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A NEW FIXED POINT APPROXIMATION METHOD FOR SOLVING THIRD-ORDER BVPs BASED ON GREEN'S FUNCTION

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Abstract. This study presents an interesting method based on Picard-Ishikwa fixed point iterative method to solve nonlinear third-order boundary value problems. We develop a sequence called Picrad-Ishikawa Green's iterative method and show that the sequence converges strongly to the fixed point of an integral operator. Our result improve many existing results in this direction.

Keywords: Picard-Ishikawa; fixed point; Green function; convergence.

2020 AMS Subject Classification: 47H10.

1. Introduction

The concept of fixed point theory is one of the vital tools of modern mathematical analysis and possesses several applications in different fields such as physics, mathematical engineering, approximation theory, game theory, economics, optimization theory, biology, chemistry, optimization theory and so on. A point c in a nonempty closed subset D of a Banach space B is called a fixed point of the mapping $T : D \to D$ if $Tc = c$. In recent years, many results have

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published in fixed point theory (see, for example [18, 19, 20, 21, 22, 35, 36, 37, 29, 30, 31] and the references in them).

Fixed point of nonlinear operators can be approximated Since the location via iteration methods. There exist several iterative methods in the literature. Some of these method include Picard [27] iteration, Krasnosel'kii [16] iteration, Mann [23] iteration, Ishikawa [6] iteration, Noor [24] iteration, Picard-Man [7] iteration, Picard-krasnosel'kii [25] iteration, Abbas [1] iteration, Agarwal [2] iteration, Thakur [32] iteration and Picard-Ishikawa iteration [26].

Very recently, Okeke [26] introduced the Picard-Ishikawa iterative method as follows:

$$\begin{cases} m_0 \in D, \\ v_k = (1-q_k)m_k + q_k T m_k, \\ u_k = (1-p_k)m_k + p_k T v_k, \\ m_{k+1} = T u_k, \end{cases} \quad k \in \mathbb{N}, \tag{1}$$

where $\{q_k\}$ and $\{p_k\}$ are sequences in [0,1]. The author showed that Picard-Ishikawa iterative method (1) converges faster than all of Picard, Krasnosel'skii, Mann, Ishikawa, Noor, Picard-Mann and Picard-Krasnosel'skii iterative methods.

A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions.

Boundary value problems emanate in many branches of physics as any physical differential equation will have them. Problems involving wave equation, are in most cases expressed as boundary value problems. Several important boundary value problem include; storm-Louisville problems.

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, or gravity driven flows. Third-order boundary value problems were discussed in many papers in recent years, for instance, see [4, 5, 8, 17, 28, 33, 34] and references therein.

Solutions of boundary value problems can sufficiently be approximated by some efficient numerical methods. Some of these numerical methods are finite difference method, standard 5-point formula, standard analytic method and iterative method.

In recent years, several fixed point iterative methods based on Green's function have been developed for solving second and third order nonlinear boundary value problems. Some of these methods are: Picard–Green's, Krasnoselskii-Green's, Mann-Green's, Ishikawa Green's and Khan–Green's iterative methods, e.g. see [3, 9, 10, 11, 12, 13, 14, 15] and the references therein.

Very recently, Khuri and Louhichi [38] presented a fascinating approach which is based on embedding Green's function into Ishikawa fixed point iterative method for solutions nonlinear third-order boundary value problems.

Motivated by the above results, in this study, we develop a strategy based on Picard-Ishikawa fixed point iterative method (1) to solve a nonlinear third order BVPs in a form of Green's function with boundary value problems. Our iterative method outperforms several well known iterative methods existing in the literature.

1.1. Brief demonstration of Green's functions. Consider the following general linear third-order BVPs,

$$Li[\eta] = g(\psi)\eta''' + w(\psi)\eta'' + z(\psi)\eta' + r(\psi)\eta = \varphi(\psi), \tag{2}$$

where $a \leq \psi \leq b$ and subject to the boundary conditions:

$$\begin{aligned} B_1[\eta] &= \mu_1\eta(a) + \mu_2\eta'(a) + \mu_3\eta''(a) = \mu, \\ B_2[\eta] &= \zeta_1\eta(b) + \zeta_2(b)\eta'(b) + \zeta_3\eta''(b) = \zeta, \\ B_3[\eta] &= \tau_1\eta(c) + \tau_2\eta'(c) + \tau_3\eta''(c) = \tau, \end{aligned} \tag{3}$$

where $c = a$ or b.

The Green's function is defined to be the solution for the following equation

$$-Li[G(\psi,s)] = \delta(\psi - s) \tag{4}$$

where δ is the Kronecker Delta which is subject to $B_1[G(\psi,s)] = B_2[G(\psi,s)] = B_3[G(\psi,s)] = 0$. It is worth mentioning that for operators that are not self adjoint, we replace the right hand side of (4) by $-\delta(\psi - s)$. For $\psi \neq s$, we solve $Li[G(\psi,s)] = 0$ and get

$$G(\psi,s) = \begin{cases} e_1\eta_1 + e_2\eta_2 + e_3\eta_3, & a < \psi < s, \\ d_1\eta_1 + d_2\eta_2 + d_3\eta_3, & s < \psi < b, \end{cases}$$

where η_1, η_2, η_3 are linearly independent solutions of $Li[\eta] = 0$ and the constants are derived through the following properties:

(V_1) G satisfies the associated homogeneous boundary conditions

$$B_1[G(\psi,s)] = B_2[G(\psi,s)] = 0, \tag{5}$$

(V_2) Continuity of G at $\psi = s$

$$e_1\eta_1(s) + e_2\eta_2(s) + e_3\eta_3(s) = d_1\eta_1(s) + d_2\eta_2(s) + d_3\eta_3(s), \tag{6}$$

(V_3) Continuity of G' at $\psi = s$:

$$e_1\eta_1'(s) + e_2\eta_2'(s) + e_3\eta_3'(s) = d_1\eta_1'(s) + d_2\eta_2'(s) + d_3\eta_3'(s), \tag{7}$$

(V_4) At $\psi = s$, G'' has a jump discontinuity:

$$\frac{1}{g(s)} + e_1\eta_1''(s) + e_2\eta_2''(s) + e_3\eta_3''(s) = d_1\eta_1''(s) + d_2\eta_2''(s) + d_3\eta_3''(s). \tag{8}$$

1.2. Picard-Ishikawa Green's fixed point iterative method.

Applying the Green's function to Picard-Ishikawa iterative method (1), the following differential equation will be considered:

$$Li[\rho] + No[\rho] = \varphi(\psi,\rho), \tag{9}$$

where $Li[\rho]$ and $No[\rho]$ are linear and nonlinear operators in ρ, respectively, and $\varphi(\psi,\rho)$ is a function in ρ which could be linear or nonlinear.

We now define the following linear integral operator in terms of Green's function as follows:

$$\Psi[\rho] = \int_a^b G(\psi,s)ds, \tag{10}$$

where G is the Green's function that is corresponding to the linear differential operator $Li[\rho]$. Observe that Ψ has a fixed point if and only if ρ is a solution of (9).

From (10), we have the following:

$$\begin{aligned}
\Psi[\rho] &= \int_a^b G(\psi,s)[Li[\rho] + No[\rho] - \varphi(s,\rho) - No[\rho] + \varphi(s,\rho)]ds \\
&= \int_a^b G(\psi,s)(Li[\rho] + No[\rho] - \varphi(\psi,s))ds + \int_a^b G(\psi,s)(\varphi(s,\rho) - No[\rho])ds
\end{aligned}$$

$$= \quad \rho + \int_a^b G(\psi,s)(Li[\rho]+No[\rho]-\varphi(s,\rho))ds.$$

Now, by applying the Picard-Ishikawa fixed point iterative method (1), we obtain

$$\begin{cases} v_k = (1-q_k)m_k + q_k\Psi[m_k], \\ u_k = (1-p_k)m_k + p_k\Psi[v_k], \\ f_{k+1} = \Psi[u_k], \end{cases} \tag{11}$$

where $\{q_k\}$ and $\{p_k\}$ are sequences in [0,1]. Then for all $k \in \mathbb{N}$, this results to

$$\begin{cases} v_k = (1-q_k)m_k + q_k[f_m + \int_a^b G(\psi,s)(Li[m_k]+No[m_k]-\varphi(s,m_k))ds], \\ u_k = (1-p_k)m_k + p_k[v_k + \int_a^b G(\psi,s)(Li[v_k]+No[v_k]-\varphi(s,\rho))ds], \\ m_{k+1} = u_k + \int_a^b G(\psi,s)(Li[u_k]+No[u_k]-\varphi(s,u_k))ds, \end{cases} \tag{12}$$

Thus, we have

$$\begin{cases} v_k = m_k + q_k \int_a^b G(\psi,s)(L_i[m_k]+N_o[m_k]-\varphi(s,m_k))ds, \\ u_k = (1-p_k)m_k + p_k[v_k + \int_a^b G(\psi,s)(Li[v_k]+No[v_k]-\varphi(s,v_k))ds], \\ m_{k+1} = u_k + \int_a^b G(\psi,s)(Li[u_k]+No[u_k]-\varphi(s,u_k))ds, \end{cases} \tag{13}$$

Remark 1.1. *The new iteration method* (13) *is independent of all Picard-Green iteration, Mann-Green iteration, Ishikawa-Green iteration, Khan-Green iteration methods which are well konwn methods in the literature.*

2. Convergence Analysis

In this section, we prove the convergence theorem of the proposed iterative method (13). Also, we show that our new method (13) convergence at a rate faster than the fixed point iterative methods based on Green's function. Without loss of generality, we will consider the convergence analysis of our method for the following nonlinear BVP:

$$-\eta^{'''(\psi)} = \varphi(\eta(\psi),\eta'(\psi),\eta''(\psi)), \text{ subject to } x(1)=Q, x''(1)=P, x(2)=W. \tag{14}$$

Solving the associated homogeneous equation $\eta^{'''} = 0$ implies

$$G(\psi,s) = \begin{cases} e_1t^2 + e_2t + e_3, \ 1 \le \psi \le s \le 2 \\ d_1t^2 + d_2t + d_3, \ 1 \le s \le \psi \le 2 \end{cases} \tag{15}$$

The unknowns e_1, e_2, e_3, d_1, d_2 and d_3 can be obtained using the properties $(V_1) - -(V_4)$. After finding the unknowns, we then have the following Green's function.

$$G(\psi, s) = \begin{cases} -\frac{1}{2}s_2 + 2s - 2 + \left(\frac{1}{2}s^2 - 2s + 2\right)t, & \text{if } 1 \leq \psi \leq s \leq 2 \\ \\ -s^2 + 2s - 2 + \left(\frac{1}{2}s^2 - 2s + 2\right)t - \frac{1}{2}t^2, & \text{if } 1 \leq s \leq \psi \leq 2. \end{cases}$$

Thus, Picard-Ishikawa Green's iteration process (13) now have the form

$$\begin{cases} v_k = m_k + q_k \int_1^2 G(\psi, s)(m_k''' - \varphi(s, m_k, m_k', m_k''))ds, \\ u_k = (1 - p_k)m_k + p_k[v_k + \int_1^2 G(\psi, s)(v_k''' - \varphi(s, v_k, v_k', v_k''))ds], \quad k \in \mathbb{N}, \\ m_{k+1} = u_k + \int_1^2 G(\psi, s)(u_k''' - \varphi(s, u_k, u_k', u_k''))ds, \end{cases} \tag{16}$$

where the initial iterate f_0 fulfilled the corresponding equation $m''' = 0$ and the boundary conditions $m_0(1) = Q$, $m_0''(1) = P$ and $m_0(2) = W$. Next, we define the operator $\Upsilon : C^2([1,2]) \to C^2([1,2])$ by

$$\Upsilon_G(m) = m + \int_1^2 G(\psi, s)(m^{(3)} - \varphi(s, m, m', m''))ds. \tag{17}$$

Then (16) reduces to the following form

$$\begin{cases} v_m = (1 - q_k)m_k + q_k \Upsilon_G(m_k), \\ u_k = (1 - p_k)m_k + p_k \Upsilon_G(v_k), \quad m \in \mathbb{N}. \\ m_{k+1} = \Upsilon_G(u_k), \end{cases} \tag{18}$$

On the other hand, by using method of integration by parts three times to evaluate $\int_1^2 G(\psi, s)m'''(s)ds$ in (17) and since $\int_1^2 \frac{\partial^3 G}{\partial^3 s^3}(\psi, s)m(s)ds = \int_1^2 \delta(x - s)m(s)ds$, we have that

$$\Upsilon_G(m) = (2 - \psi)Q + \frac{1}{2}(\psi^2 - 3\psi + 2)P + (\psi - 1)W - \int_1^2 G(\psi, s)\varphi(s, m, m', m'')ds. \tag{19}$$

Next, we prove that under some standard assumptions on the function φ, the integral operator Υ_G is a contraction on the Banach space $C^2([1,2])$ with respect to the norm $\|m\|_{C_2} = \sum_{i=0}^{2} \sup_{[1,2]} \left| m^{(i)} \right|$.

Theorem 2.1. *Suppose that the function, which appears in definition of the operator T_G fulfills the following Lipschitz condition:*

(20)
$$|\varphi(s,m,m',m'')-\varphi(s,h,h',h'')|\leq\Theta_1|m(s)-h(s)|+\Theta_2|m'(s)-h'(s)|+\Theta_3|m''(s)-h''(s)|,$$

where Θ_1,Θ_2 and Θ_3 are positive constants satisfying

$$\frac{1}{8}\max\{\Theta_1,\Theta_2,\Theta_3\}<1, \tag{21}$$

then T_G is a contraction on the Banach space $(C^2([1,2]),\|\cdot\|_{C^2})$ and the GA Green's iterative method (18) *converges strongly to the fixed point of T_G.*

Proof.

$$\begin{aligned}
|T_G(\psi_1)-T_G(\psi_2)| &= \left|\int_1^2 G(\psi,s)((\varphi(s,m_1,m_1',m_1'')-\varphi(s,m_2,m_2',m_2''))ds\right| \\
&\leq \int_1^2 |G(\psi,s)||(\varphi(s,m_1,m_1',m_1'')-\varphi(s,m_2,m_2',m_2'')|ds \\
&\leq \left(\sup_{[1,2]\times[1,2]|}|G(\psi,s)|\right)\int_1^2 |(\varphi(s,m_1,m_1',m_1'')-\varphi(s,m_2,m_2',m_2'')|ds \\
&= G\left(\frac{3}{2},1\right)\int_1^2 |(\varphi(s,m_1,m_1',m_1'')-\varphi(s,m_2,m_2',m_2'')|ds \\
&= \frac{1}{8}\int_1^2 |(\varphi(s,m_1,m_1',m_1'')-\varphi(s,m_2,m_2',m_2'')|ds \\
&\leq \frac{1}{8}\int_1^2 [\Theta_1|m_1(s)-m_2(s)|+\Theta_2|m_1'(s)-m_2'(s)|+\Theta_3|m_1''(s)-m_2''(s)|] \\
&\leq \frac{1}{8}\max\{\Theta_1,\Theta_2,\Theta_3\}\int_1^2\left(\sum_{i=0}^{2}\sup_{[1,2]}\left|m_1^{(i)}-m_2^{(i)}\right|\right) \\
&\leq \frac{1}{8}\max\{\Theta_1,\Theta_2,\Theta_3\}\|\psi_1-\psi_2\|_{C^2} \\
&= v\|\psi_1-\psi_2\|_{C^2}.
\end{aligned}$$

Where $v=\frac{1}{8}\max\{\Theta_1,\Theta_2,\Theta_3\}<1$. Thus, we know from Banach contraction principle that Υ_G is a contraction.

Next, we prove that the sequence $\{m_k\}$ defined by GA Green's iterative method (18) converges strongly to the fixed point of Υ_G. Since Υ_G is a contraction, then by Banach contraction principle, we know that T_G has a unique fixed point in $(C^2([1,2]),\|\cdot\|_{C^2})$, say ℓ. We will now show that $f_m \to c$ as $k \to \infty$. Using (18), we have

$$
\begin{aligned}
\|v_m - c\| &= \|(1-q_k)m_k + q_k\Upsilon_G(m_k) - c\| \\
&\le (1-q_k)\|m_k - c\| + q_k\|\Upsilon_G(m_k) - c\| \\
&\le (1-q_k)\|m_k - c\| + q_k\beta\|m_k - c\| \\
&= (1-(1-\beta)q_k)\|m_k - c\|. \qquad (22)
\end{aligned}
$$

And using (22), we have

$$
\begin{aligned}
\|u_k - c\| &= \|(1-p_k)m_k + p_k\Upsilon_G(v_k) - c\| \\
&\le (1-p_k)\|m_k - c\| + p_k\|\Upsilon_G(v_m) - c\| \\
&\le (1-p_k)\|m_k - c\| + p_k\beta\|v_k - c\| \\
&\le (1-(1-\beta)q_k)(1-(1-\beta)p_m)\|m_k - c\|. \qquad (23)
\end{aligned}
$$

Since $0 < \beta < 1$ and $p_k, q_k \in [0,1]$, then it follows that

$$
\begin{cases}
(1-(1-\beta)q_k) < 1, \\
(1-(1-\beta)p_k) < 1.
\end{cases} \qquad (24)
$$

Thus, using (24), then (23) becomes

$$
\|u_k - c\| \le \|m_k - c\|. \qquad (25)
$$

Finally, from (25), we get

$$
\begin{aligned}
\|m_{k+1} - v\| &= \|\Upsilon_G(u_k) - c\| \\
&\le \beta\|u_k - c\| \\
&\le \beta\|m_k - c\| \qquad (26)
\end{aligned}
$$

By induction, we have

$$
\|m_{k+1} - c\| \le \beta^{(k+1)}\|m_0 - c\|. \qquad (27)
$$

Since $0 < \beta < 1$, then we have that $\{m_k\}$ converges strongly to c. $\square$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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