})}{2}\right)\int_\Omega u^2 dx &\ge \frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}\left(\int_\Omega u^2\right)^{\frac{p+1}{2}} \\ -\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \left(b + \frac{al\max_{x,t\in Q}(k_{1,2})}{2}\right)\int_\Omega u^2 dx &\ge \frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}\left(\int_\Omega u^2\right)^{\frac{p+1}{2}}, \end{aligned}$$

which implies

$$-\frac{1}{2}\Pi'(t)+\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right)\Pi(t)=\frac{1}{(mes(\Omega))^{\frac{q+1}{2}-1}}(\Pi(t))^{\frac{q+1}{2}}.$$

Therefore, we have

$$\Pi'(t)-2\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right)\Pi(t)+\frac{2}{(mes(\Omega))^{\frac{q+1}{2}-1}}(\Pi(t))^{\frac{p+1}{2}}=0. \tag{26}$$

To solve this equation, we use the following change of variable:

$$v=\Pi^{1-q}, \tag{27}$$

where

$$q=\frac{p+1}{2}.$$

By replacing (27) in (26), we find

$$\frac{1}{q-1}v'v^{\frac{q}{1-q}}-2\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right)v^{\frac{1}{1-q}}+\frac{c_1}{(mes(\Omega))^{q-1}}v^{\frac{q}{1-q}}=0 \tag{28}$$

Multiplying equation (28) by $v^{\frac{-q}{1-q}}$ gives

$$v'-2(q-1)\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right)v+(q-1)\frac{1}{(mes(\Omega))^{q-1}}=0. \tag{29}$$

Now, we will solve the following homogeneous equation:

$$v'+2(q-1)\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right)v=0.$$

To do so, we put

$$K=2(q-1)\left(b+\frac{al\max\limits_{x,t\in Q}(k_{1,2})}{2}\right).$$

This implies that

$$v'+Kv=0$$

and

$$v_1(t)=Be^{-Kt}.$$

Now, we move on to solving the nonhomogeneous equation (29) by the method of constant variation. To do so, we set

$$v_2(t) = B(t)e^{-Kt},$$

and so, we get

$$v_2(t) = \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}.$$

Then, we obtain

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= Be^{-Kt} + \frac{1}{K}(q-1)\frac{c_1}{(mes(\Omega))^{q-1}}, \end{aligned}$$

which gives

$$\Pi(t) = \left(Be^{-Kt} + \frac{1}{K}(q-1)\frac{c_1}{(mes(\Omega))^{q-1}}\right)^{\frac{1}{1-q}}.$$

Now, for $t = 0$, we get

$$\Pi(0) = \left(B + \frac{1}{K}(q-1)\frac{c_1}{(mes(\Omega))^{q-1}}\right)^{\frac{1}{1-q}}.$$

This means that

$$B = (\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{c_1}{(mes(\Omega))^{q-1}}.$$

Finally, we can get

$$\Pi(t) = \left(\left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)e^{-Kt} + \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)^{\frac{1}{1-q}},$$

and hence, we have

$$\Pi(t) = \left(\frac{1}{\left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)e^{-Kt} + \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}}\right)^{\frac{1}{q-1}}.$$

Now, as $\dfrac{1}{q-1} > 0$, we obtain

$$\Pi \to \infty \text{ if } \left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}\right)e^{-Kt} + \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}} \to 0.$$

Therefore, we get

$$T = \frac{1}{K} \ln \left(\frac{\frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}}}{\left((\Pi(0))^{1-q} - \frac{1}{K}(q-1)\frac{1}{(mes(\Omega))^{q-1}} \right)} \right).$$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] H. Qawaqneh, Fractional analytic solutions and fixed point results with some applications, Adv. Fixed Point Theory, 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[2] I.M. Batiha, T.E. Oussaeif, A. Benguesmia, et al. A study of a superlinear parabolic Dirichlet problem with unknown coefficient, Int. J. Innov. Comput. Inform. Control. 20 (2024), 541–556.

[3] A. Fatma, I. Batiha, R. Imad, et al. Solvability and weak controllability of fractional degenerate singular problem, J. Robot. Control. 5 (2024), 542–550.

[4] I.M. Batiha, A. Benguesmia, T.E. Oussaeif, et al. Study of a superlinear problem for a time fractional parabolic equation under integral over-determination condition, IAENG Int. J. Appl. Math. 54 (2024), 187–193.

[5] Z. Chebana, T.E. Oussaeif, S. Dehilis, et al. On nonlinear Neumann integral condition for a semilinear heat problem with blow up simulation, Palestine J. Math. 12 (2023), 378–394.

[6] A. Benguesmia, I.M. Batiha, T.E. Oussaeif, et al. Inverse problem of a semilinear parabolic equation with an integral overdetermination condition, Nonlinear Dyn. Syst. Theory. 23 (2023), 249–260.

[7] I.M. Batiha, S.A. Njadat, R.M. Batyha, et al. Design Fractional-order PID controllers for single-joint robot arm model, Int. J. Adv. Soft Comput. Appl. 14 (2022), 97–114. https://doi.org/10.15849/ijasca.220720.07.

[8] I.M. Batiha, J. Oudetallah, A. Ouannas, et al. Tuning the fractional-order PID-controller for blood glucose level of diabetic patients, Int. J. Adv. Soft Comput. Appl. 13 (2021), 1–10.

[9] Z. Wu, J. Yin, H. Li, et al. Nonlinear diffusion equations, World Scientific, Singapore, 2001.

[10] J.R. Cannon, The solution of the heat equation anubject to the specification of energy, Quart. Appl. Math. 21 (1963), 155–160.

[11] A.A. Samarskii, Some problems of the theory of differential equations, Differ. Uravn. 16 (1980), 1925–1935. https://www.mathnet.ru/eng/de4116.

[12] S. Dhelis, A. Bouziani, T.E. Oussaeif, Study of solution for a parabolic integrodifferential equation with the second kind integral condition, Int. J. Anal. Appl. 16 (2018), 569–593. https://doi.org/10.28924/2291-8639-16-2018-569.

[13] T.E. Oussaeif, A. Bouziani, Inverse problem of a hyperbolic equation with an integral overdetermination condition, Electron. J. Differ. Equ. 2016 (2016), 138.

[14] A. Bouziani, T.E. Oussaeif, B.A. Leila, A mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator, J. Math. 2013 (2013), 457631. https://doi.org/10.1155/2013/457631.

[15] T.E. Oussaeif, A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, Electron. J. Differ. Equ. 2014 (2014), 179.

[16] T.E. Oussaeif, A. Bouziani, Mixed problem with an integral two-space-variables condition for a class of hyperbolic equations, Int. J. Anal. 2013 (2013), 957163. https://doi.org/10.1155/2013/957163.

[17] T.E. Oussaeif, A. Bouziani, Mixed problem with an integral two-space-variables condition for a parabolic equation, Int. J. Evol. Equ. 9 (2014), 181–198.

[18] N. Abdelhalim, A. Ouannas, I. Rezzoug, et al. A study of a high-order time-fractional partial differential equation with purely integral boundary conditions, Fract. Differ. Calc. (2023), 199–210. https://doi.org/10.7153/fdc-2023-13-13.

[19] T.E. Oussaeif, A. Bouziani, A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition, Chaos Solitons Fractals. 103 (2017), 79–89. https://doi.org/10.1016/j.chaos.2017.05.035.

[20] Y. Boukhatem, B. Benabderrahmane, A. Rahmoune, Méthode de Faedo-Galerkin pour un probléme aux limites non linéaire, An. Univ. Oradea, Fasc. Mat. XVI (2009), 167–181.

[21] S. Zheng, Nonlinear evolution equations, Chapman and Hall/CRC, London, 2004. https://doi.org/10.1201/9780203492222.

[22] T.E. Oussaeif, A. Bouziani, Solvability of nonlinear goursat type problem for hyperbolic equation with integral condition, Khayyam J. Math. 4 (2018), 198–213. https://doi.org/10.22034/kjm.2018.65161.

[23] R. Imad, T.E. Oussaeif, A. Bouziani, Solvability of a solution and controllability for nonlinear fractional differential equations, Bull. Inst. Math. 15 (2020), 237–249.

[24] T.E. Oussaeif, A. Bouziani, Solvability of nonlinear viscosity equation with a boundary integral condition, J. Nonlinear Evol. Equ. Appl. 2015 (2015), 31–45.

[25] B. Sihem, T.E. Oussaeif, A. Bouziani, Galerkin finite element method for a semi-linear parabolic equation with purely integral conditions, Bol. Soc. Paran. Mat. 40 (2022), 1–15.