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VARIOUS EXTENSIONS OF THE KANNAN FIXED POINT THEOREM IN C^*-ALGEBRA-VALUED GENERALIZED METRIC SPACES

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Abstract. This work introduce a new concept of a C^*-algebra-valued generalized metric spaces and extend some fixed-point theorems for Kannan-type mappings in this space. The C^*-algebra-valued generalized metric spaces recover various topological spaces, including standard generalized metric spaces and C^*-algebra-valued metric spaces, as well as C^*-algebra-valued b-metric spaces. Our results improve and extend a multiple recent works in the literature. We also give illustrative examples to exhibit the utility of our results. In the end, we give an application in operators equation.

Keywords: C^*-algebra-valued generalized metric space; fixed point; asymptotically regular; approximating fixed point sequence; Kannan contraction.

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1. Introduction

The Banach contraction principle plays a fundamental role in the theory of fixed points. It is used in many areas of mathematics, including the solution of nonlinear differential equations

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and integral equations, such as Volterra and Fredholm integral equations [8]. The practical significance of this principle has prompted various mathematicians to study many interesting extensions and generalizations of this principal.

Among these extensions is the work of Kannan [16], which states that in a complete metric space (X,d), if $T : X \to X$ is a mapping such that $d(Tx,Ty) \leq k(d(Tx,x)+d(y,Ty))$ for all x,y in X, where k is a constant in $[0,\frac{1}{2})$, then T has a unique fixed point x^* in X.

In recent years, several generalisations of standard metric spaces have been studied. In [12] Hitzlar and Sessa introduced the concept of dislocated spaces. The removal of the symmetry of the metric function was another way of extending this concept to quasi-metric spaces by Wilson [24]. The modification of the triangular inequality led to the creation of various types of metric spaces, such as the b-metric spaces of Czerwik and Bakhtin [1, 6, 7], the extended b-metric spaces of Kamran et al [18], the rectangular spaces of Branciari [6] and the modular metric spaces of Nakano [7]. Recently, Samet and Jleli introduced the notion of generalized metric spaces [5, 14, 15].

In 2007, Huang and Zhang [13] introduced the innovative concept of cone metric space, which represents an extension of metric spaces where the underlying space of the metric is replaced by a Banach space.

In 2013, Liu and Xu [19] developed the notion of cone metric space over Banach algebras, hence substituting the Banach space with a Banach algebra as the underlying structure. They obtained several fixed-point theorems for generalized Lipschitzian mappings using the concept of spectral radius while imposing natural and weaker constraints.

Continuing the changes of previous work, Zhenhua Ma et al. [20, 21] replaced Banach algebra in C^*-algebra and using the elements introduced the concept of C^*-algebra-valued metric space in 2014. This introduction yielded various fixed-point theorems within this framework.

Inspired by the concepts mentioned above, we introduce in this work the class of C^*-algebra-valued generalized metric space. This class allows us to generalize some spaces mentioned earlier. In this framework, we establish several fixed point theorems for a Kannan-type mapping and illustrate our contributions with many examples. Also, we have also applied our result to establish a solution of an operator equation in this new framework.

2. PRELIMINARIES

Now, let's recall some useful results on C^*-algebra that will be needed. Let $\mathbb{A}$ be a unitary Banach algebra of zero $0_{\mathbb{A}}$ and unit $1_{\mathbb{A}}$ with a continuous antilinear involution denoted * (called adjoint) such that for all u, v in $\mathbb{A}$,

$$(uv)^* = v^*u^* \text{ and } (u^*)^* = u.$$

If the additional condition on the norm $\|uu^*\| = \|u\|^2$ for all u of $\mathbb{A}$ is satisfied, then $\mathbb{A}$ is called a C^*-algebra.

Let $\mathbb{A}_h$ be the set of all elements u of $\mathbb{A}$ satisfying $u^* = u$. An element u is said to be positive if $u \in A_h$ and $\sigma(u) \subseteq \mathbb{R}_+$, where $\sigma(u)$ is the spectrum of u and we denote $0_{\mathbb{A}} \preceq u$. Using the positive elements, we define a partial order on $\mathbb{A}_h$ as follows

$$u \preceq v \text{ if and only if } 0_{\mathbb{A}} \preceq v - u.$$

Finally, we denote by $\mathbb{A}_+ = \{u \in \mathbb{A} : 0_{\mathbb{A}} \preceq u\}$ the set of all positive elements of $\mathbb{A}$.

Remark 2.1. [22] *Let $\mathbb{A}$ be a unitary C^*-algebra with a unit $1_{\mathbb{A}}$. For any $u \in \mathbb{A}_+$, we have*

$$u \preceq 1_{\mathbb{A}} \iff \|u\| \leqslant 1.$$

Lemma 2.2. [9, 22] *Let $\mathbb{A}$ be a unitary C^*-algebra with a unit $1_{\mathbb{A}}$.*

(i) For each $u \in \mathbb{A}_+$ with $\|u\| < \frac{1}{2}$, the element $1_{\mathbb{A}} - u$ is invertible and $\|(1_{\mathbb{A}} - u)^{-1}u\| < 1$.

(ii) If $u, v \in \mathbb{A}$ such that $0_{\mathbb{A}} \preceq u \preceq v$, then $\|u\| \leq \|v\|$.

(iii) Suppose that $u, v \in \mathbb{A}$ with $0_{\mathbb{A}} \preceq u, v$ and $uv = vu$, then $0_{\mathbb{A}} \preceq uv$.

(iv) If $u \in \mathbb{A}'$ and $v, w \in \mathbb{A}$ where $0_{\mathbb{A}} \preceq w \preceq v$ and $1_{\mathbb{A}} - u \in \mathbb{A}'_+$ is invertible operator, then $(1_{\mathbb{A}} - u)^{-1}w \preceq (1_{\mathbb{A}} - u)^{-1}v$, where

$$\mathbb{A}'_+ = \{u \in \mathbb{A}_+ : \forall v \in \mathbb{A},\ uv = vu\}.$$

Lemma 2.3. [20] *Let $\mathbb{A}$ be a unitary C^*-algebra with unit $1_{\mathbb{A}}$. For all $u, v \in \mathbb{A}_h$ and $w \in \mathbb{A}'_+ := \mathbb{A}^+ \cap \mathbb{A}'$, we have $u \preceq v$ implies that $wu \preceq wv$.*

Theorem 2.4. [9] *Let $\mathbb{A}$ be a C^*-algebra. We have $u^*u \in \mathbb{A}_+$, for any u in $\mathbb{A}$.*

3. MAIN RESULTS

In this section, we introduce the notion of a generalized metric space in the setting of C^*-algebra as follows.

Definition 3.1. Let X be a nonempty set, $\mathbb{A}$ be a C^*-algebra and $\mathscr{D}: X \times X \to \mathbb{A}$ be a given mapping. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-convergent to x with respect to $\mathbb{A}$ and we write $\lim_{n\to\infty} \|\mathscr{D}(x_n, x)\| = 0$, if for given $\varepsilon > 0$, there exists a positive integer N such that $\|\mathscr{D}(x_n, x)\| < \varepsilon$ for all $n > N$.

For every $x \in X$, let us define the set

$$\mathscr{C}(\mathscr{D}, X, x) = \left\{(x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to+\infty} \|\mathscr{D}(x_n, x)\| = 0\right\}.$$

Definition 3.2. Let X be a nonempty set. Suppose that the mapping $\mathscr{D}: X \times X \to \mathbb{A}$ is defined and satisfies the following properties.

- (D_1) $0_{\mathbb{A}} \preceq \mathscr{D}(x, y)$ for all x and y in X;
- (D_2) $\mathscr{D}(x, y) = 0_{\mathbb{A}} \Longrightarrow x = y$;
- (D_3) $\mathscr{D}(x, y) = \mathscr{D}(y, x)$ for all x and y in X;
- (D_4) there exists $c \in \mathbb{A}_+$ with $c \neq 0_{\mathbb{A}}$ such that if $(x, y) \in X \times X$, $(x_n)_{n\in\mathbb{N}} \in \mathscr{C}(\mathscr{D}, X, x)$ and $\limsup_{n\to\infty} \|\mathscr{D}(x_n, y)\| < \infty$, then

$$\mathscr{D}(x, y) \preceq \left(\limsup_{n\to+\infty} \|\mathscr{D}(x_n, y)\|\right).c.$$

In this case, $\mathscr{D}$ is said to be a C^*-algebra-valued generalized metric on X and $(X, \mathbb{A}, \mathscr{D})$ is said to be a C^*-algebra-valued generalized metric space.

Remark 3.3. *If the set $\mathscr{C}(\mathscr{D}, X, x)$ is empty for all $x \in X$, then we consider by convention that $(X, \mathbb{A}, \mathscr{D})$ is a C^*-algebra-valued generalized metric space if and only if the three axioms (D_1),(D_2) and (D_3) are verified.*

Proposition 3.4. *Let $(X, \mathbb{A}, \mathscr{D})$ be a C^*-algebra-valued generalized metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. If $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-convergent to an x with respect to $\mathbb{A}$, then this limit is unique.*

Proof. Suppose that sequence $(x_n)_{n\in\mathbb{N}}$ $\mathscr{D}$-converges to two elements x and y of X with respect to $\mathbb{A}$. By using the property (D_4), we have

$$\mathscr{D}(x,y) \preceq \left(\limsup_{n\to+\infty} \|\mathscr{D}(x_n,y)\|\right).c = 0_{\mathbb{A}},$$

that is, $\mathscr{D}(x,y) = 0_{\mathbb{A}}$ and $x = y$.

□

Example 3.5. Let $L^\infty(\Omega)$ be the set of bounded measurable functions, Ω is an open nonempty set of $\mathbb{R}^m$, $m \in \mathbb{N}\setminus\{0\}$ and $X = \{f \in L^\infty(\Omega) : f \text{ is positive on } \Omega\}$. Let $H = L^2(\Omega)$ be the Hilbert space of functions with integrable square. By $L(H)$ we denote the set of continuous linear operators on H. Clearly $L(H)$ is a C^*-algebra with the usual operator norm. We define $\mathscr{D} : X \times X \to L(H)$ by

$$\mathscr{D}(f,g) = \begin{cases} \frac{1}{2}\pi_{(f+g+1_X)} & \text{if } f = 0 \text{ or } g = 0 \\ \\ 2\pi_{|f-g|} & \text{otherwise,} \end{cases}$$

where $\pi_h : H \to H$ is the multiplication operator with $h \in X$,

$$\pi_h(\phi)(x) = (h.\phi)(x) = h(x)\phi(x) \text{ for all } \phi \in H \text{ and } x \in \Omega,$$

and $1_X : x \to 1$ the constant function on X.

Then, $\mathscr{D}$ is a C^*-algebra-valued generalized metric on X. Indeed, $\mathscr{D}$ satisfies the axioms (D_1), (D_2) and (D_3). To prove that $\mathscr{D}$ satisfies the axiom (D_4); let take $f \in X$ and we distinguish two cases.

- Case 1: $f \neq 0$. In this case $\mathscr{C}(\mathscr{D},X,f)$ is a nonempty set and there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of X such that $\lim_{n\to+\infty} \|\mathscr{D}(f_n,f)\| = 0$.
  If the sequence admits an infinite of zero, there exists a subsequence $(f_{\phi(n)})_{n\geq 0}$ of $(f_n)_{n\geq 0}$ such that $f_{\phi(n)} = 0$ for all $n \in \mathbb{N}$, so $\mathscr{D}(f_{\phi(n)},f) = \frac{1}{2}\pi_{(f+1_X)}$ for all $n \in \mathbb{N}$, which justifies that $\lim_{n\to\infty} \|\mathscr{D}(f_{\phi(n)},f)\| \neq 0$, which is absurd. Thus, there exists $N \in \mathbb{N}$ such that $f_n \neq 0$ for every integer $n \geq N$.

Let $g \in X \setminus \{0\}$, we have

$$\begin{aligned}\mathscr{D}(f,g) &\preceq \|\mathscr{D}(f,g)\|.1_{L(H)} = 2\|\pi_{|f-g|}\|.1_{L(H)} = 2\|f-g\|_\infty.1_{L(H)}\\ &\preceq 2\left(\|f-f_n\|_\infty + \|f_n - g\|_\infty\right).1_{L(H)}\\ &= \left(\|\mathscr{D}(f_n,f)\| + \|\mathscr{D}(f_n,g)\|\right).1_{L(H)} \text{ for any integer } n \geq N.\end{aligned}$$

Thus,

$$\mathscr{D}(f,g) \preceq \left(\limsup_{n\to+\infty} \|\mathscr{D}(f_n,g)\|\right).1_{L(H)}.$$

- Case 2: $f = 0$. Let $(h_n)_{n\in\mathbb{N}} \in \mathscr{C}(\mathscr{D},X,f)$ and consider $\mathbb{P} = \{n \in \mathbb{N} : h_n \neq 0\}$ and $\mathbb{Q} = \{n \in \mathbb{N} : h_n = 0\}$. We can identify three cases.
  - If $\mathbb{P}$ is finite, then $\mathscr{D}(0,g) = \frac{1}{2}\|\pi_{g+1_X}\|.1_{L(H)}$ and from a certain rank,

$$\mathscr{D}(0,g) = \|\mathscr{D}(h_n,g)\|.1_{L(H)} = \limsup_{n\to+\infty} \|\mathscr{D}(h_n,g)\|.1_{L(H)}.$$

  - If $\mathbb{Q}$ is finite, then from a certain rank we have,

$$\mathscr{D}(0,g) \preceq \left(\|\mathscr{D}(0,h_n)\| + \frac{1}{4}\|\mathscr{D}(h_n,g)\|\right).1_{L(H).}$$

    Thus,

$$\mathscr{D}(0,g) \preceq \limsup_{n\to+\infty} \|\mathscr{D}(h_n,g)\|.\left(\frac{1}{4}.1_{L(H)}\right).$$

  - If $\mathbb{P}$ and $\mathbb{Q}$ are infinite, then there exist two increasing functions $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have $h_{\psi(n)} \neq 0$, $h_{\sigma(n)} = 0$ and $\{h_n : n \in \mathbb{N}\} = \{h_{\psi(n)} : n \in \mathbb{N}\} \cup \{h_{\sigma(n)} : n \in \mathbb{N}\}$. We obtain

$$\begin{aligned}\mathscr{D}(0,g) &\preceq \max\{\limsup_{n\to+\infty} \|\mathscr{D}(h_{\psi(n)},g)\|, \limsup_{n\to+\infty} \|\mathscr{D}(h_{\sigma(n)},g)\|\}.1_{L(H)}\\ &\preceq \left(\limsup_{n\to+\infty} \|\mathscr{D}(h_n,g)\|\right).1_{L(H)}.\end{aligned}$$

    This inequality is also true for $g = 0$.

  Finally, for all $g \in X$, there is $c \in L(H)^+$ (simply take $c \succeq 1_{L(H)}$) such that

$$\mathscr{D}(f,g) \preceq \left(\limsup_{n\to+\infty} \|\mathscr{D}(f_n,g)\|\right).c,$$

  where $(f_n)_{n\in\mathbb{N}}$ is an element of $\mathscr{C}(\mathscr{D},X,f)$.

Therefore, $(X, L(H), \mathscr{D})$ is a C^*-algebra-valued generalized metric space.

Remark 3.6. *Any C^*-algebra-valued b-metric space is C^*-algebra-valued generalized metric space.*

Indeed, let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space with $b \succeq 1_{\mathbb{A}}$.
Then d_b satisfies the axioms (D_1), (D_2) and (D_3). For the axiom (D_4), let $(x,y) \in X \times X$ and $(x_n)_{n\in\mathbb{N}} \in \mathscr{C}(d_b, X, x)$. We have for all $n \in \mathbb{N}$,

$$\|d_b(x_n,y)\| \leqslant \|b\|\big(\|d_b(x_n,x)\| + \|d_b(x,y)\|\big),$$

So,

$$\begin{aligned}\limsup_{n\to+\infty} \|d_b(x_n,y)\| &\leqslant \|b\| \ (\limsup_{n\to+\infty}(\|(d_b(x_n,x)\| + \|d_b(x,y))\|)) \\ &= \|d_b(x,y))\| < +\infty.\end{aligned}$$

We also have

$$\begin{aligned}d_b(x,y) &\preceq b\big(d_b(x,x_n) + d_b(x_n,y)\big) \\ &\preceq \|b\|\big(\|d_b(x,x_n) + d_b(x_n,y)\|\big).1_{\mathbb{A}}.\end{aligned}$$

So,

$$\|d_b(x,y)\| \leqslant \|b\|\big(\|d_b(x,x_n)\| + \|d_b(x_n,y)\|\big),$$

which implies that

$$\|d_b(x,y)\| \leqslant \|b\| \ (\limsup_{n\to+\infty}(\|(d_b(x_n,x)\| + \limsup_{n\to+\infty} \|d_b(x,y))\|)).$$

Hence,

$$d_b(x,y) \preceq \|d_b(x,y)\|.1_{\mathbb{A}} \preceq (\limsup_{n\to+\infty} \|d_b(x_n,y)\|).(\|b\|.1_{\mathbb{A}}).$$

Therefore,

$$d_b(x,y) \preceq (\limsup_{n\to+\infty} \|d_b(x_n,y)\|).c, \text{ where } c = \|b\|.1_{\mathbb{A}}.$$

Definition 3.7. Let $(X, \mathbb{A}, \mathscr{D})$ be a C^*-algebra-valued generalized metric space.

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be a $\mathscr{D}$-Cauchy sequence with respect to $\mathbb{A}$ if for given $\varepsilon > 0$, there exists a positive integer N such that $\|\mathscr{D}(x_n,x_m)\| < \varepsilon$ for all $n,m \geq N$.

(ii) We say that $(X, \mathbb{A}, \mathscr{D})$ is a $\mathscr{D}$-complete C^*-algebra-valued generalized metric space if every $\mathscr{D}$-Cauchy sequence is $\mathscr{D}$-convergent with respect to $\mathbb{A}$ in X.

Definition 3.8. Let $(X,\mathbb{A},\mathscr{D})$ be a C^*-algebra-valued generalized metric space. A mapping $T:X\to X$ is called a Kannan mapping if there exists $u\in\mathbb{A}$ with $\|u\|<\frac{1}{2}$ such that for all $(x,y)\in X^2$,

$$\mathscr{D}(Tx,Ty)\preceq u\Big(\mathscr{D}(Tx,x)+\mathscr{D}(y,Ty)\Big). \tag{1}$$

The element u is called the constant of T.

In this section, we establish some fixed point theorems for Kannan-type mapping in the context of C^*-algebra-valued generalized metric space. In the following, $(X,\mathbb{A},\mathscr{D})$ denotes C^*-algebra-valued generalized metric space.

Theorem 3.9. *Let $(X,\mathbb{A},\mathscr{D})$ be a $\mathscr{D}$-complete C^*-algebra-valued generalized metric space of constant c and $T:X\to X$ be a Kannan mapping with constant $u\in\mathbb{A}'_+$ such that $\|u\|<\min\{\frac{1}{2},\frac{1}{\|c\|}\}$. Then T has a unique fixed point $x^*\in X$ and for any $x_0\in X$ the sequence of iterates $(T^nx_0)_{n\in\mathbb{N}}$ $\mathscr{D}$-converges to x^* with respect to $\mathbb{A}$.*

Proof. Choose $x_0\in X$ an arbitrary element and construct the sequence $(x_n)_{n\in\mathbb{N}}$ defined by the following scheme $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$.
From (1), we have

$$\begin{aligned}\mathscr{D}(x_{n+1},x_n)&=\mathscr{D}(Tx_n,Tx_{n-1})\\&\preceq u\mathscr{D}(Tx_n,x_n)+u\mathscr{D}(x_{n-1},Tx_{n-1})\\&=u\mathscr{D}(x_{n+1},x_n)+u\mathscr{D}(x_n,x_{n-1}),\end{aligned}$$

which implies that

$$(1_{\mathbb{A}}-u)\mathscr{D}(x_{n+1},x_n)\preceq u\mathscr{D}(x_n,x_{n-1}).$$

As $\|u\|<\frac{1}{2}$, then $1_{\mathbb{A}}-u$ is invertible and more $(1_{\mathbb{A}}-u)^{-1}=\sum\limits_{n=0}^{\infty}u^n$. Therefore $(1_{\mathbb{A}}-u)^{-1}\in\mathbb{A}'_+$ and by using the Lemma 2.2, we get

$$\mathscr{D}(x_{n+1},x_n)\preceq(1_{\mathbb{A}}-u)^{-1}u\mathscr{D}(x_n,x_{n-1}).$$

So,

$$\mathscr{D}(x_{n+1},x_n)\leq v\mathscr{D}(x_n,x_{n-1}),$$

where $v=(1_{\mathbb{A}}-u)^{-1}u$ and $\|v\|<1$. By using the Reasoning by recurrence, we obtain

$$\mathscr{D}(x_{n+1},x_n)\preceq v^n\mathscr{D}(x_1,x_0). \tag{2}$$

Let $n,m\in\mathbb{N}$ such that $n\geq m\geq 1$, we have

$$\begin{aligned}\mathscr{D}(x_n,x_m)&=\mathscr{D}(Tx_{n-1},Tx_{m-1})\\&\preceq u\mathscr{D}(Tx_{n-1},x_{n-1})+u\mathscr{D}(x_{m-1},Tx_{m-1})\\&=u\mathscr{D}(x_n,x_{n-1})+u\mathscr{D}(x_m,x_{m-1}).\end{aligned}$$

Hence, we get from (2)

$$\mathscr{D}(x_n,x_m)\preceq uv^{n-1}\mathscr{D}(x_1,x_0)+uv^{m-1}\mathscr{D}(x_1,x_0).$$

So,

$$\begin{aligned}\mathscr{D}(x_n,x_m)&\preceq uv^{n-1}\mathscr{D}(x_1,x_0)+uv^{m-1}\mathscr{D}(x_1,x_0)\\&\preceq\big(\|uv^{n-1}\mathscr{D}(x_1,x_0)+uv^{m-1}\mathscr{D}(x_1,x_0)\|\big).1_{\mathbb{A}}\\&\preceq\big(\|uv^{n-1}\mathscr{D}(x_1,x_0)\|+\|uv^{m-1}\mathscr{D}(x_1,x_0)\|\big).1_{\mathbb{A}}\\&\preceq\|\mathscr{D}(x_1,x_0)\|\|u\|\big(\|v\|^{n-1}+\|v\|^{m-1}\big).1_{\mathbb{A}}.\end{aligned}$$

Thus,

$$\mathscr{D}(x_n,x_m)\preceq\delta\|u\|\big(\|v\|^{n-1}+\|v\|^{m-1}\big).1_{\mathbb{A}},$$

where $\delta=\|\mathscr{D}(x_1,x_0)\|$. Since $\delta\|u\|\big(\|v\|^{n-1}+\|v\|^{m-1}\big).1_{\mathbb{A}}\to 0_{\mathbb{A}}$ as $n,m\to\infty$, then $\lim\limits_{n,m\to\infty}\|\mathscr{D}(x_n,x_m)\|=0$, which shows that $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}$-Cauchy sequence with respect to $\mathbb{A}$ in X. As $(X,\mathbb{A},\mathscr{D})$ is $\mathscr{D}$-complete with respect to $\mathbb{A}$, the sequence $(x_n)_{n\in\mathbb{N}}$ $\mathscr{D}$-converges to some x^* in X with respect to $\mathbb{A}$.

By the property (D_4),

$$\begin{aligned}\mathscr{D}(Tx^*,x^*)&\preceq\left(\limsup_{n\to\infty}\|\mathscr{D}(Tx^*,Tx_{n-1})\|\right).c\\&\preceq\left(\limsup_{n\to\infty}\|u\left(\mathscr{D}(Tx^*,x^*)+\mathscr{D}(x_{n-1},Tx_{n-1})\right)\|\right).c\\&\preceq\left(\limsup_{n\to\infty}\|u\|\left(\|\mathscr{D}(Tx^*,x^*)\|+\|\mathscr{D}(x_{n-1},x_n)\|\right)\right).c\\&\preceq\|u\|\left(\limsup_{n\to\infty}\|\mathscr{D}(Tx^*,x^*)\|+\limsup_{n\to\infty}\|\mathscr{D}(x_n,x_{n-1})\|\right).c\end{aligned}$$

$$\preceq \|u\| \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,x^*)\| + \limsup_{n\to\infty} \|v^{n-1}\mathscr{D}(x_1,x_0)\| \right).c$$

$$\preceq \|u\| \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,x^*)\| + \delta \limsup_{n\to\infty} \|v\|^{n-1} \right).c$$

$$= \|u\|\|\mathscr{D}(Tx^*,x^*)\|.c.$$

Hence,

$$\mathscr{D}(Tx^*,x^*) \preceq \big(\|u\|\|\mathscr{D}(Tx^*,x^*)\|\big).c.$$

Using the norm of $\mathbb{A}$ and the Lemma 2.2, we have

$$\|\mathscr{D}(Tx^*,x^*)\| \leq \big\| \left(\|u\|\|\mathscr{D}(Tx^*,x^*)\|\right).c\big\| \leq \|u\|\|c\|\|\mathscr{D}(Tx^*,x^*)\|.$$

Then,

$$\big(1-\|c\|\|u\|\big)\|\mathscr{D}(Tx^*,x^*)\| \leq 0.$$

Since $\|u\|\|c\| < 1$, then $\|\mathscr{D}(Tx^*,x^*)\| = 0$, which implies that $\mathscr{D}(Tx^*,x^*) = 0_{\mathbb{A}}$. It follows that $Tx^* = x^*$. Therefore x^* is a fixed point of T.

Now, let us show the uniqueness of the fixed point of T. Suppose that x^*, y^* are two fixed points of T in X. Since T is a Kannan mapping, we have

$$\mathscr{D}(x^*,x^*) = \mathscr{D}(Tx^*,Tx^*) \preceq u\big(\mathscr{D}(Tx^*,x^*) + \mathscr{D}(x^*,Tx^*)\big),$$

which gives $\mathscr{D}(x^*,x^*) \preceq 2u\mathscr{D}(x^*,x^*)$ and consequently, $\big(1-2\|u\|\big)\|\mathscr{D}(x^*,x^*)\| \leq 0$. Since $(1-2\|u\|) > 0$, then $\|D(x^*,x^*)\| = 0_{\mathbb{A}}$, it follows that $\mathscr{D}(x^*,x^*) = 0_{\mathbb{A}}$. Similarly, we have $\mathscr{D}(y^*,y^*) = 0_{\mathbb{A}}$.

On the other hand, we have

$$\begin{aligned}\mathscr{D}(x^*,y^*) &= \mathscr{D}(Tx^*,Ty^*)\\ &\preceq u\big(\mathscr{D}(Tx^*,x^*) + \mathscr{D}(y^*,Ty^*)\big)\\ &= u\big(\mathscr{D}(x^*,x^*) + \mathscr{D}(y^*,y^*)\big) = 0_{\mathbb{A}}.\end{aligned}$$

Hence $\mathscr{D}(x^*,y^*) = 0_{\mathbb{A}}$. Which implies that $x^* = y^*$. □

Example 3.10. Let $X = \mathbb{R}_+$ and $\mathbb{A} = \mathscr{M}_2(\mathbb{R})$ the set of square matrices of order 2 with $\|M\|_2 = \left(\sum_{i=1}^{4} |a_i|^2\right)^{\frac{1}{2}}$, where a_i are the coefficients of the matrix $M \in \mathbb{A}$. Then $\mathbb{A}$ is a C^*-algebra. We recall that

$$M \in \mathbb{A}_+ \Leftrightarrow (\ {}^tM = M \text{ and } \sigma(M) \subseteq \mathbb{R}_+, \text{ where } \sigma(M) \text{ is the spectrum of } M\ ).$$

Consider the mapping $\mathscr{D}$ defined by

$$\mathscr{D}(x,y) = \begin{cases} \begin{pmatrix} x+y & 0 \\ 0 & x+y \end{pmatrix} & \text{if } x=0 \text{ or } y=0 \\ \\ \begin{pmatrix} 2+\dfrac{x+y}{5} & 0 \\ 0 & 2+\dfrac{x+y}{5} \end{pmatrix} & \text{otherwise.} \end{cases}$$

We verify the axioms of the C^*-algebra-valued generalized metric space.

• It's obvious that the mapping $\mathscr{D}$ verifies the axioms (D_1), (D_2) and (D_3).

• Let us show the axiom (D_4). Let $x \in X$. We distinguish two cases:

Case 1: $x \neq 0$, then the set $\mathscr{C}(\mathscr{D},X,x)$ is empty, because otherwise there exists a sequence $(x_n)_{n\geq 0}$ of elements of X such that $\lim_{n\to\infty} \|\mathscr{D}(x_n,x)\| = 0$.

• If the sequence admits an infinite of zero, there exists a subsequence $(x_{\phi(n)})_{n\geq 0}$ of $(x_n)_{n\geq 0}$ such that $x_{\phi(n)} = 0$ for all $n \in \mathbb{N}$, so $\mathscr{D}(x_{\phi(n)},x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ for all $n \in \mathbb{N}$. As a result, $\lim_{n\to+\infty} \|\mathscr{D}(x_{\phi(n)},x)\|_2 = \sqrt{2}x = 0$, which is absurd since $x \neq 0$.

• If the sequence $(x_n)_{n\geq 0}$ does not admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for any integer $n \geq n_0$. We have $\mathscr{D}(x_n,x) = \begin{pmatrix} 2+\dfrac{x_n+x}{5} & 0 \\ 0 & 2+\dfrac{x_n+x}{5} \end{pmatrix}$ for any integer $n \geq n_0$.

As a result, $\lim_{n\to+\infty} \|\mathscr{D}(x_n,x)\|_2 = \lim_{n\to+\infty} \sqrt{2}(2+\dfrac{x_n+x}{5}) = 0$, which is absurd since $\sqrt{2}(2+\dfrac{x_n+x}{5}) > \frac{\sqrt{2}x}{5}$ for any integer $n \geq n_0$ and $x \neq 0$.

Case 2: $x = 0$. In this case $\mathscr{C}(\mathscr{D},X,x) \neq \emptyset$. Let $(u_n)_{n\geq 0} \in \mathscr{C}(\mathscr{D},X,0)$.

• If the sequence $(u_n)_{n\geq 0}$ admits an infinity of zero, there exists a subsequence $(u_{\psi(n)})_{n\geq 0}$ of $(u_n)_{n\geq 0}$ such that $u_{\psi(n)} = 0$ for all $n \in \mathbb{N}$. Let $y \in X$,

$$\lim_{n\to+\infty} \|\mathscr{D}(u_{\psi(n)},y)\|_2 = \sqrt{2}y \leq \limsup_{n\to+\infty} \|\mathscr{D}(u_n,y)\|_2 \text{ and } \mathscr{D}(x,y) = y.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It remains to be verified that $\limsup_{n\to\infty} \|\mathscr{D}(u_n,y)\| < \infty$.

- If $\mathscr{D}(u_n,y) = \begin{pmatrix} u_n+y & 0 \\ 0 & u_n+y \end{pmatrix}$, then $\mathscr{D}(u_n,y) = \mathscr{D}(u_n,0) + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$.

  So,
$$\limsup_{n\to\infty} \|\mathscr{D}(u_n,y)\| \leq \limsup_{n\to\infty} \|\mathscr{D}(u_n,0)\| + \limsup_{n\to\infty} \|\mathscr{D}(0,y)\|.$$
  Therefore $\limsup_{n\to\infty} \|\mathscr{D}(u_n,y)\| \leq \sqrt{2}y < \infty$.

- If $\mathscr{D}(u_n,y) = \begin{pmatrix} 2+\dfrac{u_n+y}{5} & 0 \\ 0 & 2+\dfrac{u_n+y}{5} \end{pmatrix}$, then $\mathscr{D}(u_n,y) = 5\mathscr{D}(u_n,0) + \begin{pmatrix} 2+\frac{y}{5} & 0 \\ 0 & 2+\frac{y}{5} \end{pmatrix}$.

  In the same way $\limsup_{n\to\infty} \|\mathscr{D}(u_n,y)\| \leq \sqrt{2}(2+\dfrac{y}{5}) < \infty$.

  So, in both cases, we have $\limsup_{n\to\infty} \|\mathscr{D}(u_n,y)\| < \infty$.

• If the sequence $(u_n)_{n\geq 0}$ does not admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $u_n \neq 0$ for any integer $n \geq n_0$. Let $y \in X \setminus \{0\}$. We have

$$\mathscr{D}(u_n,y) = \begin{pmatrix} 2+\dfrac{u_n+y}{5} & 0 \\ 0 & 2+\dfrac{u_n+y}{5} \end{pmatrix} = \frac{1}{5}\mathscr{D}(u_n,0) + \begin{pmatrix} 2+\dfrac{y}{5} & 0 \\ 0 & 2+\dfrac{y}{5} \end{pmatrix},$$

for any integer $n \geq n_0$. As a result,

$$\lim_{n\to+\infty} \|\mathscr{D}(u_n,y)\|_2 = \sqrt{2}(2+\frac{y}{5}) \text{ and } \mathscr{D}(x,y) = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

For $y=0$, $\lim_{n\to+\infty} \|\mathscr{D}(u_n,y)\|_2 = 0$ and $\mathscr{D}(x,y) = 0$.

Hence, for all $y \in X$, there are $c \in \mathbb{A}_+$ $\big($simply take $c \succeq \begin{pmatrix} \dfrac{5\sqrt{2}}{2} & 0 \\ 0 & \frac{5\sqrt{2}}{2} \end{pmatrix}\big)$ such that

$$\mathscr{D}(x,y) = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \preceq \big(\limsup_{n\to\infty} \|\mathscr{D}(x_n,y)\|\big).c.$$

Therefore, $(X,\mathbb{A},\mathscr{D})$ is be a C^*-algebra-valued generalized metric space.

Now, let show the $(X,\mathbb{A},\mathscr{D})$ is $\mathscr{D}$-complete. Let $(u_n)_{n\in\mathbb{N}}$ be a $\mathscr{D}$-Cauchy sequence of X with respect to $\mathbb{A}$.

As for all integer $n,m \in \mathbb{N}$,

$$\begin{pmatrix} 2+\dfrac{u_n+u_m}{5} & 0 \\ 0 & 2+\dfrac{u_n+u_m}{5} \end{pmatrix} \succeq 2.I_2.$$

So from certain rang of n and m,

$$\mathscr{D}(u_n,u_m) = \begin{pmatrix} u_n+u_m & 0 \\ 0 & u_n+u_m \end{pmatrix}.$$

Let $\varepsilon > 0$. Since $(u_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-Cauchy sequence of X, there exists $N_0 \in \mathbb{N}$ such that

$$\left(n,m \geqslant N_0 \implies \|\mathscr{D}(u_n,u_m)\| = \left\| \begin{pmatrix} u_n+u_m & 0 \\ 0 & u_n+u_m \end{pmatrix} \right\| = \sqrt{2}(u_n+u_m) < \varepsilon \right).$$

Then,

$$\left(n,m \geqslant N_0 \implies 0 \leqslant u_n \leqslant u_n+u_m < \frac{\varepsilon}{\sqrt{2}}\right).$$

Thus, for any $n \geqslant N_0$ we have $\|\mathscr{D}(u_n,0)\| = \left\| \begin{pmatrix} u_n & 0 \\ 0 & u_n \end{pmatrix} \right\| < \varepsilon$, which shows that the sequence $(u_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to 0 in X with respect to $\mathbb{A}$, that is, $(X,\mathbb{A},D)$ is $\mathscr{D}$-complete with respect to $\mathbb{A}$.

Consider the mapping T defined on X by $T(x) = \begin{cases} 0 & \text{if } x \in [0,21] \\ 2 & \text{if } x \in (21,+\infty). \end{cases}$

- For each $x,y \in (21,+\infty)$, we have $\mathscr{D}(Tx,Ty) = \begin{pmatrix} \frac{14}{5} & 0 \\ 0 & \frac{14}{5} \end{pmatrix}$, $\mathscr{D}(x,Tx) = \begin{pmatrix} \frac{12+x}{5} & 0 \\ 0 & \frac{12+x}{5} \end{pmatrix}$ and $\mathscr{D}(y,Ty) = \begin{pmatrix} \frac{12+y}{5} & 0 \\ 0 & \frac{12+y}{5} \end{pmatrix}$.

  Then, $\frac{1}{3}\left(\frac{12+x}{5}+\frac{12+y}{5}\right) > 4 > \frac{14}{5}$ and so,

$$\mathscr{D}(T(x),T(y)) \preceq \frac{1}{3}\left(\mathscr{D}(x,T(x))+\mathscr{D}(y,T(y))\right).$$

- For each $x \in [0,21]$ and $y \in (21,+\infty)$, we have $\mathscr{D}(Tx,Ty) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathscr{D}(x,Tx) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $\mathscr{D}(y,Ty) = \begin{pmatrix} \frac{12+y}{5} & 0 \\ 0 & \frac{12+y}{5} \end{pmatrix}$. Then, $\frac{1}{3}\left(x + \frac{12+y}{5}\right) > 2$ and so,

$$\mathscr{D}(T(x),T(y)) \preceq \frac{1}{3}\left(\mathscr{D}(x,T(x)) + \mathscr{D}(y,T(y))\right).$$

- We also have the result for $x,y \in [0,21]$.

So, for all $x,y \in X$, $\mathscr{D}(T(x),T(y)) \preceq u\left(\mathscr{D}(x,T(x)) + \mathscr{D}(y,T(y))\right)$, where $u = \frac{1}{3}.I_2$ with $\|u\| = \frac{\sqrt{2}}{3}$.

Thus, all the conditions of Theorem 3.9 are satisfied. Then T has a unique fixed point which is 0.

Corollary 3.11. *Let $(X,\mathbb{A},\mathscr{D})$ be a $\mathscr{D}$-complete C^*-algebra-valued generalized metric space and $T : X \to X$ is a mapping such that for some positive integer $N \geq 2$, T^N is a mapping such that there exists $u \in \mathbb{A}_+^{'}$ with $\|u\| < \min\{\frac{1}{2}, \frac{1}{\|c\|}\}$ satisfying for all $x,y \in X$,*

$$\mathscr{D}(T^N x, T^N y) \preceq u\big(\mathscr{D}(T^N x, x) + \mathscr{D}(y, T^N y)\big).$$

Then, T has a unique fixed point.

Proof. Since T^N is a Kannan mapping, then according to Theorem 3.9, the mapping T^N has a unique fixed point x^* in X. That is $T^N(x^*) = x^*$, then $T^{N+1}(x^*) = Tx^*$. This implies that $T^N(Tx^*) = Tx^*$ i.e. Tx^* is a fixed point of T^N. And according to the uniqueness of fixed point of T^N, we have $Tx^* = x^*$. □

Theorem 3.12. *Let $(X,\mathbb{A},\mathscr{D})$ be a C^*-algebra-valued generalized metric space, $N \geq 2$ a positive integer and $u \in \mathbb{A}_+^{'}$ with $\|u\| < \dfrac{1}{2}$. Let $T : X \to X$ be a mapping such that for all $x,y \in X$, we have*

$$\mathscr{D}(T^N x, T^N y) \preceq u\big(\mathscr{D}(Tx, x) + \mathscr{D}(y, Ty)\big). \tag{3}$$

If there is an $x^* \in X$ such that $T^N(x^*) = x^*$, then x^* is a fixed point of T.

Proof. Let $x^* \in X$ and $T^N(x^*) = x^*$. Then, from relation (3), we have

$$\mathscr{D}(x^*,Tx^*) = \mathscr{D}(T^N x^*, T^{N+1}x^*) \preceq u\big(\mathscr{D}(Tx^*,x^*) + \mathscr{D}(Tx^*,T^2x^*)\big).$$

So,

$$(1_{\mathbb{A}} - u)\mathscr{D}(x^*,Tx^*) \preceq u\mathscr{D}(Tx^*,T^2x^*).$$

Using the Lemma 2.2, we obtain

$$\mathscr{D}(x^*,Tx^*) \preceq (1_{\mathbb{A}} - u)^{-1}u\mathscr{D}(Tx^*,T^2x^*).$$

Also

$$\mathscr{D}(Tx^*,T^2x^*) = \mathscr{D}(T^{N+1}x^*,T^{N+2}x^*) \preceq u\big(\mathscr{D}(T^2x^*,Tx^*) + \mathscr{D}(T^2x^*,T^3x^*)\big).$$

So,

$$\mathscr{D}(Tx^*,T^2x^*) \preceq (1_{\mathbb{A}} - u)^{-1}u\mathscr{D}(T^2x^*,T^3x^*).$$

We have also,

$$\begin{aligned}\mathscr{D}(T^{N-1}x^*,T^Nx^*) &= \mathscr{D}(T^{N+(N-1)}x^*,T^{N+N}x^*)\\ &\preceq u\big(\mathscr{D}(T^Nx^*,T^{N-1}x^*) + \mathscr{D}(T^Nx^*,T^{N+1}x^*)\big).\end{aligned}$$

We obtain

$$\mathscr{D}(T^{N-1}x^*,T^Nx^*) \preceq (1_{\mathbb{A}} - u)^{-1}u\mathscr{D}(T^Nx^*,T^{N+1}x^*).$$

Then we deduce that,

$$\mathscr{D}(x^*,Tx^*) \preceq \big((1_{\mathbb{A}} - u)^{-1}u\big)^N\mathscr{D}(T^Nx^*,T^{N+1}x^*).$$

Hence $\mathscr{D}(x^*,Tx^*) \preceq v^N\mathscr{D}(x^*,Tx^*)$, where $v = (1_{\mathbb{A}} - u)^{-1}u$. Using the norm of $\mathbb{A}$ and the Lemma 2.2, we get

$$\|\mathscr{D}(x^*,Tx^*)\| \leq \|v^N\mathscr{D}(x^*,Tx^*)\| \quad \leq \|v\|^N\|\mathscr{D}(x^*,Tx^*)\|.$$

Thus,

$$\big(1 - \|v\|^N\big)\|\mathscr{D}(x^*,Tx^*)\| \leq 0.$$

As $\|v\| < 1$, then $\|\mathscr{D}(x^*,Tx^*)\| = 0$, this implies that $\mathscr{D}(x^*,Tx^*) = 0_{\mathbb{A}}$. Therefore $Tx^* = x^*$. □

In the follows, it is therefore useful to study the situation where $T : X \to X$ is not necessarily a Kannan mapping, but T^N is a Kannan mapping for some $N \geq 2$. We give bellow an example of a mapping where T is not a Kannan mapping, but T^2 is a Kannan mapping.

Example 3.13. Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$ be a compact Hausdorff space and $A = \mathscr{M}_3(\mathbb{C})$ equipped with the norm $\|.\|_\infty$ defined by $\|A\|_\infty = \max_{1\leq i\leq 3}(\sum_{j=1}^{3}|a_{i,j}|)$ for all $A = [a_{i,j}]_{1\leq i,j\leq 3} \in \mathbb{A}$ with $A^* = [\overline{a_{j,i}}]_{1\leq i,j\leq 3}$. Define $\mathscr{D} : X \times X \longrightarrow \mathbb{A}$ as

$$\mathscr{D}(z,z') = \begin{pmatrix} a|z-z'| & 0 & 0 \\ 0 & b|z-z'| & 0 \\ 0 & 0 & c|z-z'| \end{pmatrix}, \text{ where } a,b,c > 0 \text{ and } z,z' \in X.$$

Then $(X,\mathbb{A},\mathscr{D})$ is a C^*-algebra-valued generalized metric space since a C^*-algebra-valued metric space. Consider the mapping $T : X \to X$ defined by

$$Tz = \begin{cases} \dfrac{z}{2} & \text{if} \quad |z| < 1 \\ \\ \dfrac{1}{4} & \text{if} \quad |z| = 1 \end{cases}$$

We have

$$\mathscr{D}(T0, T\frac{1}{2}) = \big(\mathscr{D}(T0,0) + \mathscr{D}(\frac{1}{2}, T\frac{1}{2})\big).$$

We deduce that T is not a Kannan mapping. In the other side, we can see that the mapping T^2 is defined by

$$T^2z = \begin{cases} \dfrac{z}{4} & \text{if} \quad |z| < 1 \\ \\ \dfrac{1}{8} & \text{if} \quad |z| = 1. \end{cases}$$

For any $z, z' \in \mathbb{C}$,

- If $|z| < 1$ and $|z'| < 1$,

$$\mathscr{D}(T^2z, T^2z') = \begin{pmatrix} a|\frac{z}{4} - \frac{z'}{4}| & 0 & 0 \\ 0 & b|\frac{z}{4} - \frac{z'}{4}| & 0 \\ 0 & 0 & c|\frac{z}{4} - \frac{z'}{4}| \end{pmatrix}$$

$$\preceq \begin{pmatrix} a(\frac{|z|}{4} + \frac{|z'|}{4}) & 0 & 0 \\ 0 & b(\frac{|z|}{4} + \frac{|z'|}{4}) & 0 \\ 0 & 0 & c(\frac{|z|}{4} + \frac{|z'|}{4}) \end{pmatrix}$$

$$\preceq \frac{1}{3}\begin{pmatrix} a\frac{3|z|}{4} & 0 & 0 \\ 0 & b\frac{3|z|}{4} & 0 \\ 0 & 0 & c\frac{3|z|}{4} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} a\frac{3|z'|}{4} & 0 & 0 \\ 0 & b\frac{3|z'|}{4} & 0 \\ 0 & 0 & c\frac{3|z'|}{4} \end{pmatrix}$$
$$= \frac{1}{3}\big(\mathscr{D}(Tz,z) + \mathscr{D}(z',Tz')\big).$$

– If $|z| < 1$ and $|z'| = 1$,

$$\mathscr{D}(T^2z,T^2z') = \begin{pmatrix} a|\frac{z}{4}-\frac{1}{8}| & 0 & 0 \\ 0 & b|\frac{z}{4}-\frac{1}{8}| & 0 \\ 0 & 0 & c|\frac{z}{4}-\frac{1}{8}| \end{pmatrix}$$
$$\preceq \frac{1}{3}\begin{pmatrix} \frac{3a|z|}{4} & 0 & 0 \\ 0 & \frac{3b|z|}{4} & 0 \\ 0 & 0 & \frac{3c|z|}{4} \end{pmatrix} + \frac{1}{7}\begin{pmatrix} \frac{7a}{8} & 0 & 0 \\ 0 & \frac{7b}{8} & 0 \\ 0 & 0 & \frac{7c}{8} \end{pmatrix}$$
$$\preceq \frac{1}{3}\big(\mathscr{D}(T^2z,z) + \mathscr{D}(z',T^2z')\big).$$

– If $|z| = |z'| = 1$, $\mathscr{D}(T^2z,T^2z') = 0_{\mathbb{A}} \prec \frac{1}{3}\big(\mathscr{D}(T^2z,z) + \mathscr{D}(z',T^2z')\big)$.

Hence, for all $z,z' \in X$, $\mathscr{D}(T^2z,T^2z') \preceq u\big(\mathscr{D}(T^2z,z) + \mathscr{D}(z',T^2z')\big)$, where $u = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

Therefore, T^2 is a Kannan mapping.

Note that the condition $u \in \mathbb{A}'_+$ and $\|u\| < \min\{\frac{1}{2}, \frac{1}{\|c\|}\}$ is necessary in Theorem 3.9. The following example is giving to illustrate this fact.

Example 3.14. Let $X = \mathbb{R}$ and $\mathbb{A} = B(H)$ the space of all bounded linear operators on Hilbert space H. We have $\mathbb{A}$ is a C^*-algebra. Let's define the mapping
$\mathscr{D} : X \times X \longrightarrow \mathbb{A}$ by

$$\mathscr{D}(x,y) = \begin{cases} 0_{\mathbb{A}} & \text{if} \quad x = y \\ 1_{\mathbb{A}} & \text{if} \quad x \neq y. \end{cases}$$

Then $(X,\mathbb{A},\mathscr{D})$ is a $\mathscr{D}$-Complete C^*-algebra-valued generalized metric space with $c=1_{\mathbb{A}}$. Let $T:X\to X$ be a mapping such that $Tx=x^2+3$. Then we have for all $x,y\in X$,

$$\mathscr{D}(Tx,Ty)\preceq\frac{1_{\mathbb{A}}}{2}\big(\mathscr{D}(Tx,x)+\mathscr{D}(y,Ty)\big).$$

Clearly, the mapping T does not admit a fixed point.

In the following, we will be interested in Kannan-type mappings with constant u such that $u\in\mathbb{A}$ and $\|u\|<1$ and not necessarily $u\in\mathbb{A}_+^{'}$ and $\|u\|<\frac{1}{2}$. To ensure that this mapping has a fixed point, we need to extend the definition of F.E Browder's [4] concept of asymptotically regular mapping and J.Górnitchi's [11] concept of the approximate fixed point sequence.

Definition 3.15. Let $(X,\mathbb{A},\mathscr{D})$ be a C^*-algebra-valued generalized metric space and $T:X\to X$ a mapping. We say that T is $\mathscr{D}$-asymptotically regular with respect to $\mathbb{A}$, if we have
$\lim\limits_{n\to+\infty}\|\mathscr{D}(T^{n+1}x,T^nx)\|=0$ for all $x\in X$.

Definition 3.16. Let C be a nonempty subset of C^*-algebra-valued generalized metric space $(X,\mathbb{A},\mathscr{D})$ and $T:C\to C$ a mapping.Then a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be an $\mathscr{D}$-approximating fixed point sequence of T with respect to $\mathbb{A}$ if $\lim\limits_{n\to+\infty}\|\mathscr{D}(Tx_n,x_n)\|=0$.

Example 3.17. Let $X=\{z\in\mathbb{C}:|z|\leq1\}$ be a compact Hausdorff space and $\mathscr{B}(X)$ be the set of all bounded Borel functions on X, equipped with the norm $\|f\|=\sup\limits_{z\in X}|f(z)|$ with $f^*(z)=\overline{f(z)}$ for all $f\in\mathscr{B}(X)$ and $z\in X$. Define the mapping $\mathscr{D}:X\times X\to\mathscr{B}(X)$ by

$$\mathscr{D}(z,z')=\begin{cases}\dfrac{(|z|+|z'|)}{2}.f^*f & \text{if } z=0 \text{ or } z'=0\\[2ex] \dfrac{(|z|+|z'|)}{3}.f^*f & \text{otherwise,}\end{cases}$$

where $f\in\mathscr{B}(X)$ with $f\neq0_{\mathscr{B}(X)}$ fixed.

It's obvious that the mapping $\mathscr{D}$ verifies the axioms (D_1), (D_2) and (D_3). Let us show the axiom (D_4). Let $z\in X$. We distinguish two cases

Case 1: $z\neq0$, then the set $\mathscr{C}(\mathscr{D},X,z)$ is empty, because otherwise there exists a sequence $(z_n)_{n\geq0}$ of elements of X such that $\lim\limits_{n\to\infty}\|\mathscr{D}(z_n,z)\|=0$.

- If the sequence admits an infinite of zero, there exists a subsequence $(z_{\phi(n)})_{n\geq0}$ of

$(z_n)_{n\geq 0}$ such that $z_{\phi(n)} = 0$ for all $n \in \mathbb{N}$, so $\mathscr{D}(z_{\phi(n)}, z) = \frac{|z|}{2}.f^*f$ for all $n \in \mathbb{N}$. As a result $\lim_{n\to+\infty} \|\mathscr{D}(z_{\phi(n)}, z)\| = \frac{|z|}{2}\ \|f\|^2$, which is absurd since $z \neq 0$ and $f \neq 0_{\mathscr{B}(X)}$.

• If the sequence $(z_n)_{n\geq 0}$ does not admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $z_n \neq 0$ for any integer $n \geq n_0$. We have $\mathscr{D}(z_n, z) = \frac{|z_n| + |z|}{3}.f^*f$ for any integer $n \geq n_0$. As a result,

$$\lim_{n\to+\infty} \|\mathscr{D}(z_n, z)\| = \lim_{n\to+\infty} \frac{|z_n| + |z|}{3}.\|f\|^2 = 0,$$

which is absurd since $\frac{|z_n| + |z|}{3}.\|f\|^2 > \frac{|z|}{3}.\|f\|^2$ for any integer $n \geq n_0$ and $z \neq 0$, $f \neq 0_{\mathscr{B}(X)}$.

Case 2: $z = 0$. In this case $\mathscr{C}(\mathscr{D}, X, z) \neq \emptyset$. Let $(z_n)_{n\geq 0} \in \mathscr{C}(\mathscr{D}, X, 0)$.

• If the sequence $(z_n)_{n\geq 0}$ admits an infinity of zero, there exists a subsequence $(z_{\psi(n)})_{n\geq 0}$ of $(z_n)_{n\geq 0}$ such that $z_{\psi(n)} = 0$ for all $n \in \mathbb{N}$. Let $y \in X$,

$$\lim_{n\to+\infty} \|\mathscr{D}(z_{\psi(n)}, y)\| = \frac{|y|}{2}\ \|f\|^2 \leq \limsup_{n\to+\infty} \|\mathscr{D}(z_n, y)\| \text{ and } \mathscr{D}(z, y) = \frac{|y|}{2}.f^*f.$$

It remains to be verified that $\limsup_{n\to\infty} \|\mathscr{D}(z_n, y)\| < \infty$.

- If $\mathscr{D}(z_n, y) = (\frac{|z_n| + |y|}{2}).f^*f$, then $\mathscr{D}(z_n, y) = \mathscr{D}(z_n, 0) + \frac{|y|}{2}.f^*f$. So, $\limsup_{n\to\infty} \|\mathscr{D}(z_n, y)\| \leq \limsup_{n\to\infty} \|\mathscr{D}(z_n, 0)\| + \limsup_{n\to\infty} \|\mathscr{D}(0, y)\|$.
  Therefore $\limsup_{n\to\infty} \|\mathscr{D}(u_n, y)\| \leq \frac{|y|}{2}\|f\|^2 < \infty$.

- If $\mathscr{D}(z_n, y) = (\frac{|z_n| + |y|}{3}).f^*f$, then $\mathscr{D}(u_n, y) = \frac{2}{3}\mathscr{D}(z_n, 0) + \frac{|y|}{3}.f^*f$.
  In the same way $\limsup_{n\to\infty} \|\mathscr{D}(z_n, y)\| \leq \frac{|y|}{3}\|f\|^2 < \infty$.
  So, in both cases, we have $\limsup_{n\to\infty} \|\mathscr{D}(z_n, y)\| < \infty$

• If the sequence $(z_n)_n$ does not admit an infinity of zero, then there exists $n_0 \in \mathbb{N}$ such that $z_n \neq 0$ for any integer $n \geq n_0$. Let $y \in X \setminus \{0\}$. We have $\mathscr{D}(z_n, y) = \frac{|z_n| + |y|}{3}.f^*f = \frac{2}{3}\mathscr{D}(z_n, 0) + \frac{1}{3}|y|.f^*f$ for any integer $n \geq n_0$. As a result,

$$\lim_{n\to+\infty} \|\mathscr{D}(z_n, y)\| = \frac{|y|}{3}\|f\|^2 \text{ and } \mathscr{D}(z, y) = \frac{|y|}{2}f^*f.$$

For $y = 0$, $\lim_{n\to+\infty} \|\mathscr{D}(z_n, y)\| = 0$ and $\mathscr{D}(z, y) = 0$.

Hence, for all $y \in X$, there are $c \in \mathscr{B}(X)^+$ $\big($simply take $c \succeq 6.1_{\mathscr{B}(X)}\big)$ such that

$$\mathscr{D}(z,y) = \frac{|y|}{2}.f^*f \preceq (\frac{|y|}{2}\,\|f\|^2).1_{\mathscr{B}(X)} \preceq \big(\limsup_{n\to\infty}\|\mathscr{D}(z_n,y)\|\big).c.$$

Therefore, $(X,\mathscr{B}(X),\mathscr{D})$ is a C^*-algebra-valued generalized metric space.

Consider the mapping $T : X \to X$ defined by $Tz = z^2$. The sequence $(z_n)_{n\geq 1}$ defined by $z_n = \frac{i}{n}$ for all $n \in \mathbb{N}\setminus\{0\}$ with $i^2 = -1$, is an $\mathscr{D}$-approximate sequence of fixed points of T with respect to $\mathscr{B}(X)$, because

$$\|\mathscr{D}(Tz_n,z_n)\| = \frac{1}{3}(\frac{1}{n^2}+\frac{1}{n}).\|f\|^2 \to 0 \text{ as } n \to \infty.$$

Example 3.18. Under the assumptions of Example 3.13. Consider the mapping T defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if} \quad x = 0 \\ \\ \frac{x}{3} & \text{if} \quad 0 < x \leq 1. \end{cases}$$

We have, for all $x \in X$, $\lim\limits_{n\to+\infty}\|\mathscr{D}(T^{n+1}x,T^nx)\| = 0$. Therefore, T is $\mathscr{D}$-asymptotically regular with respect to $\mathbb{A}$.

Theorem 3.19. *Let $(X,\mathbb{A},\mathscr{D})$ be a $\mathscr{D}$-complete C^*-algebra-valued generalized metric space and $T : X \to X$ be a $\mathscr{D}$-asymptotically regular mapping with respect to $\mathbb{A}$ such that for all $x,y \in X$,*

$$\mathscr{D}(Tx,Ty) \preceq u\left(\mathscr{D}(Tx,x)+\mathscr{D}(y,Ty)\right),$$

where $u \in \mathbb{A}$ with $\|u\| < \min\{1,\frac{1}{\|c\|}\}$.
Then, T has a fixed point $x^* \in X$. Moreover, if $\|u\| < \frac{1}{2}$, then T has a unique fixed point.

Proof. • Let $x_0 \in X$ and construct the sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.
Let $n,m \in \mathbb{N}$ such that $n \geq m \geq 1$, we have

$$\begin{aligned}
\mathscr{D}(x_n,x_m) &= \mathscr{D}(Tx_{n-1},Tx_{m-1}) \\
&= \mathscr{D}(T^nx_0,T^mx_0) \\
&\preceq u\mathscr{D}(T^nx_0,T^{n-1}x_0) + u\mathscr{D}(T^{m-1}x_0,T^mx_0) \\
&\preceq \|u\|\big(\|\mathscr{D}(T^nx_0,T^{n-1}x_0)\| + \|\mathscr{D}(T^{m-1}x_0,T^mx_0)\|\big).1_{\mathbb{A}}.
\end{aligned}$$

Since T is $\mathscr{D}$-asymptotically regular with respect to $\mathbb{A}$, then

$$\|u\|\big(\|\mathscr{D}(T^n x_0, T^{n-1}x_0)\| + \|\mathscr{D}(T^{m-1}x_0, T^m x_0)\|\big) \to 0,$$

as $n,m \to +\infty$, which gives $\lim\limits_{n,m\to+\infty} \|\mathscr{D}(x_n,x_m)\| = 0$. Then $(x_n)_{n\in\mathbb{N}}$ is a $\mathscr{D}$-Cauchy sequence with respect to $\mathbb{A}$ in X. Since $(X,\mathbb{A},\mathscr{D})$ is $\mathscr{D}$-complete with respect to $\mathbb{A}$, so the sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathscr{D}$-converge to some x^* in X with respect to $\mathbb{A}$.

On the other hand, since $\lim\limits_{n\to+\infty} \|\mathscr{D}(x_n,x^*)\| = 0$, we have

$$\begin{aligned}
\mathscr{D}(Tx^*,x^*) &\preceq \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,x_n)\|\right).c \\
&= \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,Tx_{n-1})\|\right).c \\
&\preceq \left(\limsup_{n\to\infty} \|u\big(\mathscr{D}(Tx^*,x^*) + \mathscr{D}(x_{n-1},Tx_{n-1})\big)\|\right).c \\
&= \left(\limsup_{n\to\infty} \|u\left(\mathscr{D}(Tx^*,x^*) + \mathscr{D}(T^{n-1}x_0,T^n x_0)\right)\|\right).c \\
&\preceq \|u\| \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,x^*)\| + \limsup_{n\to\infty} \|\mathscr{D}(T^{n-1}x_0,T^n x_0)\|\right).c \\
&\preceq \|u\|\|\mathscr{D}(Tx^*,x^*)\|.c.
\end{aligned}$$

So,

$$\mathscr{D}(Tx^*,x^*) \preceq \|u\|\|\mathscr{D}(Tx^*,x^*)\|.c.$$

Using the norm of $\mathbb{A}$ and the Lemma 2.2, we get

$$\|\mathscr{D}(Tx^*,x^*)\| \leq \|u\|\|c\|\mathscr{D}(Tx^*,x^*)\|.$$

So,

$$(1-\|u\|\|c\|)\,\|\mathscr{D}(Tx^*,x^*)\| \leq 0.$$

As $\|u\|\|c\| < 1$, then $||\mathscr{D}(Tx^*,x^*)|| = 0$, which implies that $\mathscr{D}(Tx^*,x^*) = 0_{\mathbb{A}}$, therefore $Tx^* = x^*$.

• To prove the uniqueness of the fixed point of T in the case $\|u\| < \frac{1}{2}$, we use the same procedure as in the Theorem 3.9. □

Remark 3.20. *To show that there is no uniqueness in the case where $\|u\| \in \left[\frac{1}{2}, 1\right)$, we can take the following example.*

Example 3.21. Under the assumptions of Example 3.17, consider the mapping $T : X \to X$ defined by

$$Tz = \begin{cases} z & \text{if} \quad 0 \leq |z| < 1 \\ \\ 0 & \text{if} \quad |z| = 1. \end{cases}$$

- For all $z, z' \in \mathbb{C}$ such that $0 < |z| < 1$ and $0 < |z'| < 1$, we have
  $\mathscr{D}(Tz, z) = \frac{2}{3}|z|.f^*f$, $\mathscr{D}(z', Tz') = \frac{2}{3}|z'|.f^*f$ and $\mathscr{D}(Tz, Tz') = \frac{1}{3}(|z| + |z'|)f^*f$. So,
  $$\mathscr{D}(Tz, Tz') = \frac{1}{2}\big(\mathscr{D}(Tz, z) + \mathscr{D}(z', Tz')\big)$$
- For all $z = 0$ and $0 < |z| < 1$, we have
  $\mathscr{D}(T0, 0) = 0$, $\mathscr{D}(z, Tz) = \frac{2}{3}|z|.f^*f$ and $\mathscr{D}(T0, Tz) = \frac{1}{2}|z|.f^*f$. So,
  $$\mathscr{D}(T0, Tz) \preceq \frac{3}{4}\big(\mathscr{D}(T0, 0) + \mathscr{D}(z, Tz)\big).$$
- For all $0 < |z| < 1$ and $|z'| = 1$, we have
  $\mathscr{D}(Tz, z) = \frac{2}{3}|z|.f^*f$, $\mathscr{D}(z', Tz') = \frac{1}{2}|z'|.f^*f$ and $\mathscr{D}(Tz, Tz') = \frac{1}{2}|z|.f^*f$. So,
  $$\mathscr{D}(Tz, Tz') \preceq \frac{3}{4}\big(\mathscr{D}(Tz, z) + \mathscr{D}(z', Tz')\big).$$
- For all $z = 0$ and $|z| = 1$, we have
  $\mathscr{D}(T0, 0) = 0$, $\mathscr{D}(z, Tz) = \frac{1}{2}|z|.f^*f$ and $\mathscr{D}(T0, Tz) = 0$. So,
  $$\mathscr{D}(T0, Tz) \preceq \frac{3}{4}\big(\mathscr{D}(Tz, z) + \mathscr{D}(z', Tz')\big).$$
- For all $|z| = |z'| = 1$,
  $\mathscr{D}(Tz, z) = \frac{1}{2}|z|.f^*f$, $\mathscr{D}(z', Tz') = \frac{1}{2}|z'|.f^*f$ and $\mathscr{D}(Tz, Tz') = 0$. So,
  $$\mathscr{D}(Tz, Tz') \preceq \frac{3}{4}\big(\mathscr{D}(Tz, z) + \mathscr{D}(z', Tz')\big).$$
- In particular, for $z = 0$, we have $\mathscr{D}(T0, T0) \preceq \frac{3}{4}\big(\mathscr{D}(T0, 0) + \mathscr{D}(0, T0)\big)$.

Finally, for all $z, z' \in X$, we have $\mathscr{D}(Tz, Tz') \preceq \frac{3}{4}\big(\mathscr{D}(Tz, z) + \mathscr{D}(z', Tz')\big)$.
The mapping T is $\mathscr{D}$-asymptotically regular with respect to $\mathscr{B}(X)$. But T admits an infinite number of fixed points. In other words, it does not have uniqueness when $\frac{1}{2} \leq \|u\| < 1$.

Definition 3.22. Let $(X, \mathbb{A}, \mathscr{D})$ be a C^*-algebra-valued generalized metric space. A subset C of X is said to be $\mathscr{D}$-Compact with respect to $\mathbb{A}$ if any sequence $(x_n)_{n\in\mathbb{N}}$ of C has a $(x_{\phi(n)})_{n\in\mathbb{N}}$ $\mathscr{D}$-convergent subsequence with respect to $\mathbb{A}$ in C.

Theorem 3.23. *Let $(X,\mathbb{A},\mathscr{D})$ be a C^*-algebra-valued generalized metric space and C is a $\mathscr{D}$-compact subset of X with respect to $\mathbb{A}$. Let $T : C \to C$ be a map such that there exists $u \in \mathbb{A}$ with $\|u\| < \min\{1, \frac{1}{\|c\|}\}$ satisfying for all $x,y \in X$,*

$$\mathscr{D}(Tx,Ty) \preceq u\Big(\mathscr{D}(Tx,x)+\mathscr{D}(y,Ty)\Big).$$

If there is an $\mathscr{D}$-approximate sequence of fixed points of T with respect to $\mathbb{A}$, then it has a fixed point. Moreover, if $\|u\| < \frac{1}{2}$, then T has a unique fixed point.

Proof. • Let $(x_n)_{n\in\mathbb{N}} \subset C$ be an $\mathscr{D}$-approximating fixed point sequence of T with respect to $\mathbb{A}$. Since $(Tx_n)_{n\in\mathbb{N}}$ in C and C is a $\mathscr{D}$-compact of X with respect to $\mathbb{A}$, then there exists an extracted subsequence $(Tx_{\phi(n)})_{n\in\mathbb{N}}$ of $(Tx_n)_{n\in\mathbb{N}}$ such that $(Tx_{\phi(n)})_{n\in\mathbb{N}}$ is $\mathscr{D}$-converges to x^* in C with respect to $\mathbb{A}$.
As $\lim\limits_{n\to\infty} \|\mathscr{D}(Tx_{\phi(n)},x^*)\| = 0$ and using property (D_4) of C^*-algebra-valued generalized metric space, we obtain

$$\mathscr{D}(Tx^*,x^*) \preceq \big(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,Tx_{\phi(n)})\|\big).c.$$

So,

$$\begin{aligned}\mathscr{D}(Tx^*,x^*) &\preceq \left(\limsup_{n\to\infty} \|\mathscr{D}(Tx^*,Tx_{\phi(n)})\|\right).c\\ &= \left(\limsup_{n\to\infty} \|u\left(\mathscr{D}(Tx^*,x^*)+\mathscr{D}(x_{\phi(n)},Tx_{\phi(n)})\right)\|\right).c\\ &\preceq \left(\limsup_{n\to\infty} \|u\|\left(\|\mathscr{D}(Tx^*,x^*)\|+\|\mathscr{D}(x_{\phi(n)},Tx_{\phi(n)})\|\right)\right).c\\ &\preceq \|u\|\left(\|\mathscr{D}(Tx^*,x^*)\|+\limsup_{n\to\infty}\|\mathscr{D}(x_{\phi(n)},Tx_{\phi(n)})\|\right).c\\ &\preceq \|u\|\|\mathscr{D}(Tx^*,x^*)\|.c.\end{aligned}$$

Using the norm of $\mathbb{A}$ and the Lemma 2.2, we get

$$\|\mathscr{D}(Tx^*,x^*)\| \leq \|u\|\|c\|\|\mathscr{D}(Tx^*,x^*)\|.$$

So,

$$(1-\|u\|\|c\|)\,||\mathscr{D}(Tx^*,x^*)|| \leq 0.$$

Since $\|u\|\|c\| < 1$, then $\|\mathscr{D}(Tx^*,x^*)\| = 0$. Which implies that $\mathscr{D}(Tx^*,x^*) = 0_{\mathbb{A}}$, then $Tx^* = x^*$.
• To prove the uniqueness of the fixed point of T in the case $\|u\| < \frac{1}{2}$, we use the same procedure as in the Theorem 3.9 □

Remark 3.24. *In the case where* $\frac{1}{2} \leq \|u\| < 1$*, the mapping* T *of Theorem 3.23 does not have the uniqueness of the fixed point, to illustrate this point, let's take an example as follows.*

Example 3.25. Let $X = \big[0,1\big]$ and $\mathbb{A} = \mathscr{M}_2(\mathbb{R})$ with the usual norm $\|.\|_2$. Consider the mapping $\mathscr{D}$ defined by

$$\mathscr{D}(x,y) = \begin{cases} \begin{pmatrix} \frac{|x-y|}{2} & 0 \\ 0 & \frac{|x-y|}{2} \end{pmatrix} & x = 0 \text{ or } y = 0 \\ \\ \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix} & \text{otherwise.} \end{cases}$$

Then, $(X,\mathbb{A},\mathscr{D})$ is a C^*-algebra-valued generalized metric space with $c \succeq \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$. Consider the mapping $T : C \to C$ where $C = [0,1]$ defined by

$$Tx = \begin{cases} 0 & \text{if} \quad x \in \big[0,1\big) \\ 1 & \text{if} \quad x = 1. \end{cases}$$

We check that $\mathscr{D}(Tx,Ty) \preceq u\big(\mathscr{D}(Tx,x) + \mathscr{D}(y,Ty)\big)$ for all $x,y \in \big[0,1\big]$, where $u = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\|u\|_2 = \frac{\sqrt{2}}{2} \in \big[\frac{1}{2},1\big)$.
C is $\mathscr{D}$-compact with respect to $\mathbb{A}$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of $[0,1]$ and since $[0,1]$ is a compact of $\mathbb{R}$ provided with the absolute value, there exists a subsequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converging to an element x of $[0,1]$.
Let $\varepsilon > 0$. There exists $p \in \mathbb{N}$ such that for any integer n,

$$n \geqslant p \implies |x_{\sigma(n)} - x| < \sqrt{2}\varepsilon.$$

We obtain

$$\|\mathscr{D}(x_{\sigma(n)},x)\| = \frac{\sqrt{2}}{2}|x_{\sigma(n)} - x| < \varepsilon.$$

Moreover, we can find a sequence $(x_n)_{n\in\mathbb{N}}$ $\mathscr{D}$-approximate sequence of fixed points of T with respect to $\mathbb{A}$ (for example $(x_n)_{n\geq 2} \subset [0,1]$ such that $x_n = \frac{1}{n}$ for all enter $n \geqslant 2$).
Then all the conditions of the previous Theorem 3.23 are satisfied but T has two fixed points 0 and 1.

4. APPLICATIONS

The fixed point theorems have a number of applications in differential equations and especially in operator equations. Consider the following operator equation:

$$X = \sum_{n=1}^{\infty} U_n^* f(X) U_n, \tag{4}$$

where $U_1, U_2, U_3, \ldots\ldots, U_n \in L(H)$ with $L(H)$ represents be the set of continuous linear operators on a Hilbert space H and satisfy $\sum\limits_{n=1}^{\infty} \|U_n\|^2 < \dfrac{1}{3}$ and $f : L(H) \to L(H)$ a function such that $\|f(X)\| \leqslant \|X\|$ for all $X \in L(H)$.
Then, the operator equation (4) has a unique solution in $L(H)$.

Proof. Let $\lambda = \sum\limits_{n=1}^{\infty} \|U_n\|^2$. Clear that if $\lambda = 0$, then the $U_n = 0$ for all $n \in \mathbb{N}^*$ and the equation (4) has a unique solution in $L(H)$.
Now, if $\lambda > 0$, we define the mapping $\mathscr{D} : L(H) \times L(H) \to L(H)$ by

$$\mathscr{D}(X,Y) = \max\big\{\|X\|, \|Y\|\big\}.A^*A, \text{ where } A \in L(H).$$

Then, $(L(H), L(H), \mathscr{D})$ is a $\mathscr{D}$-complete C^*-algebra-valued generalized metric space. Consider the map $T : L(H) \to L(H)$ defined by $T(X) = \sum\limits_{n=1}^{\infty} U_n^* f(X) U_n$. Then,

$$\begin{aligned}
\mathscr{D}(T(X), T(Y)) &= \max\Bigg\{\|T(X)\|, \|T(Y)\|\Bigg\}.A^*A \\
&\preceq \|T(X)\|.A^*A + \|T(Y)\|.A^*A \\
&\preceq \|\sum_{n=1}^{\infty} U_n^* f(X) U_n\|.A^*A + \|\sum_{n=1}^{\infty} U_n^* f(Y) U_n\|.A^*A \\
&\preceq \left(\sum_{n=1}^{\infty} \|U_n^* f(X) U_n\|\right).A^*A + \left(\sum_{n=1}^{\infty} \|U_n^* f(Y) U_n\|\right).A^*A \\
&\preceq \left(\sum_{n=1}^{\infty} \|U_n\|^2 \|f(X)\|\right).A^*A + \left(\sum_{n=1}^{\infty} \|U_n\|^2 \|f(Y)\|\right).A^*A
\end{aligned}$$

$$\preceq \frac{1}{3}(\|X\|+\|Y\|).A^*A$$
$$\preceq \frac{1}{3}.\Big(\mathscr{D}(T(X),X)+\mathscr{D}(Y,T(Y)\Big)$$
$$\preceq Z\Big(\mathscr{D}(T(X),X)+\mathscr{D}(Y,T(Y))\Big), \text{ where } Z=\frac{1}{3}.1_{L(H)}.$$

Hence, all the conditions of Theorem 3.9 hold and hence the operator equation (4) has a unique solution $X \in L(H)$. □

5. Conclusion

This word introduces a new C^*-algebra-valued generalized metric spaces covering many different topological spaces and establishes a various extensions of Kannan-type fixed point theorems in a general context. Our findings provide a generalization of Kannan contraction, an improvement of various previous results from the literature, especially those of J. Górnicki. Finally, these results was used to solve operator equations by applying Kannan's contraction within the framework of C*-algebra.

Conflict of Interests

The authors declare that there is no conflict of interests.

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