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IDENTICALNESS OF CERTAIN RESULTS AND FIXED-POINT THEOREMS IN BS-MULTIPLICATIVE METRIC SPACE

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Abstract. In this paper, we introduce bS-multiplicative metric space and establish several fixed point theorems on contraction mappings like Banach contraction and Kannan contraction in this space. We also demonstrate fixed-point results for identicalness bS-multiplicative metric space with analogous fixed-point results in S-multiplicative space. Non-trivial example is further provided to support the hypotheses of our results.

Keywords: fixed point; S-metric spaces; S-multiplicative metric spaces; bS-multiplicative metric spaces.

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1. Introduction and Preliminaries

The Banach contraction principle was first stated explicitly in 1922 [12]. The term fixed point theory referred on those fixed points theoretic results in which geometric conditions on the underlying spaces and for mappings play a crucial role. For the past several years metric

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fixed point theory has been flourishing area for many mathematicians. Since then, topology, functional analysis, and nonlinear analysis had been reliant on metric space. This space's topological nature with applications in fixed-point theory has attracted the interest of numerous mathematicians(see [1], [5], [6], [8], [10]).

In 1906, Frechet [6] introduced metric spaces. Numerous generalisations of the concept of metric space have been constructed and different fixed point theorems have been proved in recent years. In 2008, Bashirov et al. [1] developed a different type of metric spaces which known as multiplicative metric spaces and they also established the corresponding Banach fixed point result in the same spaces. In 2012, Sedghi et al. [11] established a fixed point theorem for a self-mapping on a complete S-metric spaces. In 2019, Mustafa et al. [14] generalized S-metric spaces into S_p-metric spaces and investigated the existence of a fixed point for such mappings under various contractive conditions. Most recently in 2024, Adewale et al. [8] stated and proved some fixed-point theorems in S-multiplicative metric spaces which shows that some fixed point theorems are equivalent to those of corresponding fixed-point results in S-metric spaces.

We first present some important definitions and notations which will be used in the main results as follows:

Definition 1.1. *([6]). Let Y be a non-empty set. A mapping $d : Y \times Y \to [0,+\infty)$ is said to be metric on Y, if and only if for all $\mu, \tau, \rho \in Y$, it satisfies the following conditions:*

(1) $d(\mu,\tau) \geq 0,$

(2) $d(\mu,\tau) = 0, \ if \ and \ only \ if \ \mu = \tau,$

(3) $d(\mu,\tau) = d(\tau,\mu),$

(4) $d(\mu,\tau) \leq d(\mu,\rho) + d(\rho,\tau).$

The pair (Y,d) is called a usual metric space.

Definition 1.2. *([11]). Let Y be a non-empty set. A mapping $S : Y \times Y \times Y \to [0,+\infty)$ is said to be S-metric on Y, if and only if for all $\mu, \tau, \rho, a \in Y$, it satisfies the following conditions:*

(1) $S(\mu,\tau,\rho) = 0, \ if \ and \ only \ if \ \mu = \tau = \rho,$

(2) $S(\mu,\tau,\rho) \leq S(\mu,\mu,a) + S(\tau,\tau,a) + S(\rho,\rho,a).$

The pair (Y,S) is called a S-metric space.

Definition 1.3. *([14]). Let Y be a non-empty set. A mapping $S : Y \times Y \times Y \to [0,+\infty)$ with a strictly increasing continuous mapping $b : [0,+\infty) \to [0,+\infty)$ such that $b(t) \geq t$ for all $t > 0$ and $b(0) = 0$, is said to be S_b-metric on Y, if and only if for all $\mu, \tau, \rho, a \in Y$, it satisfies the following conditions:*

(1) $S(\mu,\tau,\rho) = 0,\ if\ and\ only\ if\ \mu = \tau = \rho,$

(2) $S(\mu,\tau,\rho) \leq b[S(\mu,\mu,a) + S(\tau,\tau,a) + S(\rho,\rho,a)].$

The pair (Y,S) is called a S_b-metric space.

Definition 1.4. *([1]). Let Y be a non-empty set. A mapping $d : Y \times Y \to [0,+\infty)$ is said to be multiplicative metric on Y, if and only if for all $\mu, \tau, \rho \in Y$, it satisfies the following conditions:*

(1) $d(\mu,\tau) \geq 1,$

(2) $d(\mu,\tau) = 1,\ if\ and\ only\ if\ \mu = \tau,$

(3) $d(\mu,\tau) = d(\tau,\mu),$

(4) $d(\mu,\tau) \leq d(\mu,\rho).d(\rho,\tau).$

The pair (Y,d) is called a multiplicative metric space.

Definition 1.5. *([8]). Let Y be a non-empty set. A mapping $S : Y \times Y \times Y \to [0,+\infty)$ is said to be S-multiplicative metric on Y, if and only if for all $\mu, \tau, \rho, a \in Y$, it satisfies the following conditions:*

(1) $S(\mu,\tau,\rho) = 1,\ if\ and\ only\ if\ \mu = \tau = \rho,$

(2) $S(\mu,\tau,\rho) \leq S(\mu,\mu,a) \times S(\tau,\tau,a) \times S(\rho,\rho,a).$

The pair (Y,S) is called a S-multiplicative metric space.

In this paper, we introduce bS-multiplicative metric space and prove several fixed point theorems for contraction mappings in this spaces. We also investigate the existence of a fixed point for such mappings under various contractive conditions.

2. Main Results

Definition 2.1. *Let Y be a non-empty set. A mapping $S : Y \times Y \times Y \to [0,+\infty)$ with $0 < b \leq 1$, is said to be bS-multiplicative metric on Y, if and only if for all $\mu,\tau,\rho,a \in Y$, it satisfies the following conditions:*

(1) $S(\mu,\tau,\rho) = 1,\ if\ and\ only\ if\ \mu = \tau = \rho,$

(2) $S(\mu,\tau,\rho) \leq \frac{1}{b}[S(\mu,\mu,a) \times S(\tau,\tau,a) \times S(\rho,\rho,a)].$

The pair (Y,S) is called a bS-multiplicative metric space.

Remark 2.2. *If $b = 1$, bS-multiplicative metric spaces reduces to S-multiplicative metric space.*

Example 2.3. *Let $Y = \mathbb{R}^+$ and define $S : Y \times Y \times Y \to [0,+\infty)$ with $0 < b \leq 1$, by*

$$S(\mu,\tau,\rho) = \begin{cases} e^{2\mu-\tau-\rho}, & \text{if } \mu = \tau = \rho \\ \log(\mu+\tau+\rho), & \text{otherwise.} \end{cases}$$

Clearly, the pair (Y,S) is a bS-multiplicative metric space.

Definition 2.4. *Let (Y,S) be a bS-multiplicative metric space. For $\mu \in Y,\ r > 0$, the bS-sphere with center μ and radius r is $S^{'}(\mu,r) = \{\tau \in Y : S(\mu,\tau,\tau) < r\}$.*

Definition 2.5. *Let (Y,S) and $(Y^{'},S^{'})$ be two bS-multiplicative metric spaces, a function $T :$ $Y \to Y^{'}$ is bS-continuous at a point $\mu \in Y$ if $T^{-1}(S_{S'}(T(\mu),r)) \in \tau(S)$, for all $r > 1$. T is bS-continuous if it is bS-continuous at all points of Y.*

Definition 2.6. *Let (Y,S) be a bS-multiplicative metric space and $\{\mu_n\}$ a sequence in Y. Then $\{\mu_n\}$ converges to μ if and only if $S(\mu_n,\mu,\mu) \to 1$ as $n \to \infty$.*

Definition 2.7. *Let (Y,S) be a bS-multiplicative metric space and $\{\mu_n\}$ a sequence in Y. Then $\{\mu_n\}$ is said to be a Cauchy sequence if and only if $S(\mu_n,\mu_m,\mu_l) \to 1$ as $n,m,l \to \infty$.*

Now, we present our main result as follows:

Theorem 2.8. *Let (Y,S) be a complete bS-multiplicative metric space with $0 < b \leq 1$ and $T : Y \to Y$ a map for which there exist the real number, α satisfying $0 \leq \alpha < 1$ such that for* *each pair* $\mu,\tau,\rho\in Y$.

$$S(T\mu,T\tau,T\rho)\leq(S(\mu,\tau,\rho))^{\alpha}. \tag{2.1}$$

Then, T has a unique fixed point.

Proof. For all $\mu,\tau\in Y$ considering (2.1),

$$S(T\mu,T\tau,T\tau)\leq(S(\mu,\tau,\tau))^{\alpha}. \tag{2.2}$$

Assume that T satisfies (2.2) and for any arbitrary point μ_0 and define a sequence μ_n by $\mu_n=T^n\mu_0$, then

$$\begin{aligned}
S(\mu_n,\mu_n,\mu_{n+1})&=S(T\mu_{n-1},T\mu_{n-1},T\mu_n)\\
&\leq(S(\mu_{n-1},\mu_{n-1},\mu_n))^{\alpha}\\
&\leq(S(\mu_{n-2},\mu_{n-2},\mu_{n-1}))^{\alpha^2}\\
&\vdots\\
&\leq(S(\mu_0,\mu_0,\mu_1))^{\alpha^n}. \qquad (2.3)
\end{aligned}$$

By using condition 2 of Definition 2.1, we have

$$S(\mu_n,\mu_m,\mu_m)\leq\frac{1}{b}[S(\mu_n,\mu_n,\mu_{n+1})(S(\mu_m,\mu_m,\mu_{n+1}))^2].$$

Continuing this process with $m>n$, we obtain

$$\begin{aligned}
S(\mu_n,\mu_m,\mu_m)\leq&\frac{1}{b^n}[S(\mu_n,\mu_n,\mu_{n+1})\times(S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^2\\
&\times(S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^4\times\ldots\times(S(\mu_{m-1},\mu_{m-1},\mu_m))^{2n}]. \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4), we have

$$\begin{aligned}
S(\mu_n,\mu_m,\mu_m)\leq&\frac{1}{b^n}[S(\mu_n,\mu_n,\mu_{n+1})^{\alpha^n}\times(S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^{\alpha^{2n}}\\
&\times(S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^{\alpha^{4n}}\times\ldots\times(S(\mu_{m-1},\mu_{m-1},\mu_m))^{\alpha^{2n^2}}].
\end{aligned}$$

Since $b\in(0,1]$, we have

$$S(\mu_n,\mu_m,\mu_m) \leq S(\mu_n,\mu_n,\mu_{n+1})^{\alpha^n} \times (S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^{\alpha^{2n}} \\ \times (S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^{\alpha^{4n}} \times ... \times (S(\mu_{m-1},\mu_{m-1},\mu_m))^{\alpha^{2n^2}}.$$

Taking the limit of $S(\mu_n,\mu_m,\mu_m)$ as $n,m \to \infty$, we have

$$\lim_{n,m\to\infty} S(\mu_n,\mu_m,\mu_m) = 1.$$

For $n,m,l \in \mathbb{N}$ with $n > m > l$,

$$S(\mu_n,\mu_m,\mu_l) \leq \frac{1}{b}[S(\mu_n,\mu_n,\mu_{n-1}) \times S(\mu_m,\mu_m,\mu_{n-1}) \times S(\mu_l,\mu_l,\mu_{n-1})].$$

Taking the limit of $S(\mu_n,\mu_m,\mu_l)$ as $n,m,l \to \infty$, we have

$$\lim_{n,m,l\to\infty} S(\mu_n,\mu_m,\mu_l) = 1.$$

Hence $\{\mu_n\}$ is a Cauchy sequence.

By the completeness of (Y,S), there exist $u^* \in Y$ such that $\{\mu_n\}$ is bS-convergent to u.

Assume that $Tu^* \neq u^*$, now

$$S(\mu_n,Tu^*,Tu^*) \leq (S(\mu_{n-1},u^*,u^*))^{\alpha}.$$

Taking the limit as $n \to \infty$ and using the fact that mapping is bS-continuous in its variables, we obtain

$$S(u^*,Tu^*,Tu^*) \leq (S(u^*,u^*,u^*))^{\alpha}.$$

Hence, $S(u^*,Tu^*,Tu^*) \leq 1$. Which is a contradiction. Hence, $Tu^* = u^*$.

Now, suppose that $u^*,v^* \in Y$ are two fixed points of T such that $u^* \neq v^*$. Then we have

$$S(Tu^*,Tv^*,Tv^*) \leq (S(u^*,v^*,v^*))^{\alpha}. \text{ Hence } S(u^*,v^*,v^*) \leq 1.$$

This is a contradiction. Hence $u^* = v^*$. □

Remark 2.9. *If (Y,S) be a bS-multiplicative metric space and a mapping $S : Y \times Y \times Y \to [0,+\infty)$ with $b = 1$, then Theorem 2.8 reduces to the Banach contraction principle in S-multiplicative metric space.*

Lemma 2.10. *Let (Y,S) be a complete bS-multiplicative metric space and μ_n a sequence in Y. Then, $S(\mu_n,\mu_{n+1},\mu_{n+1}) \leq \frac{1}{b}S(\mu_n,\mu_n,\mu_{n+1})$.*

Proof. By using condition 2 of Definition 2.1, we have

$$\begin{aligned} S(\mu_n,\mu_{n+1},\mu_{n+1}) &\leq \frac{1}{b}[S(\mu_n,\mu_n,\mu_{n+1}) \times (S(\mu_{n+1},\mu_{n+1},\mu_{n+1}))^2] \\ &\leq \frac{1}{b}S(\mu_n,\mu_{n+1},\mu_{n+1}). \end{aligned}$$

□

Theorem 2.11. *Let (Y,S) be a complete bS-multiplicative metric space with $0 < b \leq 1$ and $T : Y \to Y$ a map for which there exist the real number, k satisfying $-\infty < k < -1$ such that for each pair $\mu,\tau,\rho \in Y$.*

$$S(T\mu,T\tau,T\rho) \leq [S(\mu,T\mu,T\mu) \times S(\tau,T\tau,T\tau) \times S(\rho,T\rho,T\rho)]^k. \tag{2.5}$$

Then, T has a unique fixed point.

Proof. Assume that T satisfies (2.2) and for any arbitrary point μ_0 and define a sequence μ_n by $\mu_n = T^n\mu_0$, then

$$\begin{aligned} S(\mu_n,\mu_n,\mu_{n+1}) &= S(T\mu_{n-1},T\mu_{n-1},T\mu_n) \\ &\leq [(S(\mu_{n-1},\mu_n,\mu_n))^2 \times S(\mu_n,\mu_{n+1},\mu_{n+1})]^k \\ &\leq [(S(\mu_{n-1},\mu_n,\mu_n))^2 \times S(\mu_n,\mu_n,\mu_{n+1})]^k \\ &\leq (S(\mu_{n-1},\mu_n,\mu_n))^{\frac{2}{1-k}}. \end{aligned}$$

Using $\beta = \frac{2}{1-k}$, we have

$$S(\mu_n,\mu_n,\mu_{n+1}) \leq (S(\mu_{n-1},\mu_n,\mu_n))^{\beta}.$$

Continuing this process, we obtain

$$S(\mu_n,\mu_n,\mu_{n+1}) \leq (S(\mu_0,\mu_1,\mu_1))^{\beta^n}.$$

By using condition 2 of Definition 2.1, we have

$$S(\mu_n,\mu_m,\mu_m) \le \frac{1}{b}[S(\mu_n,\mu_n,\mu_{n+1}) \times (S(\mu_m,\mu_m,\mu_{n+1}))^2]$$
$$\le \frac{1}{b}[S(\mu_n,\mu_n,\mu_{n+1})(S(\mu_m,\mu_m,\mu_{n+1}))^2].$$

Continuing this process with $m > n$, we obtain

$$S(\mu_n,\mu_m,\mu_m) \le \frac{1}{b^n}[S(\mu_n,\mu_n,\mu_{n+1}) \times (S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^2 \times (S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^4 \times \dots \times (S(\mu_{m-1},\mu_{m-1},\mu_m))^{2n}].$$

From (2.3) and (2.4), we have

$$S(\mu_n,\mu_m,\mu_m) \le \frac{1}{b^n}[S(\mu_n,\mu_n,\mu_{n+1})^{\beta^n} \times (S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^{\beta^{2n}} \times (S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^{\beta^{4n}} \times \dots \times (S(\mu_{m-1},\mu_{m-1},\mu_m))^{\beta^{2n^2}}].$$

Since $b \in (0,1]$, we have

$$S(\mu_n,\mu_m,\mu_m) \le S(\mu_n,\mu_n,\mu_{n+1})^{\beta^n} \times (S(\mu_{n+1},\mu_{n+1},\mu_{n+2}))^{\beta^{2n}} \times (S(\mu_{n+2},\mu_{n+2},\mu_{n+3}))^{\beta^{4n}} \times \dots \times (S(\mu_{m-1},\mu_{m-1},\mu_m))^{\beta^{2n^2}}.$$

Taking the limit as $n,m \to \infty$, we have

$$\lim_{n,m\to\infty} S(\mu_n,\mu_m,\mu_m) = 1.$$

For $n,m,l \in \mathbb{N}$ with $n > m > l$,

$$S(\mu_n,\mu_m,\mu_l) \le \frac{1}{b}[S(\mu_n,\mu_n,\mu_{n-1}) \times S(\mu_m,\mu_m,\mu_{n-1}) \times S(\mu_l,\mu_l,\mu_{n-1})].$$

Taking the limit as $n,m,l \to \infty$, we have

$$\lim_{n,m,l\to\infty} S(\mu_n,\mu_m,\mu_l) = 1.$$

Hence $\{\mu_n\}$ is a Cauchy sequence.

By the completeness of (Y,S), there exist $u^* \in Y$ such that $\{\mu_n\}$ is bS-convergent to u.

Assume that $Tu^* \neq u^*$, now

$$S(\mu_n,Tu^*,Tu^*) \le [S(\mu_{n-1},\mu_n,\mu_n) \times (S(u^*,Tu^*,Tu^*))^2]^{\beta}.$$

Taking the limit as $n \to \infty$ and using the fact that mapping is bS-continuous in its variables, we obtain

$$S(u^*, Tu^*, Tu^*) \leq [S(u^*, u^*, u^*) \times (S(u^*, Tu^*, Tu^*))^2]^{\beta}$$
$$\leq [S(u^*, u^*, u^*)]^{\frac{1}{1-2\beta}}.$$

Hence, $S(u^*, Tu^*, Tu^*) \leq 1$. Which is a contradiction. Hence, $Tu^* = u^*$.

Now, suppose that $u^*, v^* \in Y$ are two fixed points of T such that $u^* \neq v^*$. Then we have

$$S(Tu^*, Tv^*, Tv^*) \leq [S(u^*, Tu^*, Tu^*) \times (S(v^*, Tv^*, Tv^*))^2]^{\beta}$$
$$\leq [S(u^*, u^*, u^*)]^{\frac{1}{1-2\beta}}.$$

Since $Tu^* = u^*$ and $Tv^* = v^*$, we have $S(u^*, Tu^*, Tu^*) \leq 1$. This is a contradiction. Hence $u^* = v^*$. □

Remark 2.12. *If (Y, S) be a bS-multiplicative metric space and a mapping $S : Y \times Y \times Y \to [0, +\infty)$ with $b = 1$, then Theorem 2.11 reduces to the Kannan contraction principle in S-multiplicative metric space.*

3. Conclusion

In this paper, bS-multiplicative metric space is introduced and some fixed point theorems are proved. We also illustrate the consistency of numerous results and fixed-point theorems for bS-multiplicative metric spaces with analogous fixed-point results in S-multiplicative metric spaces. We provide example to support our main result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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