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A GENERALIZATION OF SUZUKI'S FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS TO A SYSTEM OF MAPPINGS

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Abstract. In this paper, we present a generalization of Suzuki's fixed point theorem [J. Math. Anal. Appl., 340 (2008), 2, 1088-1095] for nonexpansive mappings to a system of mappings. Furthermore, we establish an existence result for two systems of mappings under the assumption of coordinatewise commutativity. Additionally, we provide some examples to support our findings.

Keywords: fixed point; nonexpansive mapping; Suzuki contraction; Banach space.

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1. Introduction

The theory of existence of fixed points for nonexpansive mappings was initiated by the Browder [2] Göhde [7] and Kirk [11], independently, in 1965. The nonexpansive condition (2.1) forces the mapping f to be uniformly continuous in their domain. In 2008, Suzuki [21] introduced a new class of mappings, known as Suzuki-type generalized nonexpansive mappings, which does not force the mapping f to be continuous in domain and also includes the class of

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nonexpansive mappings. A detailed study of nonexpansive mappings and their generalizations can be found in [1, 2, 6, 12, 15, 18].

Definition 1.1. [21]. Let f be a mapping on a subset Y of a Banach space E. Then, the mapping f is said to a Suzuki-type generalized nonexpansive or satisfy the condition (C) if

$$\frac{1}{2}\|u-fu\| \leq \|u-v\| \text{ implies } \|fu-fv\| \leq \|u-v\| \tag{C}$$

for all $u, v \in Y$.

Suzuki [21] established the following interesting result for Suzuki-type generalized nonexpansive mappings:

Theorem 1.2. *Let Y be a weakly compact convex subset of a uniformly convex Banach space in every direction E and f be a mapping on Y. If f satisfies condition (C) then f has a fixed point.*

In 1975, Matkowski [13, 14] generalized the celebrated Banach contraction principle by proving a fixed point theorem for a system of mappings on the product of metric spaces. The following year, Czerwik [3] proved a fixed point result for a system of multivalued mappings. He also established a generalization of Eldestein's fixed point theorem to a system of mappings in the same year (see [4]). After that, a large number of existence results for one or more than one systems of mappings have been proved by several mathematicians (see [9, 10, 16, 17, 19, 20]).

The purpose of this paper is to present a generalization of Suzuki's result [21] for nonexpansive mappings to a system of mappings. We achieve this by proving an existence and convergence theorem for a system of mappings on the product of Banach spaces. Additionally, we establish an existence result for two systems of mappings under the assumption of coordinate-wise commutativity. Our results generalize the work of Suzuki [21], Matkowski [13], Czerwik [4] and many others.

2. PRELIMINARIES

Let $(E, \|.\|)$ be a Banach space and Y be a non-empty subset of E. We denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, and the set of fixed points of mapping

$f : Y \to E$ by $F(f)$. A mapping f is called a nonexpansive if

$$\|fu - fv\| \leq \|u - v\| \text{ for all } u, v \in Y. \tag{2.1}$$

If for all $u, v \in E$ with $\|u\| = \|v\| = 1, u \neq v$, we have $\|u+v\| < 2$ then E is called strictly convex. Recall that, E is uniformly convex in every direction (UCED, for short) for $\varepsilon \in (0,2]$ and $w \in E$ with $\|w\| = 1$, if there exists $\delta(\varepsilon, w) > 0$ such that

$$\|u+v\| \leq 2(1 - \delta(\varepsilon, w))$$

for all $u, v \in E$ with $\|u\| \leq 1, \|v\| \leq 1$ and $u - v \in \{tw : t \in [-2, -\varepsilon] \cup [+\varepsilon, +2]\}$. E is said to be uniformly convex if E is UCED and for all $\varepsilon \in (0,2]$,

$$\inf\{\delta(\varepsilon, w) : \|w\| = 1\} > 0.$$

Lemma 2.1. [5]. *Let (μ_n) and (ν_n) be two bounded sequences in a Banach space E and let $t \in (0,1)$. Suppose that $\mu_{n+1} = t\nu_n + (1-t)\mu_n$ and $\|\nu_{n+1} - \nu_n\| \leq \|\mu_{n+1} - \mu_n\|$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|\nu_n - \mu_n\| = 0$.*

Lemma 2.2. [21]. *For a Banach space E the following are equivalent:*

(1) *E is UCED.*

(2) *If $\{\nu_n\}$ is a bounded sequence in E, then a function f on E defined by $f(\nu) = \limsup_{n\to\infty} \|\nu_n - \nu\|$ is strictly quasi-convex that is,*

$$f(\lambda\nu + (1-\lambda)\mu) < \max\{f(\nu), f(\mu)\}$$

for all $\lambda \in (0,1)$ and $\nu, \mu \in E$ with $\nu \neq \mu$.

Let $(B_j, \|\cdot\|_j)$, $j = 1, \ldots, n$ be Banach spaces. Define $B := B_1 \times \cdots \times B_n$ and $C := C_1 \times \cdots \times C_n$, where C_j is a non-empty subset of B_j for $j = 1, 2, \ldots, n$. Assume that $P_j, S_j : C \to C_j$ for $j = 1, 2, \ldots, n$, are mappings, and denote by $P := (P_1, \ldots, P_n)$ and $S := (S_1, \ldots, S_n)$ the systems of mappings.

We denote a point in B by $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$ and a sequence in B by $(\upsilon^m) = (\upsilon_1^m, \ldots, \upsilon_n^m)$. Two systems of mappings $(P_1, \ldots, P_n)$ and $(S_1, \ldots, S_n)$ are said to be coordinatewise commuting

[20] on C if for all $\upsilon \in C$,

$$P_j(S_1\upsilon,\dots,S_n\upsilon) = S_j(P_1\upsilon,\dots,P_n\upsilon), \quad j = 1,2,\dots,n.$$

3. Main Results

Firstly, we define a new class of a system of mappings on the product of Banach spaces.

Definition 3.1. Let $(B_j, \|\cdot\|_j)$, $j = 1,2,\dots,n$, be Banach spaces, and let C_j be a non-empty subset of B_j for each $j = 1,2,\dots,n$. Assume that $P_j : C \to C_j$ for $j = 1,2,\dots,n$, are mappings. Then, the system of mappings $(P_1,\dots,P_n)$ is said to satisfy condition (D) if there exists a non-negative matrix (a_{jk}) for $j,k = 1,2,\dots,n$, with characteristic roots $(\lambda_j,\ j = 1,2,\dots,n)$ such that

$$\max\{|\lambda_j| : j = 1,2,\dots,n\} \le 1$$

and fulfil the following inequalities:

$$\text{(D)} \qquad \frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j \le \|\upsilon_j - \vartheta_j\|_j \implies \|P_j\upsilon - P_j\vartheta\|_j \le \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k$$

for all $\upsilon, \vartheta \in C$; $\upsilon_j, \vartheta_j \in C_j$ and $j = 1,2,\dots,n$.

Notice that by considering $n = 1,\ P_1 = f,\ a_{11} = 1,\ B_1 = E$ and $C_1 = Y$ in Definition 3.1, the condition (D) reduces to the condition (C).

Example 3.2. Let $B_1 = B_2 = [0,3]$ be Banach spaces endowed with the usual norm $\|\upsilon_j - \vartheta_j\|_j = |\upsilon_j - \vartheta_j|,\ j = 1,2$. Let $P_j : B_1 \times B_2 \to B_j,\ j = 1,2$ be such that

$$P_1(\upsilon_1,\upsilon_2) = \begin{cases} 0, & \text{if } \upsilon_1 \ne 3, \\ 1, & \text{if } \upsilon_1 = 3, \end{cases} \qquad P_2(\upsilon_1,\upsilon_2) = \begin{cases} 0, & \text{if } \upsilon_2 \ne 3, \\ 1, & \text{if } \upsilon_2 = 3. \end{cases}$$

Then, the system of mappings (P_1,P_2) satisfies condition (D) for $a_{11} = a_{22} = 1, a_{12} = 1/2$ and $a_{21} = 0$. To see this, let for $\upsilon_j = 3,\ \frac{1}{2}\|\upsilon_j - P_j(\upsilon_1,\upsilon_2)\|_j = \frac{1}{2}|3-1| = 1$ and $\frac{1}{2}\|\upsilon_j - P_j(\upsilon_1,\upsilon_2)\|_j \le \|\upsilon_j - \vartheta_j\|_j$ for all $\vartheta_j \in [0,2]$. Then, $\|P_j(\upsilon_1,\upsilon_2) - P_j(\vartheta_1,\vartheta_2)\|_j = |1-0| = 1 \le \|\upsilon_j - \vartheta_j\|_j,\ j = 1,2$. Similarly, for $\upsilon_j \ne 3,\ \frac{1}{2}\|\upsilon_j - P_j(\upsilon_1,\upsilon_2)\|_j = \frac{1}{2}|\upsilon_j - 0| = \upsilon_j/2$ and for $\vartheta_j \in [0,3]$ such that $\upsilon_j/2 \leq \|\upsilon_j - \vartheta_j\|_j = |\upsilon_j - \vartheta_j|$, we have $\|P_j(\upsilon_1,\upsilon_2) - P_j(\vartheta_1,\vartheta_2)\|_j \leq \|\upsilon_j - \vartheta_j\|_j,\ j = 1,2.$

Now, we present some important lemmas which are very essential for our main findings.

Lemma 3.3. *Let C_j be a subset of a Banach space B_j for each $j = 1,2,\ldots,n$, and let $P_j : C \to C_j$ for $j = 1,2,\ldots,n$, be mappings. If the system of mappings $(P_1,\ldots,P_n)$ satisfies the condition (D), then the following statements are true for each $j = 1,2,\ldots,n$, and $\upsilon_j, \vartheta_j \in C_j$:*

(a) $\|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j \leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k$.

(b) Either $\frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j \leq \|\upsilon_j - \vartheta_j\|_j$ *or* $\frac{1}{2}\|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j \leq \|P_j\upsilon - \vartheta_j\|_j$.

(c) Either $\|P_j\upsilon - P_j\vartheta\|_j \leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k$ *or* $\|P_j(P_1\upsilon,\ldots,P_n\upsilon) - P_j\vartheta\|_j \leq \sum_{k=1}^{n} a_{jk}\|P_k\upsilon - \vartheta_k\|_k$, *where* $\upsilon, \vartheta \in C$.

Proof. Since, for each $j = 1,2,\ldots,n$, and $\upsilon_j \in C_j$, it is obvious that $\frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j \leq \|\upsilon_j - P_j\upsilon\|_j$. Then by condition (D), we have

$$\|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j \leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k.$$

To prove (b), we argue by contradiction that

$$\frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j > \|\upsilon_j - \vartheta_j\|_j \text{ and } \frac{1}{2}\|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j > \|P_j\upsilon - \vartheta_j\|_j$$

for all $\upsilon_j, \vartheta_j \in C_j$ and $j = 1,2,\ldots,n$. Then by (a) and the triangle inequality, we have

$$\begin{aligned}\|\upsilon_j - P_j\upsilon\|_j &\leq \|\upsilon_j - \vartheta_j\|_j + \|P_j\upsilon - \vartheta_j\|_j \\ &< \frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j + \frac{1}{2}\|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j \\ &\leq \frac{1}{2}\|\upsilon_j - P_j\upsilon\|_j + \frac{1}{2}\sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k,\ j = 1,2,\ldots,n.\end{aligned}$$

This implies that

$$\|\upsilon_j - P_j\upsilon\|_j < \sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k,\ j = 1,2,\ldots,n. \tag{3.1}$$

We may assume, without loss of generality, that

$$\|\upsilon_k - P_k\upsilon\|_k \leq r_k \text{ for } k = 1,2,\ldots,n.$$

From Perron-Frobenius's theorem [8, pp. 534-535], there exist positive real numbers $r_j > 0,\ j = 1,2,\ldots,n$, such that

$$\sum_{k=1}^{n} a_{jk} r_k \leq r_j,\ j = 1,2,\ldots,n.$$

This follows from (3.1) that

$$\|\upsilon_j - P_j\upsilon\|_j < r_j,\ j = 1,2,\ldots,n.$$

Since above inequities are strict, so there exists $h \in [0,1)$ such that

$$\|\upsilon_j - P_j\upsilon\|_j \leq h r_j,\ h \in [0,1) \text{ and } j = 1,2,\ldots,n.$$

Repeating this step m times we get

$$\|\upsilon_j - P_j\upsilon\|_j \leq h^m r_j,\ h \in [0,1) \text{ and } j = 1,2,\ldots,n.$$

Making $m \to \infty$, we get $\|\upsilon_j - P_j\upsilon\|_j = 0$ for each $j = 1,2,\ldots,n$, which contradict our assumption. Thus the conclusion (b) holds. From conclusion (b) and the condition (D) one can easily get conclusion (c). □

Lemma 3.4. *Let C_j be a subset of a Banach space B_j for each $j = 1,2,\ldots,n$ and $P_j : C \to C_j,\ j = 1,2,\ldots,n$ be mappings. If the system of mappings $(P_1,\ldots,P_n)$ satisfies the condition (D) then*

$$\|\upsilon_j - P_j\vartheta\| \leq 2\|\upsilon_j - P_j\upsilon\|_j + \sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k + \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k$$

holds for all $\upsilon_j, \vartheta_j \in C_j$ and $j = 1,2,\ldots,n$.

Proof. By Lemma 3.3, either

$$\|P_j\upsilon - P_j\vartheta\|_j \leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k \ \text{ or } \ \|P_j(P_1\upsilon,\ldots,P_n\upsilon) - P_j\vartheta\|_j \leq \sum_{k=1}^{n} \|P_k\upsilon - \vartheta_k\|_k$$

holds. In the first case, we have

$$\|\upsilon_j - P_j\vartheta\|_j \leq \|\upsilon_j - P_j\upsilon\|_j + \|P_j\upsilon - P_j\vartheta\|_j \leq \|\upsilon_j - P_j\upsilon\|_j + \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k.$$

In the second case, we have

$$\|\upsilon_j - P_j\vartheta\|_j \leq \|\upsilon_j - P_j\upsilon\| + \|P_j\upsilon - P_j(P_1\upsilon,\ldots,P_n\upsilon)\|_j + \|P_j(P_1\upsilon,\ldots,P_n\upsilon) - P_j\vartheta\|_j$$

$$\leq 2\|\upsilon_j - P_j\upsilon\|_j + \sum_{k=1}^{n} a_{jk}\|P_k\upsilon - \vartheta_k\|_k$$
$$\leq 2\|\upsilon_j - P_j\upsilon\|_j + \sum_{k=1}^{n} a_{jk}\|\upsilon_k - P_k\upsilon\|_k + \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k.$$

□

Lemma 3.5. *Let $B_j,\ j=1,2,\ldots,n$, be Banach spaces and $C_j \subseteq B_j,\ j=1,2,\ldots,n$, be the closed convex sets. Assume that $P_j : C \to C_j,\ j=1,2,\ldots,n$, are mappings that satisfy the condition (D). Then, the set $F(P) = \{(\rho_1,\ldots,\rho_2) \in C : P_j(\rho_1,\ldots,\rho_n) = \rho_j,\ j=1,2,\ldots,n\}$ is closed. Moreover, if $B_j,\ j=1,2,\ldots,n$, are strictly convex and $C_j,\ j=1,2,\ldots,n$, are convex, then $F(P)$ is also convex.*

Proof. Let $(\rho^m) = (\rho_1^m,\ldots,\rho_n^m)$ be a sequence in $F(P)$ converging to some point $\rho = (\rho_1,\ldots,\rho_n) \in C$. Since $\frac{1}{2}\|\rho_j^m - P_j\rho_j^m\|_j = 0 \leq \|\rho_j^m - \rho_j\|_j$ for $m \in \mathbb{N},\ j=1,2,\ldots,n$, we have

$$\|\rho_j^m - P_j\rho\|_j = \|P_j\rho^m - P_j\rho\|_j$$
$$\leq \sum_{k=1}^{n} a_{jk}\|\rho_k^m - \rho_k\|_k.$$

Making $m \to \infty$, we get

$$\lim_{m\to\infty} \|\rho_j^m - P_j\rho\|_j = 0,\ j=1,2,\ldots,n.$$

That is, $\{\rho_j^m\}$ converges to $P_j\rho$ for $j=1,2,\ldots,n$. Therefore $\rho \in F(P)$ and $F(P)$ is closed. Next, we assume that $B_j,\ j=1,2,\ldots,n$, are strictly convex and $C_j,\ j=1,\ldots,n$, are convex. We fixed $h \in (0,1)$ and $\upsilon = (\upsilon_1,\ldots,\upsilon_n),\ \vartheta = (\vartheta_1,\ldots,\vartheta_n) \in F(P)$ with $\upsilon \neq \vartheta$ and put $\rho_j = h\upsilon_j + (1-h)\vartheta_j \in C_j,\ j=1,\ldots,n$. Then, we have

$$\|\upsilon_j - \vartheta_j\|_j \leq \|\upsilon_j - P_j\rho\|_j + \|\vartheta_j - P_j\rho\|_j$$
$$\leq \|P_j\upsilon - P_j\rho\|_j + \|P_j\vartheta - P_j\rho\|_j$$
$$\leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \rho_k\|_k + \sum_{k=1}^{n} a_{jk}\|\vartheta_k - \rho_k\|_k$$
$$\leq \sum_{k=1}^{n} a_{jk}\left(\|\upsilon_k - \rho_k\|_k + \|\vartheta_k - \rho_k\|_k\right)$$

$$= \sum_{k=1}^{n} a_{jk}((1-h)\|\upsilon_k - \vartheta_k\|_k + h\|\upsilon_k - \vartheta_k\|_k)$$
$$= \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k.$$

From the strict convexity of $B_j,\ j = 1,2,\ldots,n$, there exists $\mu \in [0,1]$ such that $P_j\rho = \mu\upsilon_j + (1-\mu)\vartheta_j,\ j = 1,\ldots,n$. Since for each $j = 1,2,\ldots,n$, we have

$$(1-\mu)\|\upsilon_j - \vartheta_j\|_j = \|P_j\upsilon - P_j\rho\|_j \le \sum_{k=1}^{n} a_{jk}\|\upsilon_k - \rho_k\|_k \le (1-h)\sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k$$

and

$$\mu\|\upsilon_j - \vartheta_j\|_j = \|P_j\vartheta - P_j\rho\|_j \le \sum_{k=1}^{n} a_{jk}\|\vartheta_k - \rho_k\|_k = h\sum_{k=1}^{n} a_{jk}\|\upsilon_k - \vartheta_k\|_k.$$

By the above inequalities, we have $1-\mu \le 1-h$ and $\mu \le h$. These imply $h = \mu$. Therefore, we obtain $\rho \in F(P)$. □

Now, we state a convergence result to a system of mappings which satisfies the condition (D).

Theorem 3.6. *Let $(B_j, \|.\|_j),\ j = 1,2,\ldots,n$, be Banach spaces and $C_j \subseteq B_j,\ j = 1,2,\ldots,n$, be non-empty convex sets. Assume that $P_j : C \to C_j,\ j = 1,2,\ldots,n$, are mappings such that the system of mappings $(P_1,\ldots,P_n)$ satisfies the condition (D). Define, for each $j = 1,2,\ldots,n$, a sequence (υ_j^m) in C_j by $\upsilon_j^1 \in C_j$ and*

$$\upsilon_j^{m+1} = \alpha P_j\upsilon^m + (1-\alpha)\upsilon_j^m$$

for $m \in \mathbb{N}$ and $\alpha \in [1/2,1)$. Then

$$\lim_{m\to\infty} \|P_j\upsilon^m - \upsilon_j^m\| = 0,\ j = 1,2,\ldots n. \tag{3.2}$$

Proof. Let $\upsilon_j^1 \in C_j$ be a fixed arbitrary element for $j = 1,2,\ldots,n$ or $(\upsilon_1^1,\ldots,\upsilon_n^1) = \upsilon^1 \in C$. Define a sequence $(\upsilon_j^{m+1}) \in C_j$ for $\alpha \in [1/2,1)$ such that

$$\upsilon_j^{m+1} = \alpha P_j\upsilon^m + (1-\alpha)\upsilon_j^m,\ j = 1,2,\ldots,n,\ m \in \mathbb{N}. \tag{3.3}$$

Then, from (3.5) and the condition (D), we have

$$\frac{1}{2}\|\upsilon_j^m - P_j\upsilon^m\|_j \le \alpha\|\upsilon_j^m - P_j\upsilon^m\|_j \le \|\upsilon_j^m - \upsilon_j^{m+1}\|_j$$

implies

$$\|P_j\upsilon^m - P_j\upsilon^{m+1}\|_j \leq \sum_{k=1}^{n} a_{jk}\|\upsilon_k^m - \upsilon_k^{m+1}\|_k,\ j = 1,2,\ldots,n.$$

According to Perron-Frobenius's theorem [8], there exist $(r_1, r_2, \ldots, r_n),\ r_j > 0,\ j = 1,2,\ldots,n$ such that

$$\sum_{k=1}^{n} a_{jk} r_k \leq r_j.$$

Let $B = B_1 \times \cdots \times B_n$ and $C = C_1 \times \cdots \times C_n$. Define a norm $\|.\|$ on B as

$$\|\upsilon\| = \sum_{k=1}^{n} r_k \|\upsilon_k\|_k \ \text{ for all } \upsilon \in B.$$

It is easy to prove that $(B, \|.\|)$ forms a Banach space. Define $P : C \to C$, where $C \subset B$, such that

$$P(\upsilon) = (P_1\upsilon, \ldots, P_n\upsilon) \ \text{ for all } \ \upsilon \in C.$$

Then,

$$\begin{aligned}
\|P\upsilon^m - P\upsilon^{m+1}\| &= \sum_{j=1}^{n} r_j \|P_j\upsilon^m - P_j\upsilon^{m+1}\|_j \\
&\leq \sum_{j=1}^{n} r_j \left[\sum_{k=1}^{n} a_{jk}\|\upsilon_k^m - \upsilon_k^{m+1}\|_k\right] \\
&\leq \sum_{k=1}^{n} \left(\sum_{j=1}^{n} r_j a_{jk}\right) \|\upsilon_k^m - \upsilon_k^{m+1}\|_k \\
&\leq \sum_{k=1}^{n} r_k \|\upsilon_k^m - \upsilon_k^{m+1}\|_k = \|\upsilon^m - \upsilon^{m+1}\|.
\end{aligned}$$

It follows that

$$\|P\upsilon^m - P\upsilon^{m+1}\| \leq \|\upsilon^m - \upsilon^{m+1}\|.$$

By Lemma 2.1, we get

$$\lim_{m\to\infty} \|\upsilon^m - P\upsilon^m\| = 0$$

which implies

$$\lim_{m\to\infty} \|\upsilon_j^m - P_j\upsilon^m\|_j = 0,\ j = 1,2,\ldots,n.$$

□

Now, we state an existence result for the new class of system of mappings on the finite product of Banach space.

Theorem 3.7. *Let C_j be a weakly compact convex subset of a UCED Banach space B_j for each $j = 1,2,\dots,n$. Assume that $P : C \to C_j,\ j = 1,2,\dots n$ are mappings on C and the system of mappings $(P_1,\dots,P_n)$ satisfies the condition (D), then the system of equations*

$$P_j(\upsilon_1,\dots,\upsilon_n) = \upsilon_j,\ j = 1,2,\dots,n \tag{3.4}$$

has a solution $(\rho_1,\dots,\rho_n) \in C$.

Proof. Take an arbitrary fixed $\upsilon_j^1 \in C_j,\ j = 1,2,\dots,n$ and define a sequence $(\upsilon_j^{m+1}) \in C_j$ such that

$$\upsilon_j^{m+1} = \frac{1}{2}[P_j\upsilon^m + \upsilon_j^m],\ m \in \mathbb{N},\ j = 1,2,\dots n.$$

Then, following the proof of Theorem 3.6 for $\alpha = 1/2$, we get

$$\limsup_{m\to\infty} \|\upsilon_j^m - P_j\upsilon^m\|_j = 0,\ j = 1,2,\dots,n.$$

Define a continuous convex function $f_j : C_j \to [0,\infty)$ such that

$$f_j(v_j) = \limsup_{m\to\infty} \|\upsilon_j^m - v_j\|_j$$

for each $j = 1,2,\dots,n$ and all $v_j \in C_j,\ m \in \mathbb{N}$. Since $C_j,\ j = 1,2,\dots,n$ are weakly compact and f_j is weakly lower semi-continuous, there exists $\rho_j \in C_j$ such that

$$f_j(\rho_j) = \min\{f_j(v_j) : v_j \in C_j\}.$$

By Lemma 3.4, we have

$$\|\upsilon_j^m - P_j\rho\|_j \le 2\|\upsilon_j^m - P_j\upsilon^m\|_j + \sum_{k=1}^{n} a_{jk}\|\upsilon_k^m - P_k\upsilon^m\|_k + \sum_{k=1}^{n} a_{jk}\|\upsilon_k^m - \rho_k\|_k.$$

Making $\limsup_{m\to\infty}$ on the both side of above inequalities, we get

$$f_j(P_j\rho) \ \le\ \sum_{k=1}^{n} a_{jk}f_k(\rho_k),\ j = 1,2,\dots,n. \tag{3.5}$$

According to Perron-Frobenius's theorem [8], there exist $(r_1,r_2,\dots,r_n),\ r_j > 0,\ j = 1,2,\dots,n,$ such that

$$\sum_{k=1}^{n} a_{jk}r_k \le r_j \ \text{ for } j = 1,2,\dots,n.$$

Let $B = B_1 \times \cdots \times B_n$ and $C = C_1 \times \cdots \times C_n$. Define a norm $\|.\|$ on B as

$$\|v\| = \sum_{k=1}^{n} r_k \|v_k\|_k \text{ for all } v \in B,$$

and $P : C \to C$ and $f : C \to [0,\infty)$ such that $P := (P_1,\ldots,P_n)$ and $f := (f_1,\ldots,f_n)$ respectively. If $\rho = (\rho_1,\ldots,\rho_n),\ v^m = (v_1^m,\ldots,v_n^m) \in B$, then

$$\|\upsilon^m - \rho\| = \sum_{k=1}^{n} r_k \|\upsilon_k^m - \rho_k\|_k.$$

Taking $\limsup_{m\to\infty}$ on the both side, we get

$$f(\rho) = \limsup_{m\to\infty} \|\upsilon^m - \rho\| = \sum_{k=1}^{n} r_k \limsup_{m\to\infty} \|\upsilon_k^m - \rho_k\|_k = \sum_{k=1}^{n} r_k f_k(\rho_k).$$

Therefore,

$$\begin{aligned} f(P\rho) = \limsup_{m\to\infty} \|\upsilon^m - P\rho\| &= \sum_{j=1}^{n} r_j \limsup_{m\to\infty} \|\upsilon_j^m - P_j\rho\|_j \\ &= \sum_{j=1}^{n} r_j f_j(P_j\rho) \\ &\le \sum_{k=1}^{n} \left(\sum_{j=1}^{n} r_j a_{jk} \right) f_k(\rho_k) \ \text{ (from (3.5))} \\ &\le \sum_{k=1}^{n} r_k f_k(\rho_k) = f(\rho). \end{aligned}$$

Since $f(\rho)$ is the minimum therefore $f(P\rho) = f(\rho)$ holds. This implies that $f_j(P_j\rho) = f_j(\rho_j)$ for each $j = 1,2,\ldots,n$. If $P_j\rho \neq \rho_j$ for some j then by strict quasi-convexity, we have

$$f_j(\rho_j) \le f_j\left(\frac{\rho_j + P_j\rho}{2}\right) < \max\{f_j(\rho_j), f_j(P_j\rho)\} = f_j(\rho_j).$$

This is a contraction. Hence $P_j\rho = \rho_j$, for all $j = 1,2,\ldots,n$. □

Now, we present an illustrative example in support of our finding.

Example 3.8. Let $B_1 = B_2 = \mathbb{R}$ be Banach spaces endowed with the usual norm $\|\upsilon_j - \vartheta_j\|_j = |\upsilon_j - \vartheta_j|,\ j = 1,2$ and $C_j = [-1,1] \subset B_j,\ j = 1,2$. Let $P_j : C_1 \times C_2 \to C_j,\ j = 1,2$, be such that

$$P_1(\upsilon_1,\upsilon_2) = -\upsilon_2 \text{ and } P_2(\upsilon_1,\upsilon_2) = -\upsilon_1 \text{ for all } (\upsilon_1,\upsilon_2) \in B_1 \times B_2.$$

Then, the system of mappings (P_1,P_2) satisfies the condition (D) on $C_1\times C_2$ for $a_{11}=a_{22}=1, a_{12}=0$ and $a_{21}=0$. To see this, let for any (υ_1,υ_2), $(\vartheta_1,\vartheta_2)\in C_1\times C_2$, we have $\|P_j(\upsilon_1,\upsilon_2)-P(\vartheta_1,\vartheta_2)\|_j=|\upsilon_j-\vartheta_j|=a_{ii}\|\upsilon_j-\vartheta_j\|_j$ for $j=1,2$. Thus all the assumptions of Theorem 3.7 are satisfied and the system of mappings (P_1,P_2) has a solution $(0,0)$ in $C_1\times C_2$.

Now, we prove an existence result for two systems of mappings using Theorem 3.2.

Theorem 3.9. *Let C_j be a weakly compact convex subset of a UCED Banach space B_j for $j=1,2,\ldots,n$, and $P_j,S_j:C\to C_j,\ j=1,2,\ldots,n$, are mappings on C. Assume that $(S_1,\ldots,S_n)$ and $(P_1,\ldots,P_n)$ are two systems of coordinatewise commuting mappings satisfying the condition (D) on C. Then, the systems of equations*

$$P_j(\upsilon_1,\ldots,\upsilon_n)=\upsilon_j=S_j(\upsilon_1,\ldots,\upsilon_n),\ j=1,2,\ldots,n, \tag{3.6}$$

have a common solution in C.

Proof. Suppose that the system $(P_1,\ldots,P_n)$ satisfies the condition (D). Then by Theorem 3.7, there exists $(\vartheta_1,\ldots,\vartheta_n)\in C$ such that

$$P_j(\vartheta_1,\ldots,\vartheta_n)=\vartheta_j,\ j=1,2,\ldots,n.$$

This follows that $F(P)$, as defined in Lemma 3.5, is a nonempty and from Lemma 3.5, the set $F(P)$ is also closed and convex.

We define $A:=F(P)\cap F(S)$ then A is closed and convex. Now we will prove that $A\neq\emptyset$. Let $\vartheta\in F(P)$. Since two systems of mappings $(S_1,\ldots,S_n)$ and $(P_1,\ldots,P_n)$ are coordinatewise commuting, we have

$$P_j(S_1\vartheta,\ldots,S_n\vartheta)=S_j(P_1\vartheta,\ldots,P_n\vartheta)=S_j\vartheta,\ j=1,2,\ldots,n.$$

These imply $(S_1\vartheta,\ldots,S_n\vartheta)\in F(P)$. By coordinatewise commutativity, we also have

$$S_j\vartheta=P_j(S_1\vartheta,\ldots,S_n\vartheta)=S_j(S_1\vartheta,\ldots,S_n\vartheta),\ j=1,2,\ldots,n.$$

Hence $(S_1\vartheta,\ldots,S_n\vartheta)\in A\neq\emptyset$, meaning $(S_1\vartheta,\ldots,S_n\vartheta)$ is a common solution of the systems of equations (3.6).

□

4. Conclusion

In this paper, we have introduced a new class of systems of mappings which includes both the class of nonexpansive mappings and Suzuki-type generalized nonexpansive mappings. We have presented some existence and convergence results for this new class of systems of mappings. Our results extend the work of Matkowski [13] and Czerwik [3] to a broader class of systems of mappings and open up a scope for new research to explore additional properties of theses mappings in this direction.

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Conflict of Interest

The author declares that there is no conflict of interests.

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