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SOME NEW FIXED POINT THEOREMS OF RATIONAL TYPE CONTRACTION WITH AN APPLICATION TO COUPLED FIXED POINT THEORY

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Abstract. The aim of this article is to present several new theorems related to fixed points by considering a rational-type contractive condition on the mapping $\mathscr{A}$, in the context of complete b-metric spaces. Our results extend and improve many earlier theorems in the literature. As an application, coupled fixed point results are given in the same framework. Furthermore, to demonstrate the practical applicability of the results obtained, a few examples are showcased.

Keywords: b-metric space; Picard sequence; fixed point; coupled fixed point; rational type contraction mapping.

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1. Introduction

By demonstrating his well-known theorem, known as ”Banach’s contraction principle” [1] for a contraction mapping $\mathscr{A}$ in entire metric space, Banach set the foundation for fixed

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point theory in 1922. This served as the foundation for further research into fixed point theory. Because it was used to solve nonlinear problems in general and partial differential equations in particular, this theorem gained unmatched popularity [2, 3, 4, 5, 6, 7, 8, 9].Until now, it has developed into a crucial tool for physicists and mathematicians alike. One criticism of this theorem is that it makes the unavoidably continuous assumption that the mapping $\mathscr{A}$ is always possible. Consequently, non-continuous mappings have been included in the generalization of this traditional Banach's contraction theorem by a number of authors (notice that some of the examples provided in this article are non-continuous functions). TOne way to make this generalization is to either take different contraction conditions on the mappings (see [10, 11, 12, 13, 14, 15, 16, 17, 18]) or reduce axioms on the metric space to create new spaces such as b-metric space, quasi-metric space, metric like space, rectangular metric spaces, G-metric spaces, partial ordered metric spaces, cone metric spaces, probabilistic metric spaces, fuzzy metric spaces, (some of which can be found in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]).

Ciric [38] introduced and studied self-mappings on K so that K is a nonempty closed subset of Ω) satisfying

$$\delta(f(u), f(v)) \leq a \max\{\delta(u,v), \delta(u, f(u)), \delta(v, f(v)), \delta(u, f(v)), \delta(v, f(u))\}, \tag{1}$$

for each $u, v \in K$, where $0 < a < 1$. Edelstein [39] and Rakoch [40] in 1962 initiated investigations of mappings $\mathscr{A}$ satisfying

$$\delta(\mathscr{A}u, \mathscr{A}v) \leq k(u,v)\delta(u,v) \quad \text{for all } \; u, v \in \Omega, \tag{2}$$

for each $u, v \in K$, where $k(u,v) < 1$ and $\sup k(u,v) = 1$. Over time, Many types of contractions appeared, including the rational type expression gave interesting results in fixed point theory. Some well-known results in this direction are involved (see [10, 28, 29, 30, 31, 33, 34, 35, 36, 37, 41, 42, 43, 44]).

In 2006, Bhaskar and Laxikantham [45] first introduced the idea of coupled fixed point. The results they obtained have been used to the study of existence and uniqueness of solutions to a class of periodic boundary value problems. Later, theorems for connected fixed points have been shown for other functions (see [46, 47, 48]). In this study, we take a look at some recent findings and apply them to b-metric spaces. Drawing inspiration from Ciric's [38] expansion of the Banach contraction principle, our work expands upon findings from Khojasteh [43] and Aouine [44]. Notably, with a self mapping $\mathscr{A}$, we obtain considerably more precise dynamic results about the collection of fixed points. We present linked fixed point theorem as an application of our findings. In addition, we provide illustrated cases that support the theories.

2. Preliminary

In this section, we will review notable concepts and definitions of b-metric spaces that will be utilized in the subsequent sections.

Definition 1. [32] *Assume that Ω is a non-empty set and let s be a real number greater than or equal to 1. We define a mapping $\delta : \Omega \times \Omega \to [0,\infty)$ to be a b-metric if it satisfies the following conditions*

(b1) $\delta(u,v) = 0$ *if and only if* $u = v$,

(b2) $\delta(u,v) = \delta(v,u)$,

(b3) $\delta(u,w) \leq s[\delta(u,v) + \delta(v,w)]$,

for any $u,v,w \in \Omega$. If (Ω,δ,s) satisfies the above conditions, it is known as a b-metric space with a constant s ($s \geq 1$).

Rem 1. *As stated in the previous definition, it is apparent that the category of b-metric spaces is more extensive than that of metric spaces. This is because when $s = 1$, a b-metric space transforms into a metric space; however, the inverse is not valid. Furthermore, it's worth noting that b-metrics are not typically continuous, as demonstrated in example 4 of* [49].

Definition 2. [26] *Consider a b-metric space (Ω,δ,s). We define a sequence $\{u_n\}$ in Ω to be as*

(a) Convergent if there is an element $u \in \Omega$ that satisfies the condition $\lim_{n\to+\infty} \delta(u_n,u) = 0$ *and it is denoted as $lim_{n\to\infty} u_n = u$;*

(b) Cauchy sequence if $lim_{n,m\to\infty} \delta(u_n,u_m) = 0$.

Before starting, we would like to introduce a basic lemma that plays a crucial role in demonstrating our outcomes. This lemma established by Aghajani, A., Abbas, M., and Roshan [50].

Lemma 1. [50] *Assume (Ω,δ,s) is a b-metric space, where s is greater than or equal to 1, and $\{u_n\}$ is a convergent sequence in Ω that converges to $u \in \Omega$. For any $v \in \Omega$, the following inequalities hold:*

$$\frac{1}{s}\delta(u,v) \leq \liminf_{n\to\infty} \delta(u_n,v) \leq \limsup_{n\to\infty} \delta(u_n,v) \leq s\delta(u,v). \tag{3}$$

Definition 3. *An element $(x,y) \in \Omega\times\Omega$ is called a coupled fixed point of a mapping $\mathscr{A} : \Omega\times\Omega \longrightarrow \Omega$ if $\mathscr{A}(x,y) = x$ and $\mathscr{A}(y,x) = y$.*

3. Main Results

Firstly, we present and demonstrate our initial theorem that extends and enhances the findings of Khojasteh et al. [43] and A.C. Aouine and A. Aliouche [44].

Theorem 1. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants a,b,c,f and $e \in \mathbb{R}^+$ that ensure either $s^2a \leq \min\{c,f\}$ or $s^2b \leq \min\{c,f\}$. Additionally, the equation (4) holds for all values of u and v belonging to Ω.*

$$\delta(\mathscr{A}u,\mathscr{A}v) \leq \frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e}\max\{\delta(u,v),\delta(u,\mathscr{A}u),\delta(v,\mathscr{A}v)\}, \tag{4}$$

then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5):*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{a+b}. \tag{5}$$

Proof. Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1} = \mathscr{A}u_n$, initiated from a random element u_0 in Ω. If there is an integer n_0 ensuring that $u_{n_0} = u_{n_0+1}$, then u_{n_0} represents a fixed point for the self-map $\mathscr{A}$, this concludes the proof. However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: We will show that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1 If the inequality $s^2a \leq \min\{c,f\}$ holds, substituting $u = u_{n-1}$ and $v = u_n$ into inequality

(4) yields the following result:

$$
\begin{aligned}
\delta(u_n,u_{n+1}) &\leq \frac{a\delta(u_{n-1},u_{n+1})}{c\delta(u_{n-1},u_n)+f\delta(u_n,u_{n+1})+e}\max\{\delta(u_{n-1},u_n),\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sa\delta(u_{n-1},u_n)+sa\delta(u_n,u_{n+1})}{c\delta(u_{n-1},u_n)+f\delta(u_n,u_{n+1})+e}\max\{\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sa\delta(u_{n-1},u_n)+sa\delta(u_n,u_{n+1})}{\min\{c;f\}\big(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1})\big)+e}\max\{\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sa\delta(u_{n-1},u_n)+sa\delta(u_n,u_{n+1})}{\min\{c;f\}\big(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1})\big)+e}\delta(u_{n-1},u_n).
\end{aligned}
$$

We define the sequence (θ_n) with the general term

$$\theta_n=\frac{sa\delta(u_{n-1},u_n)+sa\delta(u_n,u_{n+1})}{\min\{c;f\}\big(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1})\big)+e},$$

for all $n\in\mathbb{N}$. Given that $s^2a\leq\min\{c,f\}$, it follows that $0\leq\theta_n<1$ for all $n\in\mathbb{N}$. Moreover, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing since

$$
\begin{aligned}
\theta_{n+1}-\theta_n &= \frac{sae\big[\delta(u_{n+1},u_{n+2})-\delta(u_{n-1},u_n)\big]}{\big[\min\{c;f\}(\delta(u_n,u_{n+1})+\delta(u_{n+1},u_{n+2}))+e\big]}\\
&\quad\times\frac{1}{\big[\min\{c;f\}(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1}))+e\big]}\\
&< 0,
\end{aligned}
$$

for all $n\in\mathbb{N}$. On the other hand, we have

$$
\begin{aligned}
\delta(u_n,u_{n+1}) &\leq \theta_n\delta(u_{n-1},u_n);\\
&\leq \theta_n\theta_{n-1}\delta(u_{n-2},u_{n-1});\\
&\vdots\\
&\leq \theta_n\theta_{n-1}\cdots\theta_1\delta(u_0,u_1);\\
&\leq \theta_1^n\delta(u_0,u_1).
\end{aligned}
$$

For all $n,,m\in\mathbb{N}$ where $m>n$, we have

$$
\begin{aligned}
\delta(u_n,u_m) &\leq \sum_{i=n}^{m-1}s^{i-n+1}\delta(u_i,u_{i+1});\\
&\leq \sum_{i=n}^{m-1}s^{i-n+1}\theta_1^i\delta(u_0,u_1);
\end{aligned}
$$

$$\leq \frac{(s\theta_1)^n-(s\theta_1)^m}{1-s\theta_1}\times\frac{1}{s^{n-1}}\delta(u_0,u_1).$$

For large enough n for the above inequality, we conclude

$$\lim_{n,m\to+\infty}\delta(u_n,u_m)=0,$$

and hence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω.

Case 2 If the inequality $s^2b\leq\min\{c,f\}$ holds, we substitute $u=u_n$ and $v=u_{n-1}$ into inequality (4) to find

$$\begin{aligned}
\delta(u_n,u_{n+1}) &\leq \frac{b\delta(u_{n-1},u_{n+1})}{c\delta(u_{n-1},u_n)+f\delta(u_n,u_{n+1})+e}\max\{\delta(u_{n-1},u_n),\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sb\delta(u_{n-1},u_n)+sb\delta(u_n,u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(u_{n-1},u_n)+e}\max\{\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sb\delta(u_{n-1},u_n)+sb\delta(u_n,u_{n+1})}{\min\{c,f\}[\delta(u_n,u_{n+1})+\delta(u_{n-1},u_n)]+e}\max\{\delta(u_{n-1},u_n),\delta(u_n,u_{n+1})\};\\
&\leq \frac{sb\delta(u_{n-1},u_n)+sb\delta(u_n,u_{n+1})}{\min\{c,f\}[\delta(u_n,u_{n+1})+\delta(u_{n-1},u_n)]+e}\delta(u_{n-1},u_n).
\end{aligned}$$

Parallel to Case 1, we may conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω. Taking into account that the b-metric space (Ω,δ,s) is complete, then the sequence (u_n) converges to an element $\dot{u}$ in Ω.

Claim 2: We will confirm that $\dot{u}$ is a fixed point of $\mathscr{A}$.

Assume that $\delta(\dot{u},\mathscr{A}\dot{u})>0$.

Case 1 If $s^2a\leq\min\{c,f\}$, then by substituting $u=u_n$ and $v=\dot{u}$ into inequality (4), we find

$$\delta(\mathscr{A}\dot{u},u_{n+1})\leq\frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(u_{n+1},\dot{u})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}\max\{\delta(u_n,\dot{u}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,u_{n+1})\}.$$

On the other hand, we have

$$\begin{aligned}
&\delta(\dot{u},\mathscr{A}\dot{u})\leq s\delta(\dot{u},u_{n+1})+s\delta(u_{n+1},\mathscr{A}\dot{u});\\
&\leq s\delta(\dot{u},u_{n+1})+s\frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(u_{n+1},\dot{u})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}\max\{\delta(u_n,\dot{u}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,u_{n+1})\}.
\end{aligned}\tag{6}$$

Taking the upper limit on both sides of previous inequality, we obtain

$$\delta(\dot{u},\mathscr{A}\dot{u})\leq\frac{sa\limsup_{n\to\infty}\delta(u_n,\mathscr{A}\dot{u})}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u}).$$

Lemma 1 yields

$$\delta(\dot{u},\mathscr{A}\dot{u}) \leq \frac{s^2 a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u})^2,$$

thus,

$$1 \leq \frac{s^2 a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u}). \tag{7}$$

Since, $s^2 a \leq \min\{c,f\}$, then $s^2 a \leq f$, then $s^2 a\delta(\dot{u},\mathscr{A}\dot{u}) < f\delta(\dot{u},\mathscr{A}\dot{u})+e$ which contradict inequality (7). Then $\delta(\dot{u},\mathscr{A}\dot{u}) = 0$ that mean $\mathscr{A}\dot{u} = \dot{u}$.

Case 2 If $s^2 b \leq \min\{c,f\}$, then by substituting $u = \dot{u}$ and $v = u_n$ into inequality (4), we find

$$\delta(\mathscr{A}\dot{u},u_{n+1}) \leq \frac{a\delta(\dot{u},u_{n+1})+b\delta(u_n,\mathscr{A}\dot{u})}{c\delta(\dot{u},\mathscr{A}\dot{u})+f\delta(u_n,u_{n+1})+e}\max\{\delta(u_n,\dot{u}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,u_{n+1})\}.$$

Parallel to Case 1, we may conclude that $\mathscr{A}\dot{u} = \dot{u}$.

Claim 3: Assume that $\mathscr{A}$ have two distinct fixed points $\dot{u},\dot{v}$ in Ω, by taking $u = \dot{u}$, $v = \dot{v}$ in inequality (4), we find

$$\delta(\dot{u},\dot{v}) \leq \frac{a\delta(\dot{u},\dot{v})+b\delta(\dot{v},\dot{u})}{c\delta(\dot{u},\dot{u})+f\delta(\dot{v},\dot{v})+e}\max\{\delta(\dot{u},\dot{v}),\delta(\dot{u},\dot{u}),\delta(\dot{v},\dot{v})\}.$$

Then, $\delta(\dot{u},\dot{v}) \geq \frac{e}{a+b}$. This completes the proof of the theorem. □

Next, we will present and show our second theorem.

Theorem 2. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants $a,b,c,f,e \in \mathbb{R}^+$ that ensure either $s^3 a \leq \frac{1}{2}\min\{c,f\}$ or $s^3 b \leq \frac{1}{2}\min\{c,f\}$ and*

$$\delta(\mathscr{A}u,\mathscr{A}v) \leq \frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e}\max\{\delta(u,\mathscr{A}v),\delta(v,\mathscr{A}u),\delta(u,v)\}, \tag{8}$$

for all $u,v \in \Omega$. Then, we have

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (9):*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{a+b}. \tag{9}$$

Proof. Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1} = \mathscr{A} u_n$, initiated from a random element u_0 in Ω. If there is an integer n_0 ensuring that $u_{n_0} = u_{n_0+1}$, then u_{n_0} represents a fixed point for the self-map $\mathscr{A}$, this concludes the proof. However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: We will show that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1. If $s^3 a \leq \frac{1}{2}\min\{c, f\}$, by substituting $u = u_{n-1}$ and $v = u_n$ into inequality (8), we find

$$\begin{aligned}
\delta(u_n, u_{n+1}) &\leq \frac{a\delta(u_{n-1}, u_{n+1})}{c\delta(u_{n-1}, u_n) + f\delta(u_n, u_{n+1}) + e}\max\{\delta(u_{n-1}, u_{n+1}), \delta(u_{n-1}, u_n)\};\\
&\leq \frac{s^2 a\delta(u_{n-1}, u_n) + s^2 a\delta(u_n, u_{n+1})}{c\delta(u_{n-1}, u_n) + f\delta(u_n, u_{n+1}) + e}[\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})];\\
&\leq \frac{s^2 a\delta(u_{n-1}, u_n) + s^2 a\delta(u_n, u_{n+1})}{\min\{c; f\}\big(\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})\big) + e}[\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})].
\end{aligned}$$

We define the sequence (θ_n) with the general term

$$\theta_n = \frac{s^2 a\delta(u_{n-1}, u_n) + s^2 a\delta(u_n, u_{n+1})}{\min\{c; f\}\big(\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})\big) + e},$$

for all $n \in \mathbb{N}$. Given that $s^3 a \leq \frac{1}{2}\min\{c, f\}$, it follows that $0 \leq \theta_n < \frac{1}{2s}$ for all $n \in \mathbb{N}$. On the other hand, we have,

$$\delta(u_n, u_{n+1}) \leq \theta_n[\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})].$$

Hence, we obtain

$$\delta(u_n, u_{n+1}) \leq \frac{\theta_n}{1-\theta_n}\delta(u_{n-1}, u_n).$$

We denote that $\lambda_n = \dfrac{\theta_n}{1-\theta_n}$ for all $n \in \mathbb{N}$.

Since $0 \leq \theta_n < \frac{1}{2}$ for all $n \in \mathbb{N}$, then $0 \leq \lambda_n < 1$, therefore, we get, $\delta(u_n, u_{n+1}) < \delta(u_{n-1}, u_n)$ for all $n \in \mathbb{N}$, then, $\delta(u_{n+1}, u_{n+2}) < \delta(u_{n-1}, u_n)$.

Moreover, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\theta_{n+1} - \theta_n &= \frac{s^2 ae\big[\delta(u_{n+1}, u_{n+2}) - \delta(u_{n-1}, u_n)\big]}{\big[\min\{c; f\}(\delta(u_n, u_{n+1}) + \delta(u_{n+1}, u_{n+2})) + e\big]}\\
&\quad\times\frac{1}{\big[\min\{c; f\}(\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})) + e\big]}\\
&< 0.
\end{aligned}$$

This indicates that the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing. Consequently, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is also decreasing, thus

$$\begin{aligned}\delta(u_n,u_{n+1}) &\leq \lambda_n\delta(u_{n-1},u_n)\\ &\leq \lambda_n\lambda_{n-1}\delta(u_{n-2},u_{n-1})\\ &\vdots\\ &\leq \lambda_n\lambda_{n-1}\cdots\lambda_1\delta(u_0,u_1)\\ &\leq \lambda_1^n\delta(u_0,u_1).\end{aligned}$$

For all $n, m \in \mathbb{N}$ where $m > n$, we have,

$$\begin{aligned}\delta(u_n,u_m) &\leq \sum_{i=n}^{m-1} s^{i-n+1}\delta(u_i,u_{i+1});\\ &\leq \sum_{i=n}^{m-1} s^{i-n+1}\theta_1^i\delta(u_0,u_1);\\ &\leq \frac{(s\theta_1)^n-(s\theta_1)^m}{1-s\theta_1}\times\frac{1}{s^{n-1}}\delta(u_0,u_1).\end{aligned}$$

For large enough n for the above inequality, we conclude,

$$\lim_{n,m\to+\infty}\delta(u_n,u_m)=0,$$

thus, the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω.

Case 2 If $s^3b \leq \frac{1}{2}\min\{c,f\}$ holds, by substituting $u=u_n$ and $v=u_{n-1}$ into inequality (8), we find,

$$\delta(u_n,u_{n+1}) \leq \frac{b\delta(u_{n-1},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(u_{n-1},u_n)+e}\max\{\delta(u_{n-1},u_{n+1}),\delta(u_{n-1},u_n)\}.$$

Parallel to Case 1, we can conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω. Taking into account that the b-metric space (Ω,δ,s) is complete, then, the sequence (u_n) converges to an element $\dot{u}$ in Ω.

Claim 2: We will confirm that $\dot{u}$ is a fixed point of $\mathscr{A}$.

Assume that $\delta(\dot{u},\mathscr{A}\dot{u})>0$.

Case 1: If $s^3a \leq \frac{1}{2}\min\{c,f\}$ holds, by substituting $u=u_n$, $v=\dot{u}$ into inequality (8), we find,

$$\delta(\mathscr{A}\dot{u},u_{n+1}) \leq \frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(\dot{u},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}\max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.$$

On the other hand, we have

(10)

$$\begin{aligned}\delta(\dot{u},\mathscr{A}\dot{u}) &\leq s\delta(\dot{u},u_{n+1})+s\delta(u_{n+1},\mathscr{A}\dot{u});\\ &\leq s\delta(\dot{u},u_{n+1})+s\frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(\dot{u},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}\max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.\end{aligned}$$

Taking the upper limit on both sides of (10), we obtain,

$$\delta(\dot{u},\mathscr{A}\dot{u}) \leq \frac{sa[\limsup_{n\to\infty}\delta(u_n,\mathscr{A}\dot{u})]^2}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}.$$

Lemma 1 yields,

$$\delta(\dot{u},\mathscr{A}\dot{u}) \leq \frac{s^2a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u})^2,$$

thus,

$$1 \leq \frac{s^2a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u}). \tag{11}$$

Since, $s^2a \leq \frac{1}{2}\min\{c,f\}$, then, $s^2a \leq f$, then, $s^2a\delta(\dot{u},\mathscr{A}\dot{u}) < f\delta(\dot{u},\mathscr{A}\dot{u})+e$ which contradict inequality (11). Then, $\delta(\dot{u},\mathscr{A}\dot{u})=0$ that mean, $\mathscr{A}\dot{u}=\dot{u}$.

Case 2 If $s^3b \leq \frac{1}{2}\min\{c,f\}$ holds, by substituting $u=\dot{u}$ and $v=u_n$ into inequality (8), we find,

$$\delta(\mathscr{A}\dot{u},u_{n+1}) \leq \frac{a\delta(\dot{u},u_{n+1})+b\delta(u_n,\mathscr{A}\dot{u})}{c\delta(\dot{u},\mathscr{A}\dot{u})+f\delta(u_n,u_{n+1})+e}\max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.$$

Parallel to Case 1, we conclude that $\mathscr{A}\dot{u}=\dot{u}$.

Claim 3: Assume that $\mathscr{A}$ have two distinct fixed points $\dot{u},\dot{v}$ in Ω, By substituting $u=\dot{u}$, $v=\dot{v}$ into inequality (8), we find,

$$\begin{aligned}\delta(\dot{u},\dot{v}) &\leq \frac{a\delta(\dot{u},\dot{v})+b\delta(\dot{v},\dot{u})}{c\delta(\dot{u},\dot{u})+f\delta(\dot{v},\dot{v})+e}\delta(\dot{u},\dot{v});\\ &\leq \frac{(a+b)\delta(\dot{u},\dot{v})}{e}\delta(\dot{u},\dot{v}).\end{aligned}$$

Thus, $\delta(\dot{u},\dot{v}) \geq \dfrac{e}{a+b}$. □

We can combine the two previous theorems to get the following result.

Theorem 3. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω, δ, s). Assume there are five constants a, b, c, f and $e \in \mathbb{R}^+$ that ensure either $s^3 a \leq \frac{1}{2}\min\{c, f\}$ or $s^3 b \leq \frac{1}{2}\min\{c, f\}$ and for all $u, v \in \Omega$,*

$$\begin{aligned} &\delta(\mathscr{A}u, \mathscr{A}v) \\ &\leq \frac{a\delta(u, \mathscr{A}v) + b\delta(v, \mathscr{A}u)}{c\delta(u, \mathscr{A}u) + f\delta(v, \mathscr{A}v) + e} \max\{\delta(x, \mathscr{A}v), \delta(v, \mathscr{A}u), \delta(u, \mathscr{A}u), \delta(v, \mathscr{A}v), \delta(u, v)\}. \end{aligned} \tag{12}$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (13):*

$$\delta(\dot{u}, \dot{v}) \geq \frac{e}{a+b}. \tag{13}$$

Proof. Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1} = \mathscr{A}u_n$, initiated from a random element u_0 in Ω. If there is an integer n_0 ensuring that $u_{n_0} = u_{n_0+1}$, then u_{n_0} represents a fixed point for the self-map $\mathscr{A}$, this concludes the proof.

However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: We will show that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1 If $s^3 a \leq \frac{1}{2}\min\{c, f\}$ holds, by substituting $u = u_{n-1}$ and $v = u_n$ into inequality (12), we find,

$$\begin{aligned} &\delta(u_n, u_{n+1}) \\ &\leq \frac{a\delta(u_{n-1}, u_{n+1})}{c\delta(u_{n-1}, u_n) + f\delta(u_n, u_{n+1}) + e} \max\{\delta(u_{n-1}, u_{n+1}), \delta(u_{n-1}, u_n), \delta(u_n, u_{n+1}), \delta(u_{n-1}, u_n)\} \\ &\leq \frac{s^2 a\delta(u_{n-1}, u_n) + s^2 a\delta(u_n, u_{n+1})}{c\delta(u_{n-1}, u_n) + f\delta(u_n, u_{n+1}) + e}[\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})] \\ &\leq \frac{s^2 a\delta(u_{n-1}, u_n) + s^2 a\delta(u_n, u_{n+1})}{\min\{c; f\}\big(\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})\big) + e}[\delta(u_{n-1}, u_n) + \delta(u_n, u_{n+1})]. \end{aligned}$$

We define the sequence (θ_n) with the general term

$$\theta_n = \frac{s^2 a\delta(u_{n-1},u_n) + s^2 a\delta(u_n,u_{n+1})}{\min\{c;f\}\big(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1})\big)+e},$$

for all $n\in\mathbb{N}$. Given that $s^3 a \leq \frac{1}{2}\min\{c,f\}$, it follows that, $0\leq \theta_n < \frac{1}{2s}$ for all $n\in\mathbb{N}$.
On the other hand, we have,

$$\delta(u_n,u_{n+1}) \leq \theta_n[\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1})].$$

Hence,

$$\delta(u_n,u_{n+1}) \leq \frac{\theta_n}{1-\theta_n}\delta(u_{n-1},u_n).$$

We denote that $\lambda_n = \dfrac{\theta_n}{1-\theta_n}$ for all $n\in\mathbb{N}$.
Since $0\leq\theta_n<\frac{1}{2s}$ for all $n\in\mathbb{N}$, then, $0\leq\lambda_n<\frac{1}{s}$, consequently, $\delta(u_n,u_{n+1})<\delta(u_{n-1},u_n)$ for all $n\in\mathbb{N}$, then, $\delta(u_{n+1},u_{n+2})<\delta(u_{n-1},u_n)$.

Moreover, for every $n\in\mathbb{N}$,

$$\begin{aligned}\theta_{n+1}-\theta_n &= \frac{s^2 ae\big[\delta(u_{n+1},u_{n+2})-\delta(u_{n-1},u_n)\big]}{\big[\min\{c;f\}(\delta(u_n,u_{n+1})+\delta(u_{n+1},u_{n+2}))+e\big]}\\ &\quad\times\frac{1}{\big[\min\{c;f\}(\delta(u_{n-1},u_n)+\delta(u_n,u_{n+1}))+e\big]}\\ &< 0.\end{aligned}$$

This indicates that the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing. Consequently, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is also decreasing, thus,

$$\begin{aligned}\delta(u_n,u_{n+1}) &\leq \lambda_n\delta(u_{n-1},u_n);\\ &\leq \lambda_n\lambda_{n-1}\delta(u_{n-2},u_{n-1});\\ &\vdots\\ &\leq \lambda_n\lambda_{n-1}\cdots\lambda_1\delta(u_0,u_1);\\ &\leq \lambda_1^n\delta(u_0,u_1).\end{aligned}$$

For all $n,m\in\mathbb{N}$ where $m>n$, we have,

$$\delta(u_n,u_m) \leq \sum_{i=n}^{m-1} s^{i-n+1}\delta(u_i,u_{i+1});$$

$$\leq \sum_{i=n}^{m-1} s^{i-n+1}\theta_1^i \delta(u_0,u_1);$$
$$\leq \frac{(s\theta_1)^n-(s\theta_1)^m}{1-s\theta_1}\times\frac{1}{s^{n-1}}\delta(u_0,u_1).$$

For large enough n for the above inequality, we conclude,

$$\lim_{n,m\to+\infty}\delta(u_n,u_m)=0,$$

thus, the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω.

Case 2 If $s^3b\leq\frac{1}{2}\min\{c,f\}$ holds, by substituting $u=u_n$ and $v=u_{n-1}$ into inequality (12), we find,

$$\delta(u_n,u_{n+1})\leq\frac{b\delta(u_{n-1},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(u_{n-1},u_n)+e}\max\{\delta(u_{n-1},u_{n+1}),\delta(u_n,u_{n+1}),\delta(u_{n-1},u_n)\}.$$

Parallel to Case 1, we conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω.

Taking into account that the b-metric space (Ω,δ,s) is complete, then, the sequence (u_n) converges to an element $\dot{u}$ in Ω.

Claim 2: We will confirm that $\dot{u}$ is a fixed point of $\mathscr{A}$.

Assume that $\delta(\dot{u},\mathscr{A}\dot{u})>0$.

Case 1 If $s^3a\leq\frac{1}{2}\min\{c,f\}$ holds, by substituting $u=u_n$, $v=\dot{u}$ into inequality (12), we find,

$$\delta(\mathscr{A}\dot{u},u_{n+1})\leq\frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(\dot{u},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}$$
$$\times\max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,u_{n+1}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.$$

On the other hand, we have,

$$\begin{aligned}\delta(\dot{u},\mathscr{A}\dot{u}) &\leq s\delta(\dot{u},u_{n+1})+s\delta(u_{n+1},\mathscr{A}\dot{u});\\ &\leq s\delta(\dot{u},u_{n+1})+s\frac{a\delta(u_n,\mathscr{A}\dot{u})+b\delta(\dot{u},u_{n+1})}{c\delta(u_n,u_{n+1})+f\delta(\dot{u},\mathscr{A}\dot{u})+e}\\ &\times\max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,u_{n+1}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.\end{aligned}\tag{14}$$

Taking the upper limit on both sides of (14), we obtain,

$$\delta(\dot{u},\mathscr{A}\dot{u})\leq\frac{sa\limsup_{n\to\infty}\delta(u_n,\mathscr{A}\dot{u})}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\max\{\limsup_{n\to\infty}\delta(u_n,\mathscr{A}\dot{u}),\delta(\dot{u},\mathscr{A}\dot{u})\}.$$

Lemma 1 yields,

$$\delta(\dot{u},\mathscr{A}\dot{u})\leq\frac{s^2a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u})^2,$$

thus,

$$1 \leq \frac{s^2 a}{f\delta(\dot{u},\mathscr{A}\dot{u})+e}\delta(\dot{u},\mathscr{A}\dot{u}). \tag{15}$$

Since, $s^2a \leq \frac{1}{2}\min\{c,f\}$, then $s^2a \leq f$, then $s^2a\delta(\dot{u},\mathscr{A}\dot{u}) < f\delta(\dot{u},\mathscr{A}\dot{u})+e$ which contradict inequality (15). Then $\delta(\dot{u},\mathscr{A}\dot{u}) = 0$, that mean, $\mathscr{A}\dot{u} = \dot{u}$.

Case 2 If $s^3b \leq \frac{1}{2}\min\{c,f\}$ holds, by substituting $u = \dot{u}$ and $v = u_n$ into inequality (12), we find,

$$\delta(\mathscr{A}\dot{u},u_{n+1}) \leq \frac{a\delta(\dot{u},u_{n+1})+b\delta(u_n,\mathscr{A}\dot{u})}{c\delta(\dot{u},\mathscr{A}\dot{u})+f\delta(u_n,u_{n+1})+e} \times \max\{\delta(\dot{u},u_{n+1}),\delta(u_n,\mathscr{A}\dot{u}),\delta(u_n,u_{n+1}),\delta(\dot{u},\mathscr{A}\dot{u}),\delta(u_n,\dot{u})\}.$$

Parallel to Case 1, we conclude that $\mathscr{A}\dot{u} = \dot{u}$.

Claim 3: Assume that $\mathscr{A}$ have two distinct fixed points $\dot{u},\dot{v}$ in Ω and we find the distance between these two fixed points.

By substituting $u = \dot{u}$, $v = \dot{v}$ into inequality (12), we find,

$$\begin{aligned} \delta(\dot{u},\dot{v}) &\leq \frac{a\delta(\dot{u},\dot{v})+b\delta(\dot{v},\dot{u})}{c\delta(\dot{u},\dot{u})+f\delta(\dot{v},\dot{v})+e}\delta(\dot{u},\dot{v}); \\ &\leq \frac{(a+b)\delta(\dot{u},\dot{v})}{e}\delta(\dot{u},\dot{v}). \end{aligned}$$

Thus, $\delta(\dot{u},\dot{v}) \geq \dfrac{e}{a+b}$. □

We can improve this result to the following forms:

Theorem 4. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants a,b,c,f and $e \in \mathbb{R}^+$ that ensure either $s^3a \leq \frac{1}{2}\min\{c,f\}$ or $s^3b \leq \frac{1}{2}\min\{c,f\}$ and for all $u,v \in \Omega$,*

$$\delta(\mathscr{A}u,\mathscr{A}v) \leq \frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e} \times \max\{\delta(u,\mathscr{A}v),\delta(v,\mathscr{A}u),\delta(u,\mathscr{A}u)+\delta(v,\mathscr{A}v),\delta(u,v)\}. \tag{16}$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*

3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality* (17)*:*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{a+b}. \tag{17}$$

Proof. The procedure for proving the above theorem is the same as for proving Theorem3. □

Theorem 5. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants a,b,c,f and $e \in \mathbb{R}^+$ that ensure either $s^3a \leq \frac{1}{2}\min\{c,f\}$ or $s^3b \leq \frac{1}{2}\min\{c,f\}$ and for all $u,v \in \Omega$,*

$$\begin{aligned}\delta(\mathscr{A}u,\mathscr{A}v) \leq &\frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e}\\ &\times\max\{\delta(u,\mathscr{A}v)+\delta(v,\mathscr{A}u),\delta(u,\mathscr{A}u),\delta(v,\mathscr{A}v),\delta(u,v)\}.\end{aligned} \tag{18}$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality* (19)*:*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{2(a+b)}. \tag{19}$$

Proof. The procedure for proving the above theorem is the same as for proving Theorem3. □

Theorem 6. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants a,b,c,f and $e \in \mathbb{R}^+$ that ensure either $s^3a \leq \frac{1}{2}\min\{c,f\}$ or $s^3b \leq \frac{1}{2}\min\{c,f\}$ and for all $u,v \in \Omega$,*

$$\begin{aligned}\delta(\mathscr{A}u,\mathscr{A}v) \leq &\frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e}\\ &\times\max\{\delta(u,\mathscr{A}v)+\delta(v,\mathscr{A}u),\delta(u,\mathscr{A}u)+\delta(v,\mathscr{A}v),\delta(u,v)\}.\end{aligned} \tag{20}$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*

3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality* (21)*:*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{2(a+b)}. \tag{21}$$

Proof. The procedure for proving the above theorem is the same as for proving Theorem3. □

Theorem 7. *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Ω,δ,s). Assume there are five constants a,b,c,f and $e \in \mathbb{R}^+$ that ensure either $s^2a \leq \frac{1}{5}\min\{c,f\}$ or $s^2b \leq \frac{1}{5}\min\{c,f\}$ and for all $u,v \in \Omega$,*

$$\begin{aligned}\delta(\mathscr{A}u,\mathscr{A}v) \leq \frac{a\delta(u,\mathscr{A}v)+b\delta(v,\mathscr{A}u)}{c\delta(u,\mathscr{A}u)+f\delta(v,\mathscr{A}v)+e} \\ \times[\delta(u,\mathscr{A}v)+\delta(v,\mathscr{A}u)+\delta(u,\mathscr{A}u)+\delta(v,\mathscr{A}v)+\delta(u,v)].\end{aligned} \tag{22}$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{u}$ within the space Ω for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality* (23)*:*

$$\delta(\dot{u},\dot{v}) \geq \frac{e}{3(a+b)}. \tag{23}$$

Proof. The procedure for proving the above theorem is the same as for proving Theorem3. □

Rem 2. *If rang$\mathscr{A}$ is a closed sub set of Ω, the inequalities* (4),(8)(12), (16), (18), (20) *and* (22) *can be restricted for all $u,v \in$ rang$\mathscr{A}$, and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.*

4. Illustrative Examples

The subsequent example explains and affirms Theorem 1.

Example 1. *Let $\Omega = \{0,1,2\}$ be a set equipped with a b-metric δ defined as follows: $\delta(0,1) = 0.2$, $\delta(0,2) = 1$ and $\delta(1,2) = 1.5$. Additionally, $\delta(u,v) = \delta(v,u)$ and $\delta(u,u) = 0$ for every $u,v \in \Omega$.*

Consider a self-map $\mathscr{A}$ acting on Ω defined by $\mathscr{A}(0) = 0$, $\mathscr{A}(1) = 0$ and $\mathscr{A}(2) = 2$.

It may be readily inferred that the b-metric space (Ω, δ) *is a complete b-metric space with the constant* $s = \sqrt{2}$*. Over more, the inequality* (4) *has been hold for every* $u, v \in \Omega$*, with constant* $a = 2$*,* $b = 1$*,* $c = f = 2$ *and* $e = 3$*.*

Theorem 1 leads us to the conclusion that $\mathscr{A}$ *has a minimum of one fixed point. (precisely,* $\mathscr{A}$ *possesses two distinct fixed points* 2 *and* 0*). Furthermore, it can be observed that the distance separating each other, denoted as* $\delta(0,2)$*, is greater than or equal to one.*

The following examples illustrates and supports Theorem 2.

Example 2. *We can process the previous example by taking the constants* $a = 2$*,* $b = 1$*,* $c = f = 10$ *and* $e = 3$*, so that equation* (8) *is fulfilled, and according to Theorem 2, the mapping* $\mathscr{A}$ *accepts at least one fixed point.(precisely,* $\mathscr{A}$ *possesses two distinct fixed points* 2 *and* 0*). Furthermore, it can be observed that the distance separating each other, denoted as* $\delta(0,2)$*, is greater than or equal to one.*

Example 3. *Let* $\Omega = [0;0.2] \cup [1;2]$ *associated with a b-metric* δ *such that* $\delta(u,v) = (u-v)^2$ *for all* $u, v \in \Omega$*.*

Let $\mathscr{A} : \Omega \to \Omega$ *be a self mapping given by,*

$$\mathscr{A}u = \begin{cases} 0 & \text{if } u \in [0;0.2], \\ 1 & \text{if } u \in [1;2]. \end{cases}$$

It is easy to conclude that (Ω, δ) *is a complete b-metric space with the constant* $s = 2$*.*

Let $u, v \in \Omega$ *and denote by m the function,*

$$m(u,v) = -\delta(\mathscr{A}u, \mathscr{A}v) + \frac{\delta(u, \mathscr{A}v) + \delta(v, \mathscr{A}u)}{16\delta(u, \mathscr{A}u) + 16\delta(v, \mathscr{A}v) + 1} \max\{\delta(u, \mathscr{A}v), \delta(v, \mathscr{A}u), \delta(u,v)\}.$$

If $u = v$*, the equation is obviously verified;*

If $u, v \in [0;0.2]$ *or* $u, v \in [1;2]$*, the inequality is obviously verified;*

If $u \in [0;0.2]$ *and* $v \in [1;2]$ *or* $u \in [1;2]$ *and* $v \in [0;0.2]$*, We plot the graph of the function m over this domain.*

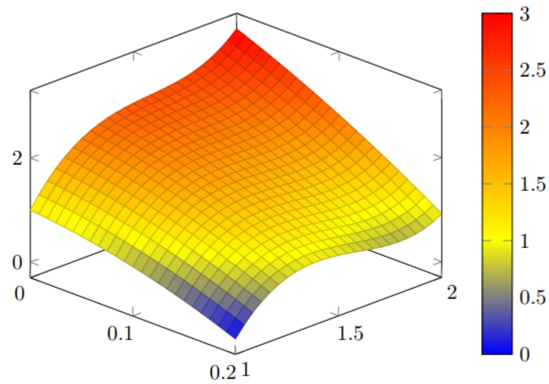

FIGURE 1. Plot of the function m.

It is worth noting that the inequality (8) *holds true on this domain for the constants* $a = b = 1$, $c = f = 16$, *and* $e = 1$. *By applying Theorem 2, we can deduce that* $\mathscr{A}$ *possesses no less than one fixed point. The function has precisely two fixed points, either* 0 *and* 1. *Furthermore, it can be observed that the distance separating each other, denoted as* $\delta(0,1)$, *is greater than or equal to one-half.*

The following example illustrates and supports Theorem 3.

Example 4. *Let* $\Omega = \{\alpha, \beta, \gamma\}$ *be a set equipped with a b-metric* δ *defined as follows:* $\delta(\alpha, \beta) = 0.2$, $\delta(\alpha, \gamma) = 1$ *and* $\delta(\beta, \gamma) = 1.5$. *Additionally,* $\delta(u, v) = \delta(v, u)$ *and* $\delta(u, u) = 0$ *for all* $u, v \in \Omega$.

Consider a self-map $\mathscr{A}$ *acting on* Ω *defined by* $\mathscr{A}(\alpha) = \alpha$, $\mathscr{A}(\beta) = \beta$ *and* $\mathscr{A}(\gamma) = \alpha$.

It may be readily inferred that the b-metric space (Ω, δ) *is a complete b-metric space with the constant* $s =^3 \sqrt{2}$. *Over more, the inequality* (12) *has been hold for every* $u, v \in \Omega$, *with constant* $a = 2$, $b = 1$, $c = f = 4$ *and* $e = 0.6$.

Theorem 3 leads us to the conclusion that $\mathscr{A}$ *has a minimum of one fixed point. (precisely,* $\mathscr{A}$ *possesses two distinct fixed points* α *and* β*). Furthermore, it can be observed that the distance separating each other, denoted as* $\delta(\alpha, \beta)$, *is greater than or equal to* 0.2.

5. Application to Coupled Fixed Point Theory

Theorem 8. *Let (Ω,δ,s) be a complete b-metric space and let $\mathscr{A}:\Omega\times\Omega\longrightarrow\Omega$ be a mapping. Assume there exist five positive real number $a,b,c,f,\ e\in\mathbb{R}^+$, such that $s^2a\leq\min\{c,f\}$ or $s^2b\leq\min\{c,f\}$ and for all $(x_1,y_1),(x_2,y_2)\in\Omega\times\Omega$,*

$$\begin{aligned}&\delta(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2))\\ &\leq\frac{a\delta(x_1,\mathscr{A}(x_2,y_2))+b\delta(y_2,\mathscr{A}(y_1,x_1))}{c\delta(x_1,\mathscr{A}(x_1,y_1))+c\delta(y_1,\mathscr{A}(y_1,x_1))+f\delta(x_2,\mathscr{A}(x_2,y_2))+f\delta(y_2,\mathscr{A}(y_2,x_2))+e}\times\\ &\max\{\delta(x_1,x_2),\delta(y_1,y_2),\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}\end{aligned}\tag{24}$$

Then, $\mathscr{A}$ has at least one coupled fixed point $(\dot{x},\dot{y})\in\Omega\times\Omega$, over more, $\delta(\dot{x},\dot{y})=0$ or $\delta(\dot{x},\dot{y})\geq\dfrac{e}{a+b}$.

Proof. Let (Ω,δ,s) be a complete b-metric space, then, $(\Omega\times\Omega,\delta,s)$ is also a complete b-metric space where δ is a b-metric distance defined on $\Omega\times\Omega$, as follow,

$$\delta((x_1,y_1),(x_2,y_2))=\max\{\delta(x_1,x_2),\delta(y_1,y_2)\}$$

Let $\mathscr{A}:\Omega\times\Omega\longrightarrow\Omega$ be a mapping. We note by F the self mapping $F:\Omega\times\Omega\longrightarrow\Omega\times\Omega$ defined as follow, $F(x,y)=(\mathscr{A}(x,y),\mathscr{A}(y,x))$ for all couple $(x,y)\in\Omega\times\Omega$.

According to inequality (24), we have,

$$\begin{aligned}&\delta(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2))\leq\\ &\frac{a\max\{\delta(x_1,\mathscr{A}(x_2,y_2)),\delta(y_1,\mathscr{A}(y_2,x_2)\}+b\max\{\delta(y_2,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_1,y_1))\}}{c\max\{\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1))\}+f\max\{\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}+e}\\ &\max\{\delta(x_1,x_2),\delta(y_1,y_2),\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}\end{aligned}\tag{25}$$

and

$$\begin{aligned}&\delta(\mathscr{A}(y_1,x_1),\mathscr{A}(y_2,x_2))\leq\\ &\frac{a\max\{\delta(x_1,\mathscr{A}(x_2,y_2)),\delta(y_1,\mathscr{A}(y_2,x_2)\}+b\max\{\delta(y_2,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_1,y_1))\}}{c\max\{\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1))\}+f\max\{\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}+e}\\ &\max\{\delta(x_1,x_2),\delta(y_1,y_2),\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}\end{aligned}\tag{26}$$

from inequalities (25) and (26), we get

$$\begin{aligned}&\max\{\delta(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2)),\delta(\mathscr{A}(y_1,x_1),\mathscr{A}(y_2,x_2)))\}\leq\\ &\frac{a\max\{\delta(x_1,\mathscr{A}(x_2,y_2)),\delta(y_1,\mathscr{A}(y_2,x_2)\}+b\max\{\delta(y_2,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_1,y_1))\}}{c\max\{\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1))\}+f\max\{\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}+e}\end{aligned}$$

$$\max\{\delta(x_1,x_2),\delta(y_1,y_2),\delta(x_1,\mathscr{A}(x_1,y_1)),\delta(y_1,\mathscr{A}(y_1,x_1)),\delta(x_2,\mathscr{A}(x_2,y_2)),\delta(y_2,\mathscr{A}(y_2,x_2))\}$$

that mean,

$$\delta(Fu,Fv) \leq \frac{a\delta(u,Fv)+b\delta(v,Fu)}{c\delta(u,Fu)+f\delta(v,Fv)+e}\max\{\delta(u,v),\delta(u,Fu),\delta(v,Fv)\},$$

according Theorem 1, we conclude that F has at least one fixed point, that mean, $\mathscr{A}$ has a coupled fixed point at least noted by $(\dot{x},\dot{y}) \in \Omega\times\Omega$.

Since $(\dot{y},\dot{x})$ is also a coupled fixed point for $\mathscr{A}$, then, by choosing $(x_1,y_1)=(\dot{x},\dot{y})$ and $(x_2,y_2)=(\dot{y},\dot{x})$ in inequality (24), we have,

$$\delta(\dot{x},\dot{y}) \leq \frac{a+b}{e}d^2(\dot{x},\dot{y}),$$

that mean, $\delta(\dot{x},\dot{y})=0$ or $\delta(\dot{x},\dot{y}) \geq \frac{e}{a+b}$. This complete the proof of our theorem. □

Rem 3. *If rang*$\mathscr{A}$ *is a closed sub set of* Ω*, the inequality* (24) *can be restricted to rang*$\mathscr{A}\times$ *rang*$\mathscr{A}$*, and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.*

In order to prove the possibility of fulfilling the conditions of the previous theory, we provide the following example.

Example 5. *Let* $\Omega=\{0,1,2\}$ *associated with a b-metric* δ *such that* $\delta(x,y)=(x-y)^2$*. Let* $\mathscr{A}$ *be a mapping defined in* $\Omega\times\Omega$ *such that*

$$\mathscr{A}(x,y)=\begin{cases} 0 & if\, x=0 or\, x=2\ or\, y=2, \\ 1 & otherwise\ . \end{cases}$$

It is easy to conclude that (Ω,δ,s) *is a complete b-metric space with the constant* $s=2$ *and the inequality* (24) *was verified for all* $(x_1,y_1),(x_2,y_2)\in$ *rang*$\mathscr{A}\times$*rang*$\mathscr{A}$ *with constant* $a=b=e=1$ *and* $c=f=4$*. According to Theorem 8, we conclude that* $\mathscr{A}$ *has at least one coupled fixed point* $(\dot{x},\dot{y})\in\Omega\times\Omega$*. (exactly, it has four coupled fixed point* $(0,0)$*,* $(1,0)$*,* $(0,1)$ *and* $(1,1)$*). Moreover, the distance between each couple is* $\delta(0,0)=\delta(1,1)=0$ *and* $\delta(0,1)=\delta(1,0)\geq\frac{1}{2}$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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