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FIXED POINT THEOREMS FOR β-CONTRACTION MAPPING IN SMYTH COMPLETE QUASI-METRIC SPACES

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Abstract. In this article, we begin by recalling the concept of quasi-metric spaces, providing all the important definitions, and explaining the different types of completeness. Following this foundational overview, we introduce a new type of contraction called the β-contraction. We then prove the existence and uniqueness of fixed points for such contractions in Smyth's complete quasi-metric spaces. To illustrate our results, we present a relevant example. Finally, we conclude by generalizing the last theorem to a broader context, demonstrating the robustness and applicability of our findings.

Keywords: fixed point; Smyth complete; β-contraction.

2020 AMS Subject Classification: 47H10, 47H09.

1. Introduction

Fixed point theory is a cornerstone in domains like nonlinear analysis, operator theory, and differential equations. Stefan Banach's seminal work in 1922 [15] laid the foundation by introducing the concept of contraction mappings, marking a significant milestone in mathematical

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theory. Since then, numerous scholars have expanded upon Banach's original principle, resulting in many intriguing extensions and generalizations (see [15, 16, 1, 11, 12, 4, 10]).

Quasi-metric spaces, a natural generalization of metric spaces, have garnered significant interest due to their flexibility in modeling asymmetric distance functions. The concept of quasi-metric spaces was first introduced by Wilson [18] in 1931, providing a broader framework for analyzing problems where the symmetry of distance does not hold. These spaces allow for a broader framework to analyze problems in various scientific fields, including computer science, information theory, and topology.

Understanding different notions of completeness within quasi-metric spaces is essential for advancing fixed point theory. Notions such as Smyth completeness, d-completeness, and b-completeness play a pivotal role in determining the behavior and existence of fixed points. Each type of completeness provides a unique perspective on the convergence and stability of sequences within these spaces, thereby influencing the applicability of fixed point theorems.

Motivated by the need to extend fixed point theory to more general settings, this article embarks on a unique journey, exploring the application of fixed point theory within quasi-metric spaces. We navigate through different notions of completeness, analyzing their impact on fixed point behavior. Recent advancements in this realm are illuminated, alongside the identification of existing challenges and open inquiries.

In this article, we introduce a new type of contraction, which we call the β-contraction. This new contraction parameter aims to address specific limitations observed in existing theories and provides a more generalized approach to fixed point analysis. We prove the existence and uniqueness of fixed points for β-contractions in Smyth complete quasi-metric spaces, thereby extending the applicability of fixed point theorems to a broader class of spaces.

To illustrate our theoretical findings, we present a relevant example that demonstrates the practical implications of our results. Additionally, we conclude by generalizing the last theorem, offering insights into how our new approach can be adapted and extended to other contexts within quasi-metric spaces.

Through this exploration, we aim to offer a fresh perspective on fixed point theory and its implications within the realm of quasi-metric spaces, contributing to the ongoing development and enrichment of this fascinating field.

2. PRELIMINARIES

Definition 2.1. [5]Let$E$ be a nonempty set and let $d : E \times E \to [0, +\infty[$ be a function satisfying the following conditions:

1) $d(x,y) = d(y,x) = 0$ implies $x = y$

2) $d(x,z) \leq d(x,y) + d(y,z)$

Then, d a quasi-metric on E, If a quasi metric d satisfies the additional condition $d(x,y) = 0 \Longrightarrow x = y$

Then d is said to be the T1-quasi metric. It is clear that every metric is a T1-quasi metric, every T1-quasi metric is a quasi metric. A quasi (resp. T1-quasi) metric space is a pair (E,d) such that X is a nonempty set and d is a quasi (resp. T1-quasi) metric.

Example 2.2. [5]Let $E = R$ and $d(x,y) = max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$.

Then, (E,d) it's a quasi-metric space but it's not a metric space.

Definition 2.3. [17]For a quasi-metric d, its conjugate (or dual) quasi-metric d^* is defined for all d,

$$d^*(x,y) = d(y,x)$$

For all $x, y \in E$.

Definition 2.4. [17]Let (E,d) be a quasi-metric space. The fonction $d^s : E \times E \to R^+$ defined by:

$$d^s(x,y) = max\{d(x,y), d(y,x)\}$$

It's a metric.

Every quasi metric d on E generates a natural topology τ_d on X. This topology on X generated by the family of open balls $B^L_d(x, \varepsilon)$ for all $x \in E$.

$$B^L_d(x,\varepsilon) = \{y \in E : d(x,y) < \varepsilon\}$$

Definition 2.5. [13]Let (E,d) be a quasi-metric space and (a_n) a sequence in E and $x \in E$.

1. (a_n) is said to be right convergent to x iff

$$\lim_{n\to\infty} d(a_n,x) = 0$$

2. (a_n) is said to be left convergent to x iff

$$\lim_{n\to\infty} d(x,a_n) = 0$$

3. (a_n) is said to be convergent to x iff

$$\lim_{n\to\infty} d(a_n,x) = \lim_{n\to\infty} d(x,a_n) = 0$$

Definition 2.6. [13]Let (E,d) be a quasi-metric space and (a_n) be a sequence on E. we said that:

1. Left K-Cauchy , if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$$\forall k,n, n \geq k \geq n_0 \;\; d(a_k,a_n) < \varepsilon$$

2. Right K-Cauchy , if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that:

$$\forall k,n, k \geq n \geq n_0 \; d(a_k,a_n) < \varepsilon$$

3. d^s-Cauchy , if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that

$$\forall n,k \geq n_0 \; d(a_k,a_n) < \varepsilon$$

Definition 2.7. [13]Let (E,d) be a quasi-metric space . we said that:

1. (E,d) sequentially complete , if every left K-Cauchy sequence in (E,d) converges for the topology τ_{d^*}

2.(E,d) left K-sequentially complete, if every left K-Cauchy sequence in (E,d) converges for the topology τ_d

3. (E,d) d-sequentially complete, if every Cauchy sequence in (E,d^s) converges for the topology τ_d

4. (E,d) est Smyth complete, if every left K-Cauchy sequence in (E,d) converges for the topology τ_{d^s}.

Remark 2.8. [13]The following implications are obvious for a quasi-metric space (E,d):

Smyth complete $\Rightarrow$ left K-sequentially complete $\Rightarrow$ d-sequentially complete.

Example 2.9. [13]Let $\mathbb{R}$ be the set of all real numbers and let d be the T_1 quasi-metric on $\mathbb{R}$ defined by:

$$d(x,y) = \begin{cases} y-x & \text{if } x \leq y, \\ 1 & \text{if } x > y. \end{cases}$$

Then, $(\mathbb{R},d)$ is d-sequentially complete because the Cauchy sequences in the metric space $(\mathbb{R},d_s)$ eventually become constant. However, it is not left K-sequentially complete since the sequence $(-1/n)_{n\in\mathbb{N}}$ is left K-Cauchy but does not converge with respect to τ_{d_s}. Note that τ_d corresponds to the well-known Sorgenfrey topology on $\mathbb{R}$.

Inspired by the following theorem:

Theorem 2.10. [9]*Let (E,d) be a complete metric space and T a mapping from E into itself satisfying the following condition*

$$d(Tx,Ty) \leq \frac{d(x,Ty)+d(y;Tx)}{d(x,Tx)+d(y;Ty)+1}\max(d(x,Tx),d(y,Ty)) \tag{2.1}$$

For all $x,y \in E$

T has a unique fixed point $z \in E$,

3. Main Results

With these preliminaries, we can now move towards defining β-contractions and proving our main results.

Definition 3.1. Let (E,d) be a quasi-metric space, and let $T : E \to E$ be a mapping. We say that T is a β-contraction if there exists a value β such that

$$d(Tx,Ty) \leq \beta\max(d(x,Tx),d(y,Ty))$$

for all $x,y \in E$, where

$$\beta = \frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}.$$

Our main result it's a generalisation of the last theorem in a quasi-metric space Smyth complete.

The following lemma has an important role in our proof:

Lemma 3.2. *Let (E,d) be a quasi-metric space and a_n $(n \in \mathbb{N})$ such that:*

$$d(a_n,a_{n+1}) \leq \beta_n d(a_{n-1},a_n) \quad \forall n \in \mathbb{N}^* \tag{3.1}$$

With

$$\beta_n = \frac{d(a_{n-1},a_n)\ + d(a_n,a_{n+1})}{d(a_{n-1},a_n)\ + d(a_n,a_{n+1}) + 1}$$

Then $\{a_n\}$ is left K-Cauchy sequence in the quasi-metric space (E,d).

Proof. Let $\{a_n\}$ be a sequence in E

such that $a_{n+1} \neq a_n \quad \forall n > 1$

And set $d_n = d(a_{n-1};a_n)$

$$\beta_n = \frac{d_n + d_{n+1}}{d_n + d_{n+1} + 1}$$

Since $0 < \beta_n < 1$

And

$$d(a_n;a_{n+1}) \leq \beta_n d(a_{n-1};a_n)$$

Then

$$d_{n+1} \leq \beta_n d_n < d_n \quad \forall n \in \mathbb{N} \tag{3.2}$$

We will prove that $\forall n \geq 1 \qquad \beta_n < \beta_{n-1}$

Is equivalent to

$$\frac{d_n + d_{n+1}}{d_n + d_{n+1} + 1} < \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1}$$

$$\Leftrightarrow d_{n+1} < d_{n-1} \text{ already proved in (3.2)}$$

Then

$$\beta_n < \beta_1 \quad \forall n \geq 1$$

hence

$$d_{n+1} < \beta_1 d_1 \quad \forall n \in \mathbb{N}$$

Then

$$d_n \leq \beta_1 d_{n-1} \leq \beta_1^n d_{\ 0} \quad \forall n \in \mathbb{N}$$

For all $k, n \in \mathbb{N}$ with $n > k$

$$d(a_k; a_n) \leq \sum_{i=k}^{n-1} d_i \leq \sum_{i=k}^{n-1} \beta_1^i d_0.$$

Thus (a_n) is left K-Cauchy sequence in the quasi-metric space (E, d) □

Theorem 3.3. *Let (E, d) be a quasi-metric space Smyth complete, and T a mapping from E into itself satisfying the following condition:*

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y; Ty) + 1} \max(d(x, Tx), d(y; Ty)) \tag{3.3}$$

For all $x, y \in E$

Then T has a unique fixed point

Proof. Let $a_{n+1} = Ta_n$

$$\begin{aligned} d(a_n, a_{n+1}) &= d(Ta_{n-1}; Ta_n) \\ &\leq \frac{d(a_{n-1}; a_{n+1}) + d(a_n; a_n)}{d(a_{n-1}; a_n) + d(a_n; a_{n+1}) + 1} \max\{d(a_{n-1}; a_n); d(a_n; a_{n+1})\} \\ &\leq \frac{d(a_{n-1}; a_n) + d(a_n; a_{n+1})}{d(a_{n-1}; a_n) + d(a_n; a_{n+1}) + 1} \max\{d(a_{n-1}; a_n); d(a_n; a_{n+1})\} \\ &= \beta_n d(a_{n-1}; a_n) \end{aligned}$$

Because $d(a_n; a_{n+1}) \leq d(a_{n-1}; a_n)$

Applying Lemma , we deduce that $\{a_n\}$ is left K-Cauchy sequence in the quasi-metric space(E, d) Since (E, d) is smyth complete here exists a point $z \in E$ such that $\lim_{n \to \infty} a_n = z$, then z z is a fixed point of T because if $Tz \neq z$ using the inequality (3.3) we get:

$$d(Ta_n, Tz) \leq \frac{d(a_n, Tz) + d(z, a_{n+1})}{d(a_n, a_{n+1}) + d(z; Tz) + 1} \max(d(a_n, a_{n+1}), d(z; Tz))$$

Taking the limit as $n \to \infty$ we get:

$$d(z, Tz) \leq \frac{d(z, Tz)^2}{d(z, Tz) + 1} < d(z; Tz)$$

Thus $Tz = z$.

For the uniqueness, assume that $w \neq z$ is another fixed point of T using the inequality (3.3)

$$d(z;w) \leq d(Tz;Tw) \leq 0$$

Hence, z is unique. □

Example 3.4. Let $E = [0,1]$ and $d : E \times E \to \mathbb{R}_+$ define by $d(x,y) = \max\{x-y,0\}$

(E,d) is a quasi-metric smyth complet. Define $T : E \to E$ by:

$$\begin{cases} T(x) = \frac{1}{8} \; si \;\; x \in [0,\frac{1}{2}] \\ T(x) = 0 \; si \;\; x \in]\frac{1}{2},1] \end{cases}$$

The cases $x,y \in [0,\frac{1}{2}]$, and $x,y \in]\frac{1}{2},1]$ are obvious .

For the case $x \in [0,\frac{1}{8}]$ and $y \in]\frac{1}{2},1]$. we have $Tx = \frac{1}{8}, Ty = 0$, $d(x,Ty) = x$, $d(y,Tx) = y - \frac{1}{8}$, $d(x,Tx) = 0$ and $d(y,Ty) = y$

$$d(Tx,Ty) = \frac{1}{8} and \; \frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y;Ty)+1} \max(d(x,Tx),d(y;Ty)) = \frac{(x+y-\frac{1}{8})y}{y+1}$$

It is enough to prove that

$$\begin{aligned} \frac{1}{8} &\leq \frac{(y-\frac{1}{8})y}{y+1} \\ &\iff \frac{1}{8}(y+1) \leq y^2 - \frac{1}{8}y \\ &\iff (y-\frac{1}{8})^2 \geq \frac{9}{64} \\ &\iff y \geq \frac{1}{2} \end{aligned}$$

For the case $x \in [\frac{1}{8},\frac{1}{2}]$ and $y \in]\frac{1}{2},1]$. we have $Tx = \frac{1}{8}, Ty = 0$, $d(x,Ty) = x$, $d(y,Tx) = y - \frac{1}{8}$, $d(x,Tx) = x - \frac{1}{8}$ et $d(y,Ty) = y$

$$d(Tx,Ty) = \frac{1}{8} and \frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y;Ty)+1} \max(d(x,Tx),d(y;Ty)) = \frac{(x+y-\frac{1}{8})y}{x+y+\frac{7}{8}}$$

Let's prove that

$$\frac{1}{8} \leq \frac{(x+y-\frac{1}{8})y}{x+y+\frac{7}{8}}$$

It's enough to prove that

$$\begin{aligned} \frac{1}{8} &\leq \frac{y^2}{x+y+\frac{7}{8}} \\ &\Longleftrightarrow \frac{1}{8}(x+y+\frac{7}{8}) \leq y^2 \\ &\Longleftrightarrow \frac{1}{8}(x+\frac{7}{8}) \leq y^2 - \frac{1}{8}y \end{aligned}$$

Since

$$\frac{1}{8}(x+\frac{7}{8}) \leq \frac{1}{8}(\frac{1}{2}+\frac{7}{8}) = \frac{11}{64} \quad \text{and} \quad y^2 - \frac{1}{8}y = (y-\frac{1}{16})^2 - \frac{1}{16^2} \geq (\frac{1}{2}-\frac{1}{16})^2 - \frac{1}{16^2} = \frac{48}{256}$$

Then

$$\frac{1}{8}(x+\frac{7}{8}) \leq y^2 - \frac{1}{8}y$$

Which yield to

$$d(Tx,Ty) \leq \frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y;Ty)+1}\max(d(x,Tx),d(y;Ty))$$

Then T satisfy all conditions of Theorem 3.3, so T has a unique fixed point $x = \frac{1}{8}$.

Now, we prove another fixed point theorem with new contraction mapping in quasi-metric space Smyth complete.

Theorem 3.5. *Let (E,d) be a quasi-metric space Smyth complete, and T a mapping from E into itself satisfying the following condition:*

$$d(Tx,Ty) \leq \frac{d(x,Ty)d(y,Tx)}{d(x,Tx)d(y;Ty)+1}\max(d(x,y),d(x,Tx),d(y;Ty)) \tag{3.4}$$

For all $x,y \in E$ Then T has a unique fixed point

To prove the theorem, we first prove the lemma.

Lemma 3.6. *Let (E,d) be a quasi-metric space and $(a_n)_{(n\in\mathbb{N})}$ such that:*

$$d(a_n,a_{n+1}) \leq \beta'_n d(a_{n-1},a_n) \quad \forall n \in \mathbb{N}^* \tag{3.5}$$

With

$$\beta'_n = \frac{d(a_{n-1},a_n)d(a_n,a_{n+1})}{d(a_{n-1},a_n)d(a_n,a_{n+1})+1}$$

Then $\{a_n\}$ is left K-Cauchy sequence in the quasi-metric space (E,d).

Proof. Let $\{a_n\}$ be a sequance in E

such that $a_{n+1} \neq a_n \quad \forall n > 1$

And set $d_n = d(a_{n-1}; a_n)$

$$\beta'_n = \frac{d_n d_{n+1}}{d_n d_{n+1} + 1}$$

Since $0 < \beta'_n < 1$

And

$$d(a_n; a_{n+1}) \leq \beta'_n d(a_{n-1}; a_n)$$

Then

$$d_{n+1} \leq \beta'_n d_n < d_n \quad \forall n \in \mathbb{N} \tag{3.6}$$

We will prove that $\forall n \geq 1 \qquad \beta'_n < \beta'_{n-1}$

Is equivalent to

$$\frac{d_n d_{n+1}}{d_n d_{n+1} + 1} < \frac{d_{n-1} d_n}{d_{n-1} d_n + 1}$$

$$\Leftrightarrow d_{n+1} < d_{n-1} \text{ already proved in (3.6)}$$

Then

$$\beta'_n < \beta'_1 \quad \forall n \geq 1$$

hence

$$d_{n+1} < \beta'_1 d_1 \quad \forall n \in \mathbb{N}$$

Then

$$d_n \leq \beta'_1 d_{n-1} \leq (\beta'_1)^n d_{\ 0} \quad \forall n \in \mathbb{N}$$

For all $k, n \in \mathbb{N}$ with $n > k$

$$d(a_k; a_n) \leq \sum_{i=k}^{n-1} d_i \leq \sum_{i=k}^{n-1} (\beta'_1)^i d_0.$$

Thus (a_n) is left K-Cauchy sequence in the quasi-metric space (E, d)

□

Now we return to proving the theorem.

Proof. Let $a_{n+1} = Ta_n$

$$\begin{aligned} d(a_n, a_{n+1}) &= d(Ta_{n-1}; Ta_n) \\ &\leq \frac{d(a_{n-1}; a_{n+1})d(a_n; a_n)}{d(a_{n-1}; a_n)d(a_n; a_{n+1}) + 1} \max\{d(a_{n-1}; a_n); d(a_{n-1}; a_n); d(a_n; a_{n+1})\} \\ &\leq \frac{d(a_{n-1}; a_n)d(a_n; a_{n+1})}{d(a_{n-1}; a_n)d(a_n; a_{n+1}) + 1} \max\{d(a_{n-1}; a_n); d(a_n; a_{n+1})\} \\ &= \beta'_n d(a_{n-1}; a_n) \end{aligned}$$

Applying Lemma , we deduce that $\{a_n\}$ is left K-Cauchy sequence in the quasi-metric space(E,d) Since (E,d) is smyth complete there exists a point $z \in E$ such that $\lim_{n\to\infty} a_n = z$, then z z is a fixed point of T because if $Tz \neq z$ using the inequality (3.4) we get:

$$d(Ta_n, Tz) \leq \frac{d(a_n, Tz)d(z, a_{n+1})}{d(a_n, a_{n+1})d(z; Tz) + 1} \max(d(a_n, z), d(a_n, a_{n+1}), d(z; Tz))$$

Taking the limit as $n \to \infty$ we get:

$$d(z, Tz) \leq 0$$

Thus $Tz = z$.

For the uniqueness, assume that $w \neq z$ is another fixed point of T using the inequality (3.4)

$$d(z; w) \leq d(Tz; Tw) \leq 0$$

Hence, z is unique. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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