Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:49
https://doi.org/10.28919/afpt/8710
ISSN: 1927-6303

EXISTENCE OF COINCIDENCE AND COMMON FIXED POINTS FOR EXTENDED RECTANGULAR B-METRIC SPACES

PHEERACHATE BUNPATCHARACHAROEN, PAIWAN WONGSASINCHAI*, ANANTACHAI PADCHAROEN

Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand

Abstract. In this paper, we combine the (α,β)-admissible mappings and the simulation function to produce a generalized version of the Suzuki generalized rational type $\mathbb{Z}$-contraction mapping. This concept is also used in the definition of extended rectangular b-metric spaces to obtain several popular fixed point theorems.

Keywords: (α,β)-admissible mappings; Suzuki-type; extended rectangular b-metric spaces.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Bakhtin [1] explicitly presented and investigated the b-metric space, an intriguing generalized metric space, in 1989. Since then, other researchers have expanded and developed fixed point theorems in b-metric spaces. Recent work on fixed point theorems in b-metric spaces can be found in [2, 3, 4, 5].

A novel class of contractive type mappings called α-ψ contractive type mapping was presented by Samet et al. [6] expanded upon and generalized the fixed point conclusions that have

*Corresponding author
E-mail address: paiwan.w@rbru.ac.th
Received June 15, 2024

already been published in the literature, particularly the Banach contraction principle. Furthermore, different fixed point theorems for this generalized class of contractive mappings were derived by Karapinar and Samet [7] by generalizing the α-ψ-contractive type mappings.

Much research has been done on common fixed points of mappings that satisfy specific contractive requirements. In this study, some coincidence theorems and common fixed point theorems for the $\mathbb{Z}_{(\alpha,\beta)}$-contractive pair of mappings are obtained.

2. PRELIMINARIES

Throughout this paper, we will refer to $\mathbb{R}$ as the set of all real numbers and $\mathbb{N}$ as the set of all non-negative integers.

Definition 2.1. [1] *Let Ω be a nonempty set and the mapping $d:\Omega\times\Omega\to[0,+\infty)$ satisfies*:

(1) $d(\kappa,\varpi)=0$*, if and only if* $\kappa=\varpi$ *for all* $\kappa,\varpi\in\Omega$;

(2) $d(\kappa,\varpi)=d(\varpi,\kappa)$ *for all* $\kappa,\varpi\in\Omega$;

(3) *there exist a real number* $s\geq 1$ *such that* $d(\kappa,\varpi)\leq s[d(\kappa,u)+d(u,\varpi)]$ *for all* $\kappa,\varpi,u\in\Omega$.

Then d is called a b-metric on Ω and (Ω,d) is called a b-metric space with coefficient s.

Kamran et al. [8] proposed a binary function in 2017, which was used to introduce a new metric-type space.

Definition 2.2. [8] *Let Ω be a nonempty set, $\theta:\Omega\times\Omega\to[1,+\infty)$ and let $d:\Omega\times\Omega\to[0,\infty)$ satisfies*:

(1) $d(\kappa,\varpi)=0$*, if and only if* $\kappa=\varpi$ *for all* $\kappa,\varpi\in\Omega$;

(2) $d(\kappa,\varpi)=d(\varpi,\kappa)$ *for all* $\kappa,\varpi\in\Omega$;

(3) $d(\kappa,\varpi)\leq\theta(\kappa,\varpi)[d(\kappa,u)+d(u,\varpi)]$ *for all* $\kappa,\varpi,u\in\Omega$.

Then d is called an extended b-metric on Ω and (Ω,d) is called an extended b-metric space with θ.

Branciari [9] provided a generalized metric in 2000, replacing the triangle inequality with quadrilateral inequality.

Definition 2.3. [9] *Let Ω be a nonempty set and let $d : \Omega \times \Omega \to [0, +\infty]$ be a mapping such that for all $\kappa, \varpi \in \Omega$:*

(1) $d(\kappa, \varpi) = 0$*, if and only if* $\kappa = \varpi$;

(2) $d(\kappa, \varpi) = d(\varpi, \kappa)$;

(3) $d(\kappa, \varpi) \leq d(\kappa, u) + d(u, v) + d(v, \varpi)$ *for all distinct points* $u, v \in \Omega \setminus \{\kappa, \varpi\}$.

Then d is called a rectangular metric on Ω and (Ω, d) is called a rectangular metric space.

George et al. [10] introduced the rectangular b-metric, a hybrid of the b-metric and the rectangular metric, in 2015.

Definition 2.4. [10] *Let Ω be a nonempty set, $s \geq 1$ be a given real number, and let $d : \Omega \times \Omega \to [0, +\infty]$ be a mapping such that for all $\kappa \in \Omega$:*

(1) $d(\kappa, \varpi) = 0$*, if and only if* $\kappa = \varpi$;

(2) $d(\kappa, \varpi) = d(\varpi, \kappa)$;

(3) $d(\kappa, \varpi) \leq s[d(\kappa, u) + d(u, v) + d(v, \varpi)]$ *for all distinct points* $u, v \in \Omega \setminus \{\kappa, \varpi\}$.

Then d is called a rectangular metric on Ω and (Ω, d) is called a rectangular b-metric space.

Asim et al. [11] presented a more generalized metric space called extended rectangular b-metric space.

Definition 2.5. [11] *Let Ω be a nonempty set, $\varphi : \Omega \times \Omega \to [1, +\infty)$ and let $d : \Omega \times \Omega \to [0, +\infty]$ be a mapping such that for all $\kappa, \varpi \in \Omega$:*

(1) $d(\kappa, \varpi) = 0$*, if and only if* $\kappa = \varpi$;

(2) $d(\kappa, \varpi) = d(\varpi, \kappa)$;

(3) $d(\kappa, \varpi) \leq \varphi(\kappa, \varpi)[d(\kappa, u) + d(u, v) + d(v, \varpi)]$ *for all distinct points* $u, v \in \Omega \setminus \{\kappa, \varpi\}$.

Then d is called an extended rectangular b-metric on Ω and (Ω, d_φ) is called an extended rectangular b-metric space.

Definition 2.6. [11] *Let (Ω, d_φ) be an extended rectangular b-metric space.*

(1) *a sequence $\{\kappa_n\}$ in Ω is said to be a Cauchy sequence if* $\lim_{n,m \to \infty} d(\kappa_n, \kappa_m) = 0$;

(2) *a sequence $\{\kappa_n\}$ in Ω is said to be convergent to z if* $\lim_{n \to \infty} d(\kappa_n, z) = 0$;

(3) *(Ω, d_φ) is said to be complete if every Cauchy sequence in Ω convergent to some point in Ω.*

Definition 2.7. [12] *Let (Ω, d_φ) be an extended rectangular b-metric space and $\{\kappa_n\}$ be a sequence in Ω. Self-mapping V and Q on Ω is said to be compatible, if $d_\varphi(V\kappa_n, z) \to 0$ and $d_\varphi(Q\kappa_n, z) \to 0$, then $d_\varphi(VQ\kappa_n, QV\kappa_n) \to 0$, as $n \to +\infty$.*

Definition 2.8. [13] *Let Ω be a nonempty set and $V, Q : \Omega \to \Omega$ be self-mappings. $\{V, Q\}$ is called weakly compatible, for every $z \in \Omega$, if $Vz = Qz$ then $QVz = VQz$.*

Definition 2.9. [12] *Let (Ω, d_φ) and (Y, d_φ) be the extended rectangular b-metric spaces. A mapping $V : \Omega \to Y$ is said to be continuous on Ω if and only if every sequence $\{\kappa_n\}$ that is convergent to z, then the sequence $\{V\kappa_n\}$ is convergent to Vz.*

Definition 2.10. [14] *Let Ω be a non-empty set, N is a natural number such that $N \geq 2$ and $V_1, V_2, \ldots, V_N : \Omega \to \Omega$ are given self-mappings on Ω. If $w = V_1 z = V_2 z = \cdots = V_N z$ for some $z \in \Omega$, then z is called a coincidence point of $V_1, V_2, \ldots, V_{N-1}$ and V_N, and w is called a point of coincidence of $V_1, V_2, \ldots, V_{N-1}$ and T_N. If $w = z$, then z is called a common fixed point of $V_1, V_2, \ldots, V_{N-1}$ and T_N.*

Let $V, Q : \Omega \to \Omega$ be two mappings. We denote by $C(V, Q)$ the set of coincidence points of Q and V; that is,

$$C(V, Q) = \{z \in \Omega : Vz = Qz\}.$$

Next, we introduce the simulation function was introduced by Khojasteh et al. [15].

Definition 2.11. [15] *A function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a simulation function, if it satisfies the following conditions:*

(1) $\eta(0,0) = 0$;

(2) $\eta(\kappa, \varpi) < \varpi - \kappa$, *for* $\kappa, \varpi > 0$;

(3) *if* $\{\kappa_n\}, \{\varpi_n\}$ *are sequences in* $(0, +\infty)$ *such that* $\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \varpi_n > 0$, *then*

$$\limsup_{n\to\infty}(\kappa_n, \varpi_n) < 0.$$

We denote the set of all simulation functions by $\mathbb{Z}$.

Definition 2.12. [15] *Let (Ω, d) be a metric space, $V : \Omega \to \Omega$ be a mapping and $\eta \in \mathbb{Z}$. Then, V is called a $\mathbb{Z}$-contraction with respect to η if the following condition holds*:

$$\eta(d(V\kappa, V\varpi), d(\kappa, \varpi)) > 0,$$

where $\kappa, \varpi \in \Omega$, with $\kappa \neq \varpi$.

Theorem 2.1. [15] *Every $\mathbb{Z}$-contraction on a complete metric space has a unique fixed point.*

Definition 2.13. [16] *We say that $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a η-simulation function, there exists $\psi \in \Psi$ such that*

(η_1) $\eta(\varpi, \kappa) < \psi(\kappa) - \psi(\varpi)$ *for* $\kappa, \varpi > 0$;

(η_2) *If* $\{\varpi_n\}$ *and* $\{\kappa_n\}$ *are sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \kappa_n > 0$, *then*

$$\limsup_{n\to\infty} \eta(\varpi_n, \kappa_n) < 0.$$

Let $\mathbb{Z}_\psi$ is the set of all η-simulation function.

Definition 2.14. [6] *Let V be a self map on a nonempty space Ω and $\alpha : \Omega \times \Omega \to [0, +\infty)$. We say that V is α admissible if, for all $\kappa, \varpi \in \Omega$, we have*

$$\alpha(\kappa, \varpi) \geq 1 \quad \textit{implies} \quad \alpha(V\kappa, V\varpi) \geq 1.$$

Definition 2.15. [17] *Let V, Q be self maps on a nonempty space X and $\alpha : \Omega \times \Omega \to [0, +\infty)$. We say that V is Q-α admissible if, for all $\kappa, \varpi \in \Omega$, we have*

$$\alpha(Q\kappa, Q\kappa) \geq 1 \quad \textit{implies} \quad \alpha(V\kappa, V\varpi) \geq 1.$$

Definition 2.16. [17] *Let Ω be a non empty set, $V : \Omega \to \Omega$ and $\alpha, \beta : \Omega \times \Omega \to [0, +\infty)$, we say that V is an (α, β)-admissible mapping if $\alpha(\kappa, \varpi) \geq 1$ and $\beta(\kappa, \varpi) \geq 1$ implies $\alpha(V\kappa, V\varpi) \geq 1$ and $\beta(V\kappa, V\varpi) \geq 1$ for all $\kappa, \varpi \in \Omega$.*

Also, we denote Ψ and Φ the sets of functions $\psi, \phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions, respectively

(1) $\psi(t) = \phi(t) = 0$ if and only if $t = 0$;

(2) $\psi(t), \phi(t) > 0$ for all $t > 0$;

(3) $\liminf_{\tau\to t}\psi(\tau)$ and $\limsup_{\tau\to t}\phi(\tau)$ exist for all $t>0$.

Let

$$\Psi=\{\psi:[0,\infty)\to[0,\infty)\mid \psi \text{ is continuous and } \psi(t)=0\leftrightarrow t=0\}.$$

Also, we denote

$$\Phi=\{\phi:[0,\infty)\to[0,\infty)\mid \phi \text{ is continuous and } \phi(t)=0\leftrightarrow t=0\}.$$

Lemma 2.1. [18] *If a sequence $\{\kappa_n\}$ in Ω is not Cauchy, then there exist $\varepsilon>0$ and two subsequence $\{\kappa_{m(k)}\}$ and $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ such that $m(k)$ is smallest index for which $m(k)>n(k)>k$,*

$$d(\kappa_{m(k)},\kappa_{n(k)})\geq\varepsilon \tag{2.1}$$

and

$$d(\kappa_{m(k)-1},\kappa_{n(k)})<\varepsilon. \tag{2.2}$$

Moreover, suppose that $\lim_{n\to\infty}d(\kappa_n,\kappa_{n+1})=0$. Then we have:

(1) $\lim_{k\to\infty}d(\kappa_{m(k)},\kappa_{n(k)})=\varepsilon$;
(2) $\lim_{k\to\infty}d(\kappa_{m(k)-1},\kappa_{n(k)-1})=\varepsilon$;
(3) $\lim_{k\to\infty}d(\kappa_{m(k)},\kappa_{n(k)-1})=\varepsilon$;
(4) $\lim_{k\to\infty}d(\kappa_{m(k)-1},\kappa_{n(k)})=\varepsilon$.

3. MAIN RESULTS

Definition 3.1. *Let (Ω,d_φ) be an extended rectangular b-metric space with function $\varphi:\Omega\times\Omega\to[1,+\infty)$ and $\alpha,\beta:\Omega\times\Omega\to[0,+\infty)$. Let V and Q be two self-maps on Ω. We say that the pair (V,Q) is Suzuki generalized rational type $\mathbb{Z}_{(\alpha,\beta)}$-contraction if for any $\kappa,\varpi\in\Omega$ and $L\geq 0$ such that*

$$\begin{aligned}&\frac{1}{2}\min\{d_\varphi(V\kappa,Q\kappa),d_\varphi(V\varpi,Q\varpi)\}\leq\max\{d_\varphi(Q\kappa,Q\varpi),d_\varphi(V\kappa,V\varpi)\}\quad \textit{implies}\\ &\eta(\alpha(Q\kappa,Q\varpi)B(\kappa,\varpi),A(\kappa,\varpi))\geq 0,\end{aligned} \tag{3.1}$$

where $\eta\in\mathbb{Z}_\psi$,

$$B(\kappa,\varpi)=\beta(Q\kappa,Q\varpi)d_\varphi(V\kappa,V\varpi)$$

and

$$A(\kappa,\varpi)=\max\Big\{d_\varphi(Q\kappa,Q\varpi),d_\varphi(Q\kappa,V\kappa),d_\varphi(V\varpi,Q\varpi),$$
$$\frac{G(\kappa,\varpi)+H(\kappa,\varpi)}{1+d_\varphi(Q\kappa,V\kappa)+d_\varphi(V\varpi,Q\varpi)},\frac{G(\kappa,\varpi)+H(\kappa,\varpi)}{1+d_\varphi(V\kappa,V\varpi)+d_\varphi(Q\kappa,Q\varpi)}\Big\}$$
$$+L\min\{d_\varphi(V\kappa,Q\kappa),d_\varphi(V\varpi,Q\varpi),d_\varphi(Q\kappa,Q\varpi),d_\varphi(V\kappa,Q\varpi)\}$$

with

$$G(\kappa,\varpi)=d_\varphi(V\kappa,Q\varpi)d_\varphi(Q\kappa,Q\varpi)$$

and

$$H(\kappa,\varpi)=d_\varphi(V\kappa,Q\kappa)d_\varphi(V\kappa,V\varpi).$$

Theorem 3.1. *Let (Ω,d_φ) be a complete extended rectangular b-metric space and $V,Q:\Omega\to\Omega$ be a compatible pair of self-maps such that $V(\Omega)\subseteq Q(\Omega)$. Assume that the pair (V,Q) is Suzuki generalized rational type $\mathbb{Z}_{(\alpha,\beta)}$-contraction and satisfy the following conditions:*

1. *V is α-admissible with respect to Q;*
2. *there exists $\kappa_0\in\Omega$ such that $\alpha(Q\kappa_0,V\kappa_0)\geq 1$ and $\beta(Qx_0,Vx_0)\geq 1$;*
3. *If $\{Q\kappa_n\}$ is a sequence in Ω such that $\alpha(Q\kappa_n,Q\kappa_{n+1})\geq 1$ for all n and $Q\kappa_n\to Qz\in Q(\Omega)$ as $n\to+\infty$, then there exists a subsequence $\{Q\kappa_{n(k)}\}$ of $\{Q\kappa_n\}$ such that $\alpha(Q\kappa_{n(i)},Qz)\geq 1$ for all k;*
4. *$Q(\Omega)$ is closed.*

Then V and Q have a unique coincidence point in Ω.

Proof. In view of condition (2), let $\kappa_0\in\Omega$ be such that $\alpha(Q\kappa_0,V\kappa_0)\geq 1$. Because $V(\Omega)\subseteq Q(\Omega)$, we can choose a point $\kappa_1\in\Omega$ such that $V\kappa_0=Q\kappa_1$. Continuing this process having chosen $\kappa_1,\kappa_2,\kappa_3,\ldots,\kappa_n$, we choose κ_{n+1} in Ω such that

$$V\kappa_n=Q\kappa_{n+1},\quad n=0,1,2,\ldots. \tag{3.2}$$

From condition (1), V is α-admissible with respect to Q, we have

$$\alpha(Q\kappa_0,V\kappa_0)=\alpha(Q\kappa_0,Q\kappa_1)\geq 1\quad\text{implies}\quad\alpha(V\kappa_0,V\kappa_1)=\alpha(Q\kappa_1,Q\kappa_2)\geq 1.$$

Using mathematical induction, we get

$$\alpha(Q\kappa_n,Q\kappa_{n+1})\geq 1,\quad n=0,1,2,\ldots. \tag{3.3}$$

Similarly, we obtain

$$\beta(Q\kappa_n, Q\kappa_{n+1}) \geq 1, \quad n = 0,1,2,\dots. \tag{3.4}$$

If $V\kappa_{n+1} = V\kappa_n$ for some n, then by (3), we have $Q\kappa_{n+1} = V\kappa_{n+1}$, so that κ_{n+1} is a coincidence point of V and Q and the proof is completed. For this, we suppose that $d_\varphi(V\kappa_n, V\kappa_{n+1}) > 0$ for all n. Applying the inequality (3.1) and using (3.3), (3.4), we obtain

$$\frac{1}{2}\min\{d_\varphi(V\kappa_n, Q\kappa_n), d_\varphi(V\kappa_{n+1}, Q\kappa_{n+1})\} \leq \max\{d_\varphi(Q\kappa_n, Q\kappa_{n+1}), d_\varphi(V\kappa_n, V\kappa_{n+1})\} \quad \text{implies}$$

$$\eta(\alpha(Q\kappa_n, Q\kappa_{n+1})B(\kappa_n, \kappa_{n+1}), A(\kappa_n, \kappa_{n+1})) \geq 0, \tag{3.5}$$

and

$$\psi(A(\kappa_n, \kappa_{n+1})) - \psi(\alpha(Q\kappa_n, Q\kappa_{n+1})B(\kappa_n, \kappa_{n+1})) > 0.$$

Hence,

$$\psi(A(\kappa_n, \kappa_{n+1})) > \psi(\alpha(Q\kappa_n, Q\kappa_{n+1})B(\kappa_n, \kappa_{n+1})).$$

From definition of ψ, we have

$$A(\kappa_n, \kappa_{n+1}) > \alpha(Q\kappa_n, Q\kappa_{n+1})B(\kappa_n, \kappa_{n+1}), \tag{3.6}$$

which

$$B(\kappa_n, \kappa_{n+1}) = \beta(Q\kappa_n, Q\kappa_{n+1})d_\varphi(V\kappa_n, V\kappa_{n+1}) \tag{3.7}$$

and

$$\begin{aligned} A(\kappa_n, \kappa_{n+1}) = \max\Big\{ & d_\varphi(Q\kappa_n, Q\kappa_{n+1}), d_\varphi(Q\kappa_n, V\kappa_n), d_\varphi(V\kappa_{n+1}, Q\kappa_{n+1}), \\ & \frac{G(\kappa_n, \kappa_{n+1}) + H(\kappa_n, \kappa_{n+1})}{1 + d_\varphi(Q\kappa_n, V\kappa_n) + d_\varphi(V\kappa_{n+1}, Q\kappa_{n+1})}, \frac{G(\kappa_n, \kappa_{n+1}) + H(\kappa_n, \kappa_{n+1})}{1 + d_\varphi(V\kappa_n, V\kappa_{n+1}) + d_\varphi(Q\kappa_n, Q\kappa_{n+1})}\Big\} \\ & + L\min\{d_\varphi(V\kappa_n, Q\kappa_n), d_\varphi(V\kappa_{n+1}, Q\kappa_{n+1}), d_\varphi(Q\kappa_n, Q\kappa_{n+1}), d_\varphi(V\kappa_n, Q\kappa_{n+1})\} \end{aligned} \tag{3.8}$$

with

$$G(\kappa_n, \kappa_{n+1}) = d_\varphi(V\kappa_n, Q\kappa_{n+1})d_\varphi(Q\kappa_n, Q\kappa_{n+1}) \tag{3.9}$$

and

$$H(\kappa_n, \kappa_{n+1}) = d_\varphi(V\kappa_n, Q\kappa_n)d_\varphi(V\kappa_n, V\kappa_{n+1}). \tag{3.10}$$

Therefore,

$$
\begin{aligned}
&A(\kappa_n,\kappa_{n+1}) = \max\Big\{d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_{n+1},V\kappa_n),\\
&\frac{G(\kappa_n,\kappa_{n+1})+H(\kappa_n,\kappa_{n+1})}{1+d_\varphi(V\kappa_{n-1},V\kappa_n)+d_\varphi(V\kappa_{n+1},V\kappa_n)}, \frac{G(\kappa_n,\kappa_{n+1})+H(\kappa_n,\kappa_{n+1})}{1+d_\varphi(V\kappa_n,V\kappa_{n+1})+d_\varphi(V\kappa_{n-1},V\kappa_n)}\Big\}\\
&+L\min\{d_\varphi(V\kappa_n,V\kappa_{n-1}), d_\varphi(V\kappa_{n+1},V\kappa_n), d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_n,V\kappa_n)\}
\end{aligned}
\tag{3.11}
$$

with

$$
G(\kappa_n,\kappa_{n+1}) = d_\varphi(V\kappa_n,V\kappa_n)d_\varphi(V\kappa_{n-1},V\kappa_n) = 0 \tag{3.12}
$$

and

$$
H(\kappa_n,\kappa_{n+1}) = d_\varphi(V\kappa_n,V\kappa_{n-1})d_\varphi(V\kappa_n,V\kappa_{n+1}). \tag{3.13}
$$

From (3.10), (3.11) and (3.12), we obtain

$$
\begin{aligned}
&A(\kappa_n,\kappa_{n+1}) = \max\Big\{d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_{n+1},V\kappa_n),\\
&\frac{G(\kappa_n,\kappa_{n+1})+H(\kappa_n,\kappa_{n+1})}{1+d_\varphi(V\kappa_{n-1},V\kappa_n)+d_\varphi(V\kappa_{n+1},V\kappa_n)}, \frac{d_\varphi(V\kappa_n,V\kappa_{n-1})d_\varphi(V\kappa_n,V\kappa_{n+1})}{1+d_\varphi(V\kappa_n,V\kappa_{n+1})+d_\varphi(V\kappa_{n-1},V\kappa_n)}\Big\}\\
&+L\min\{d_\varphi(V\kappa_n,V\kappa_{n-1}), d_\varphi(V\kappa_{n+1},V\kappa_n), 0\}.
\end{aligned}
\tag{3.14}
$$

Since $d_\varphi(V\kappa_n,V\kappa_{n+1}) \leq 1+d_\varphi(V\kappa_n,V\kappa_{n+1})+d_\varphi(V\kappa_{n-1},V\kappa_n)$, from (3.13), we have

$$
A(\kappa_n,\kappa_{n+1}) = \max\Big\{d_\varphi(V\kappa_{n-1},V\kappa_n), d_\varphi(V\kappa_{n+1},V\kappa_n)\Big\}. \tag{3.15}
$$

If $A(\kappa_n,\kappa_{n+1}) = d_\varphi(V\kappa_{n+1},V\kappa_n)$ and (3.6), we obtain

$$
d_\varphi(V\kappa_{n+1},V\kappa_n) < d_\varphi(V\kappa_{n+1},V\kappa_n), \tag{3.16}
$$

a contradiction. Thus, for all $n \geq 1$, we have

$$
A(\kappa_n,\kappa_{n+1}) = d_\varphi(V\kappa_{n-1},V\kappa_n). \tag{3.17}
$$

From (3.6), we have

$$
\alpha(V\kappa_{n-1},V\kappa_n)\beta(V\kappa_{n-1},V\kappa_n)d_\varphi(V\kappa_n,V\kappa_{n+1}) < d_\varphi(V\kappa_{n-1},V\kappa_n). \tag{3.18}
$$

So,

$$
d_\varphi(V\kappa_{n+1},V\kappa_n) < d_\varphi(V\kappa_{n-1},V\kappa_n). \tag{3.19}
$$

The sequence $\{d_\varphi(V\kappa_{n+1}, V\kappa_n)\}$ is non increasing i.e. decreasing. So, there exist $r \geq 0$ such that

$$\lim_{n\to\infty} d_\varphi(V\kappa_{n-1}, V\kappa_n) = r.$$

We prove that

$$\lim_{n\to\infty} d_\varphi(V\kappa_{n-1}, V\kappa_n) = 0. \tag{3.20}$$

Now, we assume on the contrary such that $r > 0$. By (3.16), we have

$$\lim_{n\to\infty} \alpha(V\kappa_{n-1}, V\kappa_n)\beta(V\kappa_{n-1}, V\kappa_n)d_\varphi(V\kappa_{n+1}, V\kappa_n) = r. \tag{3.21}$$

Since $r > 0$ and letting $\varpi_n = \alpha(V\kappa_{n-1}, V\kappa_n)\beta(V\kappa_{n-1}, V\kappa_n)d_\varphi(V\kappa_{n+1}, V\kappa_n)$ and $\kappa_n = d_\varphi(V\kappa_{n+1}, V\kappa_n)$ such that $\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \kappa_n = r$, then by (η_2), we obtain

$$\limsup_{n\to\infty} \eta(\varpi_n, \kappa_n) < 0.$$

Since $\eta(\varpi_n, \kappa_n) \geq 0$, so

$$0 \leq \limsup_{n\to\infty} \eta(\varpi_n, \kappa_n) < 0,$$

which is contradiction. So our assumption is false. Hence $r = 0$. Again we show that $\{\kappa_n\}$ is a Cauchy sequence in (Ω, d_φ) i.e.

$$\lim_{n,m\to\infty} d_\varphi(V\kappa_n, V\kappa_m) = 0. \tag{3.22}$$

Suppose on the contrary i.e. $\{\kappa_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ for which we can assume subsequences $\kappa_{n(k)}$ and $\kappa_{n(k)}$ of $\{\kappa_n\}$ with $n(k) > m(k) > k$ such that for every k,

$$d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)}) \geq \varepsilon \tag{3.23}$$

and $n(k)$ is the smallest number such that (3.23) holds. From (3.23), we obtain

$$d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)}) < \varepsilon. \tag{3.24}$$

Then by triangular inequality and (3.22), we have

$$\begin{aligned} \varepsilon &\leq d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)}) \\ &\leq d_\varphi(V\kappa_{n(k)}, V\kappa_{n(k)-1}) + d_\varphi(V\kappa_{n(k)-1}, V\kappa_{m(k)}) \\ &< d_\varphi(V\kappa_{n(k)}, V\kappa_{n(k)-1}) + \varepsilon. \end{aligned} \tag{3.25}$$

Taking $n \to \infty$ in above equation and applying (3.20), we obtain

$$\varepsilon \leq \lim_{n\to\infty} d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)}) < \varepsilon.$$

Thus,

$$\lim_{n\to\infty} d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)}) = \varepsilon. \tag{3.26}$$

Similarly, it is easy to show that

$$\lim_{n\to\infty} d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)+1}) = \varepsilon \tag{3.27}$$

and

$$\lim_{n\to\infty} d_\varphi(V\kappa_{n(k)-1}, V\kappa_{m(k)}) = \varepsilon. \tag{3.28}$$

Using (3.6) and (η_2), we obtain

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \eta(\alpha(V\kappa_{n(k)-1}, V\kappa_{m(k)})\beta(V\kappa_{n(k)-1}, V\kappa_{m(k)})d_\varphi(V\kappa_{n(k)}, V\kappa_{m(k)+1}), d_\varphi(V\kappa_{n(k)-1}, V\kappa_{m(k)})) \\ &< 0, \end{aligned} \tag{3.29}$$

which is contradict due to our assumption. So $\{\kappa_n\}$ is a Cauchy sequence. Therefore $\{V\kappa_n\} = \{Q\kappa_{n+1}\}$ is a Cauchy sequence in Ω. Since $Q(\Omega)$ is closed there exists $z \in \Omega$ such that

$$\lim_{n\to\infty} Q\kappa_n = \lim_{n\to\infty} V\kappa_{n+1} = Qz. \tag{3.30}$$

We now show that z is a coincidence point of V and Q. On contrary, assume that $d_\varphi(Vz, Qz) > 0$. Using condition (3) and (3.30), we have $\alpha(Q\kappa_{n(k}, Qz) \geq 1$ for all k. Therefore

$$\frac{1}{2}\min\{d_\varphi(V\kappa_{n(k)}, Q\kappa_{n(k)}), d_\varphi(Vz, Qz)\} \leq \max\{d_\varphi(Q\kappa_{n(k)}, Q\kappa_{n(k)}), d_\varphi(V\kappa_{n(k)}, Vz)\}.$$

Letting $k \to \infty$ in the above inequality yields

$$\begin{aligned}\frac{1}{2}\min\{d_{\varphi}(Vz,Qz),d_{\varphi}(Vz,Qz)\} &\leq \max\{d_{\varphi}(Qz,Qz),d_{\varphi}(Vz,Vz)\} \\ &= \max\{0,0\} \\ &= 0,\end{aligned}$$

which is a contradiction. Hence, our supposition is wrong and $d_{\varphi}(Vz,Qz)=0$, that is, $Vz=Qz$. This shows that V and Q have a coincidence point. □

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for all $\rho,\sigma \in C(V,Q)$, if $\alpha(Q\rho,Q\sigma)\geq 1$ and the pair (V,Q) is weakly compatible. Then V and Q have a unique common fixed point.*

Proof. From the proof of Theorem 3.1, we have $\{Q\kappa_n\}$ is a non decreasing sequence and converges to Qz and $Vz=Qz$. Also, since V and Q are weakly compatible, we have

$$Vz = VQz = QVz = Qz.$$

Let $u = Vz = Qz$, we obtain

$$u = Vu = Qu.$$

So that V and Q have a common fixed point. To prove uniqueness, let u and u' be two common fixed points of V and Q i.e.,

$$u = Vu = Qu \quad \text{and} \quad u' = Vu' = Qu'.$$

Since

$$\begin{aligned}&\frac{1}{2}\min\{d_{\varphi}(Vu,Qu),d_{\varphi}(Vu',Qu')\}=0 \\ &\leq \max\{d_{\varphi}(Qu,Qu'),d_{\varphi}(Vu,Vu')\}\end{aligned} \tag{3.31}$$

implies

$$\eta(\alpha(Qu,Qu')B(u,u'),A(u,u'))\geq 0 \tag{3.32}$$

and

$$\psi(A(u,u'))-\psi(\alpha(Qu,Qu')B(u,u'))>0. \tag{3.33}$$

So,

$$\psi(A(u,u')) > \psi(\alpha(Qu,Qu')B(u,u')). \tag{3.34}$$

From definition of ψ, we have

$$A(u,u') > \alpha(Qu,Qu')B(u,u'). \tag{3.35}$$

which

$$B(u,u') = \beta(Qu,Qu')d_\varphi(Vu,Vu') \tag{3.36}$$

and

$$\begin{aligned} A(u,u') = \max\Big\{ & d_\varphi(Qu,Qu'), d_\varphi(Qu,Vu), d_\varphi(Vu',Qu'), \\ & \frac{G(u,u')+H(u,u')}{1+d_\varphi(Qu,Vu)+d_\varphi(Vu',Qu')}, \frac{G(u,u')+H(u,u')}{1+d_\varphi(Vu,Vu')+d_\varphi(Qu,Qu')}\Big\} \\ & + L\min\{d_\varphi(Vu,Qu), d_\varphi(Vu',Qu'), d_\varphi(Qu,Qu'), d_\varphi(Vu,Qu')\} \end{aligned} \tag{3.37}$$

with

$$G(u,u') = d_\varphi(Vu,Qu')d_\varphi(Qu,Qu') \tag{3.38}$$

and

$$H(u,u') = d_\varphi(Vu,Qu)d_\varphi(Vu,Vu') = 0. \tag{3.39}$$

From (3.36), (3.37), (3.36) and (3.39), we obtain

$$\begin{aligned} A(u,u') = \max\Big\{ & d_\varphi(Qu,Qu'), 0, 0, \\ & d_\varphi(Vu,Qu')d_\varphi(Qu,Qu'), \frac{d_\varphi(Vu,Qu')d_\varphi(Qu,Qu')}{1+d_\varphi(Vu,Vu')+d_\varphi(Qu,Qu')}\Big\} \\ & + L\min\{0,0,d_\varphi(Qu,Qu'), d_\varphi(Vu,Qu')\}. \end{aligned} \tag{3.40}$$

Since $d_\varphi(Vu,Qu') \leq 1+d_\varphi(Vu,Vu')+d_\varphi(Qu,Qu')$, from (3.40), we have

$$\begin{aligned} A(u,u') = \max\Big\{ & d_\varphi(Qu,Qu'), 0, 0, \\ & d_\varphi(Vu,Qu')d_\varphi(Qu,Qu'), d_\varphi(Qu,Qu')\Big\} \\ & + L\min\{0,0,d_\varphi(Qu,Qu'), d_\varphi(Vu,Qu')\}. \end{aligned} \tag{3.41}$$

If $A(u,u') = d_\varphi(Qu,Qu')$ and (3.35), we obtain

$$d_\varphi(Vu,Vu') < d_\varphi(Qu,Qu') = d_\varphi(Vu,Vu'), \tag{3.42}$$

a contradiction. Thus, for all $n \geq 1$, we have

$$A(u,u') = d_\varphi(Vu,Qu')d_\varphi(Qu,Qu'). \tag{3.43}$$

From (3.43), and(3.36) and (3.35), we obtain

$$\alpha(Qu,Qu')\beta(Qu,Qu')d_\varphi(Vu,Vu') < d_\varphi(Vu,Qu')d_\varphi(Qu,Qu'). \tag{3.44}$$

Therefore V and Q have a unique common fixed point in Ω.

CONCLUSION

This work derives new relation-theoretic coincidence and common fixed point conclusions for some mappings of V and Q using Suzuki generalized rational type $\mathbb{Z}_{(\alpha,\beta)}$-contraction in extended rectangular b-metric space. We improve and broaden a number of recent discoveries.

□

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

ACKNOWLEDGMENT

This project was supported by the Research and Development Institute, Rambhai Barni Rajabhat University (Grant no. 2217/2566).

REFERENCES

[1] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Ulianowsk Gos. Ped. Inst. 30 (1989), 26–37.

[2] D. Kitkuan, P. Sukprasert, Set-valued contractions in b-metric spaces with application, WSEAS Trans. Math. 20 (2021), 312–318. https://doi.org/10.37394/23206.2021.20.32.

[3] J. Limprayoon, D. Kitkuan, Fixed point results for Geraghty contraction type mappings in b-metric spaces, J. Math. Comput. Sci. 11 (2021), 5167–5180. https://doi.org/10.28919/jmcs/5993.

[4] D. Kitkuan, P. Bunpatcharacharoen, Coincidence point theorems for multi-valued mappings in b-metric spaces via digraphs, Adv. Math., Sci. J. 10 (2021), 2785–2797. https://doi.org/10.37418/amsj.10.6.7.

[5] C. Khaofong, P. Saipara, A. Padcharoen, Some results of generalized Hardy-Roger mappings in rectangular b-metric spaces, Nonlinear Funct. Anal. Appl. 28 (2023), 1097–1113. https://doi.org/10.22771/NFAA.2023.28.04.16.

[6] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal.: Theory Methods Appl. 75 (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014.

[7] E. Karapinar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal. 2012 (2012), 793486. https://doi.org/10.1155/2012/793486.

[8] T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[9] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57 (2000), 31–37.

[10] R. George, S. Radenovic, K.P. Reshma, et al. Rectangular b-metric space and contraction principles, J. Nonlinear Sci. Appl. 6 (2015), 1005–1013.

[11] M. Asim, M. Imdad, S. Radenovic, Fixed point results in extended rectangular b-metric spaces with an application, U.P.B. Sci. Bull. Ser. A, 81 (2019), 43–50.

[12] B. Nurwahyu, M.K. Saeed, N. Fabiano, et al. Common fixed point on generalized weak contraction mappings in extended rectangular b-metric spaces, Filomat 35 (2021), 3621–3633. https://doi.org/10.2298/fil2111621n.

[13] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), 771–779. https://doi.org/10.1155/S0161171286000935.

[14] H. Aydi, H.K. Nashine, B. Samet, H. Yazidi, Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations, Nonlinear Anal.: Theory Methods Appl. 74 (2011), 6814–6825. https://doi.org/10.1016/j.na.2011.07.006.

[15] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theory for simulation functions, Filomat 29 (2015), 1189–1194. https://doi.org/10.2298/fil1506189k.

[16] G.H. Joonaghany, A. Farajzadeh, M. Azhini, et al. A new common fixed point theorem for Suzuki type contractions via generalized Ψ-simulation functions, Sahand Commun. Math. Anal. 16 (2019), 129–148. https://doi.org/10.22130/scma.2018.78315.359.

[17] G.V.R. Babu, K.K.M. Sarma, V.A. Kumari, Common fixed point of (α, η)-Geraghty contraction maps, J. Adv. Res. Pure Math. 6 (2014), 9–23.

[18] S.H. Cho, J.S. Bae, Fixed points of weak α-contraction type maps, Fixed Point Theory Appl. 2014 (2014), 175. https://doi.org/10.1186/1687-1812-2014-175.