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EXISTENCE AND APPROXIMATION OF COMMON FIXED POINTS OF TWO MAPPINGS IN A CLASS OF GENERALIZED NONEXPANSIVE MAPPINGS ON BANACH SPACE

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Abstract. In this paper, we introduce the generalized Suzuki type nonexpansive mappings and study existence and approximation of common fixed point of this class of generalized nonexpansive mappings. We use the three step iteration process of Abbas-Nazir for two mappings satisfying the generalized Suzuki type nonexpansive on nonempty subset of a Banach space. We prove some results related to strong and weak convergence of the iteration scheme to get the common fixed point of two mappings satisfying the generalized Suzuki type nonexpansive. Finally, we give an example of two mappings satisfying the given conditions.

Keywords: generalized nonexpansive mapping; condition $B_{\gamma,\mu}$; condition (C); fixed point.

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1. Introduction

The generalization of nonexpansive mappings and the study of related fixed point theorems with different practical applications in nonlinear functional analysis have found great importance during the recent decades [2, 5, 6, 9, 11, 12, 15, 17, 19, 20, 23, 31, 33, 34, 37, 41]. Several

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prominent authors [7, 8, 10, 20, 24, 27, 30, 32, 33, 38, 39, 40] have contributed immensely in this field, and different new classes of mappings with interesting properties have been developed in this context.

In 2008, Suzuki [39] introduced a new class of generalized nonexpansive mapping which is the extension of non- expansive mapping. In 2011, Falset et al. [17] extended condition (C) in to the condition (C_λ), $\lambda \in (0,1)$. In 2016, Lael and Heidapour [27] introduced monotone (C_λ)-condition which generalize the condition (C_λ). Recently in 2018, Patir et al. [33] introduced a new class of generalized nonexpansive mapping (or condition $B_{\gamma,\mu}$), and this new class of generalized nonexpansive mapping is wider than that of condition (C).

Fixed point theory and its application played an important role in many areas of applied science and solved many problems rising in engineering, mathematical economics and optimization. Several authors have studied iterative methods for approximating fixed points of non-expansive mappings (for example, see Ishikawa [22], Senter and Dotson [36], Dugundji [16], Goebel and Kirk [18], Sangago [35], Zegeye and Shahzad [44], Ullah et al. [43], and Abbas and Nazir [1]).

In 1953, Mann [29] introduced the iterative scheme

$$u_{n+1} = \beta_n u_n + (1-\beta_n)\mathscr{G}u_n,\ \beta_n \in (0,1),\ n \geq 0, \tag{1.1}$$

and when $\beta_n = \theta$ we call it the Krasnoselskii-Mann's iterative method and is reduced to

$$u_{n+1} = \theta u_n + (1-\theta)\mathscr{G}u_n,\ \theta \in (0,1),\ n \geq 0, \tag{1.2}$$

which was introduced by Krasnoselskii [26].

In 2007, Agrawal et al. [3] introduced S- iteration defined by

$$\begin{cases} u_1 \in \mathscr{K} \\ u_{n+1} = (1-\alpha_n)\mathscr{G}u_n + \alpha_n \mathscr{G}v_n \\ v_n = (1-\theta_n)u_n + \theta_n \mathscr{G}u_n \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$.

In 2014, Gursoy et al. [21] introduced the Picard-S-iteration process defined as:

$$\begin{cases} u_1 \in \mathscr{K} \\ u_{n+1} = \mathscr{G} w_n \\ w_n = (1-\beta_n)\mathscr{G} u_n + \beta_n \mathscr{G} v_n \\ v_n = (1-\theta_n) u_n + \theta_n \mathscr{G} u_n \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$.

The following is Abbas and Nazir [1] iteration process defined as:

$$\begin{cases} u_1 \in \mathscr{K} \\ u_{n+1} = (1-\alpha_n)\mathscr{G} w_n + \alpha_n \mathscr{G} v_n \\ w_n = (1-\beta_n)\mathscr{G} u_n + \beta_n \mathscr{G} v_n \\ v_n = (1-\theta_n) u_n + \theta_n \mathscr{G} u_n, \end{cases} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$.

It is to be mentioned that all the above algorithms are related to fixed point of single mapping and there are few iteration process that are concerned with fixed point of two or more. Among that, the commonly utilized one is Liu et al. [28] iteration process defined by

$$\begin{cases} u_1 \in \mathscr{K} \\ u_{n+1} = (1-\alpha_n)\mathscr{G}_1 u_n + \alpha_n \mathscr{G}_2 v_n \\ v_n = (1-\theta_n)\mathscr{G}_1 u_n + \theta_n \mathscr{G}_2 u_n, \end{cases} \tag{1.6}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$.

Motivated and inspired by the above results and the works of Patir et al. [33], we extend the existence of fixed points and convergence of iterative schemes to common fixed points of two mappings satisfying condition $B_{\gamma,\mu}$. The other objective of this article is to approximate a common fixed point of two mappings satisfying the $B_{\gamma,\mu}$ condition by using Abbas-Nazir iteration scheme with some technique. We use such three-step iteration process to find some weak and strong convergence results.

2. PRELIMINARIES

The following definitions, facts and lemmas will be useful in proving our main results. Throughout this article, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{R}$ for the set of real numbers, $\mathscr{B}$ for a Banach space with its dual space $\mathscr{B}^*$, except if it is specified. For a self-mapping $\mathscr{G}$ on a set X, we denote the set of all fixed points of $\mathscr{G}$ by $Fix(\mathscr{G})$.

Let $\mathscr{K}$ be a nonempty subset of a Banach space $\mathscr{B}$. A mapping $\mathscr{G}:\mathscr{G}\longrightarrow\mathscr{G}$ is called contraction [10] , if there exists $t\in[0,1)$ such that

$$\|\mathscr{G}u-\mathscr{G}v\|\leq t\,\|u-v\| \text{ for all } u,v\in\mathscr{K}. \tag{2.1}$$

If (2.1) holds at $t=1$, then it is called nonexpansive mapping. The mapping $\mathscr{G}$ is called quasi-nonexpansive [14] if for each $p\in Fix(\mathscr{G})$ and $u\in\mathscr{K}$, we have

$$\|\mathscr{G}u-p\|\leq\|u-p\|. \tag{2.2}$$

A mapping $\mathscr{G}:\mathscr{K}\longrightarrow\mathscr{K}$ is said to be Suzuki generalized nonexpansive mapping (or satisfy condition (C)) [39] if for all $u,v\in\mathscr{K}$

$$\frac{1}{2}\|u-\mathscr{G}u\|\leq\|u-v\|\Rightarrow\|\mathscr{G}u-\mathscr{G}v\|\leq\|u-v\|. \tag{2.3}$$

Remark 2.1. *Every nonexpansive mapping satisfies the condition (C) on $\mathscr{K}$. But there are also some noncontinuous mappings satisfying the condition (C); (see* [39]*).*

A mapping $\mathscr{G}:\mathscr{K}\longrightarrow\mathscr{K}$ is said to satisfy condition (C_λ) [17], if for all $u,v\in\mathscr{K}$

$$\lambda\,\|u-\mathscr{G}u\|\leq\|u-v\|\Rightarrow\|\mathscr{G}u-\mathscr{G}v\|\leq\|u-v\|. \tag{2.4}$$

A mapping $\mathscr{G}:\mathscr{K}\longrightarrow\mathscr{K}$ is said to satisfy condition $B_{\gamma,\mu}$ [33] if there exists $\gamma\in[0,1]$, $\mu\in[0,\frac{1}{2}]$ with $2\mu\leq\gamma$ such that $u,v\in\mathscr{K}$

$$\gamma\|u-\mathscr{G}u\|\leq\|u-v\|+\mu\,\|v-\mathscr{G}v\|, \tag{2.5}$$

implies that

$$\|\mathscr{G}u-\mathscr{G}v\|\leq(1-\gamma)\,\|u-v\|+\mu(\|u-\mathscr{G}v\|+\|v-\mathscr{G}u\|). \tag{2.6}$$

A Banach space $\mathscr{B}$ is said to be uniformly convex [4], if for every $\varepsilon \in (0,2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequalities $\|u\| \leq 1,\ \|v\| \leq 1$ and $\|u-v\| \geq \varepsilon$ imply $\left\| \frac{u+v}{2} \right\| \leq 1-\delta$. A Banach space $\mathscr{B}$ is said to be smooth [4] if for each $u \in S_{\mathscr{B}}$ there exists a unique functional $j_{u^*} \in \mathscr{B}^*$ such that $\langle u, j_{u^*}\rangle = \|x\|$ and $\|j_{u^*}\| = 1$. The norm of a Banach space $\mathscr{B}$ is said to be Fréchet differentiable at $u \in \mathscr{B}$ if for all $v \in S_{\mathscr{B}}$

$$\lim_{k \to 0} \frac{\|u+kv\| - \|u\|}{k}, \tag{2.7}$$

exists, where $S_{\mathscr{B}} = \{u \in \mathscr{B} : \|u\| = 1\}$. The norm of $\mathscr{B}$ is Fréchet differentiable if for each $u \in \mathscr{B}$, the limit (2.7) exists uniformly for $v \in S_{\mathscr{B}}$. In this case

$$\frac{1}{2}\|u\|^2 + \langle v, J(u)\rangle \leq \frac{1}{2}\|u+v\|^2 \leq \frac{1}{2}\|u\|^2 + \langle v, J(u)\rangle + h(\|u\|), \tag{2.8}$$

for all $u, v \in \mathscr{B}$, where $J(u)$ is the Fréchet derivative of functional $\frac{1}{2}\|.\|^2$ and h is an increasing function on $[0,\infty)$ such that $\lim_{k \to 0} \frac{h(k)}{k} = 0$. Moreover, for each $\varepsilon \in [0,2]$, the modulus $\delta_{\mathscr{B}}(\varepsilon)$ of convexity of a Banach space $\mathscr{B}$ is defined by

$$\delta_{\mathscr{B}}(\varepsilon) = \inf\{1 - \frac{u+v}{2} : \|u\| \leq 1, \|v\| \leq 1, \|u-v\| \geq \varepsilon\}. \tag{2.9}$$

Let $\mathscr{K}$ be a nonempty subset of a Banach space $\mathscr{B}$ and $\mathscr{G} : \mathscr{K} \longrightarrow \mathscr{B}$ a mapping. Then $\mathscr{G}$ is said to be demiclosed at $v \in \mathscr{B}$ [4], if for any sequence $\{u_n\}$ in $\mathscr{K}$ the following implication holds;

$$u_n \rightharpoonup u \text{ and } \mathscr{G}u_n \longrightarrow v \Rightarrow \mathscr{G}u = v. \tag{2.10}$$

Let $\mathscr{K}$ be a nonempty closed convex and bounded subset of a Banach space $\mathscr{B}$. If a self mapping $\mathscr{G}$ on $\mathscr{K}$ satisfies condition $B_{\gamma,\mu}$ on $\mathscr{K}$, then there exists a sequence $\{u_n\}$ such that $\lim_{n\to\infty} \|\mathscr{G}u_n - u_n\| = 0$. Such a sequence is called almost fixed point sequence for $\mathscr{G}$.

Definition 2.2. *(see [4]). Let $\mathscr{B}$ be a Banach space, $\mathscr{K}$ be a nonempty subset of $\mathscr{B}$ and $\{u_n\}$ be a bounded sequence in $\mathscr{B}$. Then, for each $p \in \mathscr{B}$*

(1) asymptotic radius of $\{u_n\}$ at p is determined and defined by

$$r(p, \{u_n\}) = \limsup_{n\to\infty} \|u_n - p\| \tag{2.11}$$

(2) asymptotic radius of $\{u_n\}$ relative to $\mathscr{K}$ is determined and defined by

$$r(\mathscr{K},\{u_n\}) = \inf\{r(p,\{u_n\}) : p \in \mathscr{K}\} \tag{2.12}$$

(3) asymptotic center of $\{u_n\}$ relative to $\mathscr{K}$ is determined and defined by

$$A(\mathscr{K}\{u_n\}) = \{p \in \mathscr{K} : r(p,\{u_n\}) = r(\mathscr{K},\{u_n\})\}. \tag{2.13}$$

Note that $A(\mathscr{K},\{u_n\})$ is nonempty. If $\mathscr{B}$ is uniformly convex Banach space, then $A(\mathscr{K},\{u_n\})$ has exactly one point.

A Banach space $\mathscr{B}$ is said to satisfy the Opial condition, if for each sequence $\{u_n\}$ in $\mathscr{B}$ with $u_n \rightharpoonup p$, we have

$$\liminf_{n\to\infty} \|u_n - p\| < \liminf_{n\to\infty} \|u_n - q\|, \tag{2.14}$$

whenever $p \neq q$.

For a sequence $\{u_n\}$ of $\mathscr{B}$ and a point u in $\mathscr{B}$, the strong convergence of $\{u_n\}$ to u is denoted by $u_n \longrightarrow u$ and the weak convergence of $\{u_n\}$ to u is denoted by $u_n \rightharpoonup u$.

Lemma 2.3. *(see [25]). Let $\mathscr{B}$ be a uniformly convex space and $\{\alpha_n\}$ is a sequence in $(0,1)$ for all $n \in \mathbb{N}$. If $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathscr{B}$ such that $\lim sup_{n\to\infty} \|u_n\| \leq r$, $\lim sup_{n\to\infty} \|v_n\| \leq r$, and $\lim_{n\to\infty} \|\alpha_n u_n + (1-\alpha_n)v_n\| = r$ for some $r \geq 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 2.4. *(see [33]). Let $\mathscr{G}$ be a self mapping on a subset $\mathscr{K}$ of a Banach space $\mathscr{B}$ with the Opial's Condition. Assume that $\mathscr{G}$ satisfies condition $B_{\gamma,\mu}$. If $\{u_n\}$ converges weakly to p and $\lim_{n\to\infty} \|u_n - \mathscr{G}u_n\| = 0$, then $\mathscr{G}p = p$. That is $I - \mathscr{G}$ is demiclosed at zero.*

Definition 2.5. *(see [13]). Let $\mathscr{K}$ be a nonempty subset of a Banach space $\mathscr{B}$. The mappings $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ with $Fix(\mathscr{G}) = Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$ are said to satisfy condition (A), if there exists a non-decreasing function $h : [0,\infty) \longrightarrow [0,\infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{2}(\|u - \mathscr{G}_1 u\| + \|u - \mathscr{G}_2 u\|) \geq h(d(u, Fix(\mathscr{G})))$ for all $u \in \mathscr{K}$.*

Definition 2.6. *(see)[13]). Let $\mathscr{K}$ be a nonempty subset of a Banach space $\mathscr{B}$. Two mappings $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ with $Fix(\mathscr{G}) = Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$ are said to satisfy condition (B), if* *there exists a non-decreasing function* $h : [0,\infty) \longrightarrow [0,\infty)$ *with* $h(0) = 0$ *and* $h(r) > 0$ *for all* $r \in (0,\infty)$ *such that* $\max\{(\|u-\mathscr{G}_1 u\|, \|u-\mathscr{G}_2 u\|\} \geq h(d(u, Fix(\mathscr{G})))$ *for all* $u \in \mathscr{K}$.

Methodology: Well developed analytic as well as fixed point theoretical methods to prove our results are implemented. Mainly the key existing methods in the literature to prove our results are taken from [1, 33, 37, 38, 43] and references therein.

3. MAIN RESULTS

In this section, we prove the existence and approximation theorems of common fixed point of two mappings satisfying the generalized Suzuki nonexpansive mappings. Also we use the three-step iterative process due to Abbas-Nazir for two mappings $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$, where $\mathscr{K}$ is a non-empty subset of a Banach space $\mathscr{B}$, which is as follows:

$$\begin{cases} u_1 \in \mathscr{K} \\ u_{n+1} = (1-\alpha_n)\mathscr{G}_1 w_n + \alpha_n \mathscr{G}_2 v_n \\ w_n = (1-\beta_n)\mathscr{G}_1 u_n + \beta_n \mathscr{G}_2 v_n \\ v_n = (1-\theta_n)u_n + \theta_n \mathscr{G}_1 u_n, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\theta_n\}$ are sequence in $(0,1)$.

First, we construct the following class of mappings.

Definition 3.1. *Let* $\mathscr{K}$ *be a nonempty subset of a Banach space* $\mathscr{B}$. *Let* $\gamma \in [0,1]$ *and* $\mu \in [0,\frac{1}{2}]$ *such that* $2\mu \leq \gamma$. *The mappings* $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ *are said to satisfy* $B_{\gamma,\mu}$ *condition if for all* $u, v \in \mathscr{K}$

$$\gamma \min\{\|u-\mathscr{G}_1 u\|, \|u-\mathscr{G}_2 u\|\} \leq \|u-v\| + \mu \max\{\|v-\mathscr{G}_1 v\|, \|v-\mathscr{G}_2 v\|\} \tag{3.2}$$

implies that

$$\begin{aligned} &\max\{\|\mathscr{G}_1 u-\mathscr{G}_1 v\|, \|\mathscr{G}_2 u-\mathscr{G}_2 v\|\} \\ &\leq (1-\gamma)\|u-v\| + \mu \min\{\|u-\mathscr{G}_1 v\| + \|v-\mathscr{G}_1 u\|, \|u-\mathscr{G}_2 v\| + \|v-\mathscr{G}_2 u\|\}. \end{aligned} \tag{3.3}$$

Example 3.2. *Let $\mathscr{K} = [0,2]$, $\mathscr{G}_1$ and $\mathscr{G}_2$ be mappings on $\mathscr{K}$ defined by*

$$\mathscr{G}_1 x = \begin{cases} 0, \; if \; x \neq 2 \\ \frac{9}{10}, \; if \; x = 2. \end{cases} \tag{3.4}$$

$$\mathscr{G}_2 x = \begin{cases} 0, \; if \; x \neq 2 \\ 1, \; if \; x = 2. \end{cases} \tag{3.5}$$

Proof. Then we need to show $\mathscr{G}_1$ and $\mathscr{G}_2$ are satisfying the $B_{\gamma,\mu}$ condition but not condition (C). Now first we need to show the mappings are not satisfying the condition (C). Let $x = 1.2$ and $y = 2$. Then $\frac{1}{2}\|\mathscr{G}_1 x - x\| = 0.6 \leq 0.8 = \|x-y\|$, but $\|\mathscr{G}_1 x - \mathscr{G}_1 y\| = 0.9 \leq 0.8 = \|x-y\|$ is false, thus $\mathscr{G}_1$ does not satisfy the condition (C). Similarly $\frac{1}{2}\|\mathscr{G}_2 x - x\| = 0.6 \leq 0.8 = \|x-y\|$, but $\|\mathscr{G}_2 x - \mathscr{G}_2 y\| = 1 \leq 0.8 = \|x-y\|$ is false, thus $\mathscr{G}_2$ does not satisfy the condition (C).

Next we need to show both $\mathscr{G}_1$ and $\mathscr{G}_2$ are satisfying the $B_{\gamma,\mu}$ condition, where $\gamma = 1$ and $\mu = \frac{1}{2}$. The condition $\gamma\|x - \mathscr{G}_1 x\| \leq \|x-y\| + \mu\|y - \mathscr{G}_1 y\|$ is satisfied only when x and y satisfy the conditions mentioned in Case i and Case ii.

Case i. $x \in [0, 1.275]$ and $y = 2$.

Then we have

$$\begin{aligned} \gamma\|x - \mathscr{G}_1 x\| &= \|x - \mathscr{G}_1 x\| = \|x\| \\ &\leq \|x-y\| + \mu\|y - \mathscr{G}_1 y\| = \|x-2\| + \frac{1}{2}\|2 - 0.9\| \\ &= \|x-2\| + 0.55. \end{aligned}$$

For all such x and $y = 2$, the inequality

$$\|\mathscr{G}_1 x - \mathscr{G}_1 y\| = \frac{9}{10} \leq (1-\gamma)\|x-y\| + \mu(\|x - \mathscr{G}_1 y\| + \|y - \mathscr{G}_1 x\|) = \frac{1}{2}\left\|x - \frac{9}{10}\right\| + 1$$

holds.

Case ii. $x \neq 2,\; y \neq 2$ and $y \geq \frac{4}{3}x$.

Then we have

$$\gamma\|x - \mathscr{G}_1 x\| = \|x\| \leq y - x + \frac{1}{2}y = \|x-y\| + \frac{1}{2}\|y - \mathscr{G}_1 y\| = \|x-y\| + \frac{1}{2}\|y\|. \tag{3.6}$$

It is obvious in this case that

$$\|\mathscr{G}_1 x-\mathscr{G}_1 y\| = 0 \le (1-\gamma)\|x-y\| + \mu(\|x-\mathscr{G}_1 y\| + \|y-\mathscr{G}_1 x\|)$$
$$= \frac{1}{2}(\|x\| + \|y\|).$$

The condition $\gamma\|x-\mathscr{G}_2 x\| \le \|x-y\| + \mu\|y-\mathscr{G}_2 y\|$ is satisfied only when x and y satisfy the conditions mentioned in Case iii and Case iv.

Case iii. $x \in [0, 1.25]$ and $y = 2$.

Then we have

$$\gamma\|x-\mathscr{G}_2 x\| = \|x-\mathscr{G}_2 x\| = \|x\|$$
$$\le \|x-y\| + \mu\|y-\mathscr{G}_2 y\| = \|x-2\| + 0.50.$$

For all such x and $y = 2$, the inequality

$$\|\mathscr{G}_2 x-\mathscr{G}_2 y\| = 1 \le (1-\gamma)\|x-y\| + \mu(\|x-\mathscr{G}_1 y\| + \|y-\mathscr{G}_1 x\|) = \frac{1}{2}\|x-1\| + 1$$

holds.

Case iv. $x \neq 2,\ y \neq 2$ and $y \ge \frac{4}{3}x$.

Then we have

$$\gamma\|x-\mathscr{G}_2 x\| = \|x\| \le \|x-y\| + \frac{1}{2}\|y-\mathscr{G}_2 y\| = \|x-y\| + \frac{1}{2}\|y\|. \tag{3.7}$$

It is obvious in this case that

$$\|\mathscr{G}_2 x-\mathscr{G}_2 y\| = 0 \le (1-\gamma)\|x-y\| + \mu(\|x-\mathscr{G}_2 y\| + \|y-\mathscr{G}_2 x\|) = \frac{1}{2}(\|x\| + \|y\|).$$

holds

Therefore, it follows from the above four cases that $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy the $B_{\gamma,\mu}$ condition. This completes the proof. □

The following lemma shows that $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfying the $B_{\gamma,\mu}$ condition are both quasi-nonexpansive.

Lemma 3.3. *Let $\mathscr{K}$ be a nonempty subset of the Banach space $\mathscr{B}$ and $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be mappings satisfying the $B_{\gamma,\mu}$ condition. If $p \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$ on $\mathscr{K}$, then for all $u \in \mathscr{K}$*

$$\|p - \mathscr{G}_1 u\| \leq \|p - u\| \text{ and } \|p - \mathscr{G}_2 u\| \leq \|p - u\|. \tag{3.8}$$

Proof. Since $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ are two mappings satisfying the $B_{\gamma,\mu}$ condition and $p \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$, then we have the following

$$0 = \gamma \min\{\|p - \mathscr{G}_1\||, \|p - \mathscr{G}_2 p\|\} \leq \|p - u\| + \mu \max\{\|u - \mathscr{G}_1\|, \|u - \mathscr{G}_2 u\|\}. \tag{3.9}$$

So, by Definition 3.1

$$\begin{aligned} &\max\{\|\mathscr{G}_1 p - \mathscr{G}_1 u\|, \|\mathscr{G}_2 p - \mathscr{G}_2 u\|\} \\ &\leq (1-\gamma)\|p - u\| + \mu \min\{\|p - \mathscr{G}_1 u\| + \|u - \mathscr{G}_1 p\|, \|p - \mathscr{G}_2 u\| + \|u - \mathscr{G}_2 p\|\}. \end{aligned} \tag{3.10}$$

Now, we consider the following different cases;

Case I. Let $\|\mathscr{G}_1 p - \mathscr{G}_1 u\| \leq \|\mathscr{G}_2 p - \mathscr{G}_2 u\|$ and $\|p - \mathscr{G}_2 u\| + \|u - \mathscr{G}_2 p\|$ be minimum. Then, from (3.10) we get we get

$$\begin{aligned} \|\mathscr{G}_1 p - \mathscr{G}_1 u\| &\leq (1-\gamma)\|p - u\| + \mu(\|p - \mathscr{G}_1 u\| + \|u - \mathscr{G}_1 p\|) \\ &= (1-\gamma)\|p - u\| + \mu(\|\mathscr{G}_1 p - \mathscr{G}_1 u\| + \|u - p\|) \\ &= (1-\gamma+\mu)\|p - u\| + \mu\|\mathscr{G}_1 p - \mathscr{G}_1 u\|, \end{aligned}$$

since $p = \mathscr{G}_1 p$, $\dfrac{(1-\gamma+\mu)}{(1-\mu)} \leq 1$, then we have the following

$$\begin{aligned} (1-\mu)\|\mathscr{G}_1 p - \mathscr{G}_1 u\| &\leq (1-\gamma+\mu)\|p - u\| \\ \Rightarrow \|\mathscr{G}_1 p - \mathscr{G}_1 u\| &\leq \frac{(1-\gamma+\mu)}{(1-\mu)}\|p - u\| \\ &\leq \|p - u\|. \end{aligned}$$

Assume that $\|\mathscr{G}_2 p - \mathscr{G}_2 u\| \leq \|\mathscr{G}_1 p - \mathscr{G}_1 u\|$. Then, from (3.10) we get

$$\begin{aligned} \|\mathscr{G}_2 p - \mathscr{G}_2 u\| &\leq (1-\gamma)\|p - u\| + \mu(\|p - \mathscr{G}_1 u\| + \|u - \mathscr{G}_1 p\|) \\ &= (1-\gamma)||p - u|| + \mu(\|\mathscr{G}_1 p - \mathscr{G}_1 u\| + \|u - p\|) \\ &= (1-\gamma+\mu)\|p - u\| + \mu\|\mathscr{G}_1 p - \mathscr{G}_1 u\|, \end{aligned}$$

then we have the following

$$\|\mathscr{G}_2 p-\mathscr{G}_2 u\|-\mu\|\mathscr{G}_1 p-\mathscr{G}_1 u\| \leq(1-\gamma+\mu)\|p-u\|, \tag{3.11}$$

since $\|\mathscr{G}_2 p-\mathscr{G}_2 u\| \leq\|\mathscr{G}_1 p-\mathscr{G}_1 u\|$ and $-\|\mathscr{G}_1 p-\mathscr{G}_1 u\| \leq-\|\mathscr{G}_2 p-\mathscr{G}_2 u\|$, then we obtain

$$\begin{aligned}
\|\mathscr{G}_2 p-\mathscr{G}_2 u\|-\mu\|\mathscr{G}_2 p-\mathscr{G}_2 u\| & \leq(1-\gamma+\mu)\|p-u\|, \\
\Rightarrow\|\mathscr{G}_2 p-\mathscr{G}_2 u\| & \leq \frac{(1-\gamma+\mu)}{(1-\mu)}\|p-u\| \\
& \leq\|p-u\| .
\end{aligned}$$

$$\|p-\mathscr{G}_1 u\| \leq\|p-u\| \text{ and } \|p-\mathscr{G}_2 u\| \leq\|p-u\| \tag{3.12}$$

Case II. If $\|\mathscr{G}_1 p-\mathscr{G}_1 u\| \leq\|\mathscr{G}_2 p-\mathscr{G}_2 u\|$ and $\|p-\mathscr{G}_1 u\|+\|u-\mathscr{G}_1 p\|$ is minimum of (3.10) , then we obtain

$$\begin{aligned}
\|\mathscr{G}_2 p-\mathscr{G}_2 u\| & \leq(1-\gamma)\|p-u\|+\mu(\|p-\mathscr{G}_2 u\|+\|u-\mathscr{G}_2 p\|) \\
& =(1-\gamma)\|p-u\|+\mu(\|G_2 p-\mathscr{G}_2 u\|+\|u-p\|) \\
& =(1-\gamma+\mu)\|p-u\|+\mu\|\mathscr{G}_2 p-\mathscr{G}_2 u\|,
\end{aligned}$$

since $p=\mathscr{G}_2 p$, $\frac{(1-\gamma+\mu)}{(1-\mu)} \leq 1$, then we have the following

$$\begin{aligned}
(1-\mu)\|\mathscr{G}_2 p-\mathscr{G}_2 u\| & \leq(1-\gamma+\mu)\|p-u\| \\
\Rightarrow\|\mathscr{G}_2 p-\mathscr{G}_2 u\| & \leq \frac{(1-\gamma+\mu)}{(1-\mu)}\|p-u\| \\
& \leq\|p-u\| .
\end{aligned}$$

And if $\|\mathscr{G}_2 p-\mathscr{G}_2 u\| \leq\|\mathscr{G}_1 p-\mathscr{G}_1 u\|$ and $\|p-\mathscr{G}_1 u\|+\|u-\mathscr{G}_1 p\|$ is minimum, then from (3.10) we obtain

$$\begin{aligned}
\|\mathscr{G}_1 p-\mathscr{G}_1 u\| & \leq(1-\gamma)\|p-u\|+\mu(\|p-\mathscr{G}_2 u\|+\|u-\mathscr{G}_2 p\|). \\
& =(1-\gamma)\|p-u\|+\mu(\|G_2 p-\mathscr{G}_2 u\|+\|-p\|) \\
& =(1-\gamma+\mu)\|p-u\|+\mu\|\mathscr{G} G_2 p-\mathscr{G}_2 u\|,
\end{aligned}$$

from the above inequality we get

$$\|\mathscr{G}_1 p - \mathscr{G}_1 u\| - \mu \|\mathscr{G}_2 p - \mathscr{G}_2 u\| \leq (1 - \gamma + \mu) \|p - u\|, \tag{3.13}$$

since $\|\mathscr{G}_1 p - \mathscr{G}_1 u\| \leq \|\mathscr{G}_2 p - \mathscr{G}_2 u\|$ and $-\|\mathscr{G}_2 p - \mathscr{G}_2 u\| \leq -\|\mathscr{G}_1 p - \mathscr{G}_1 u\|$, then we have the following

$$\begin{aligned}(1-\mu)\|\mathscr{G}_1 p - \mathscr{G}_1 u\| &\leq (1-\gamma+\mu)\|p-u\| \\ \Rightarrow \|\mathscr{G}_1 p - \mathscr{G}_1 u\| &\leq \frac{(1-\gamma+\mu)}{(1-\mu)}\|p-u\| \\ &\leq \|p-u\|.\end{aligned}$$

Hence, for $p \in Fix(\mathscr{G})$ and for all $u \in \mathscr{K}$

$$\|p - \mathscr{G}_1 u\| \leq \|p - u\| \text{ and } \|p - \mathscr{G}_2 u\| \leq \|p - u\|. \tag{3.14}$$

This shows that $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi nonexpansive. This completes the proof. □

However, the converse of Lemma 3.3 does not hold in general.

Example 3.4. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be mappings on $[0,5]$ defined by*

$$\mathscr{G}_1 x = \begin{cases} 0, \; if \; x \in [0,4) \\ 3.5, \; if \; x \in [4,5]. \end{cases} \tag{3.15}$$

$$\mathscr{G}_2 x = \begin{cases} 0, \; if \; x \in [0,4) \\ 3.4, \; if \; x \in [4,5]. \end{cases} \tag{3.16}$$

Proof. Since $\mathscr{G}_1$ has a fixed point at $x = 0$, and also

$$\|\mathscr{G}_1(0) - \mathscr{G}_1(x)\| = \|\mathscr{G}_1(x)\| \leq \|x\|, \; \forall \, x \in [0,5]. \tag{3.17}$$

Similarly, for $\mathscr{G}_2$, we have

$$\|\mathscr{G}_2(0) - \mathscr{G}_2(x)\| = \|\mathscr{G}_2(x)\| \leq \|x\|, \; \forall \, x \in [0,5]. \tag{3.18}$$

Hence, $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi non-nonexpansive.

We need to show $\mathscr{G}_1$ and $\mathscr{G}_2$ did not satisfying the $B_{\gamma,\mu}$ condition. Let $x = 4.5$ and $y = 3.5$. Then

$$\gamma\|x-\mathscr{G}_1(x)\| = \gamma \leq 1+3.5\mu \tag{3.19}$$

But

$$\|\mathscr{G}_1x-\mathscr{G}_1y\| = 3.5, \text{ and} \tag{3.20}$$

$$\begin{aligned}(1-\gamma)\|x-y\|+\mu(\|x-\mathscr{G}_1y\|+\|y-\mathscr{G}_1x\|) &= 1-\gamma+4.5\mu \\ &\leq 1-\gamma+2.25\gamma \\ &< 3.5 = \|\mathscr{G}_1x-\mathscr{G}_1y\|,\end{aligned}$$

it is impossible. And

$$\gamma\|x-\mathscr{G}_2(x)\| = 1.1\gamma \leq 1+3.5\mu \leq 1+1.75\gamma \tag{3.21}$$

But

$$\|\mathscr{G}_2x-\mathscr{G}_2y\| = 3.4, \text{ and} \tag{3.22}$$

$$\begin{aligned}(1-\gamma)\|x-y\|+\mu(\|x-\mathscr{G}_2y\|+\|y-\mathscr{G}_2x\|) &= 1-\gamma+4.6\mu \\ &\leq 1-\gamma+2.3\gamma \\ &< 3.4 = \|\mathscr{G}_2x-\mathscr{G}_2y\|,\end{aligned}$$

also it is impossible. Therefore, $\mathscr{G}_1$ and $\mathscr{G}_2$ did not satisfying the $B_{\gamma,\mu}$ condition. This completes the proof. □

Next, we prove some basic properties of two mappings satisfy the $B_{\gamma,\mu}$ condition.

Proposition 3.5. *Let $\mathscr{B}$ be a Banach space and $\mathscr{K}$ be a nonempty subset of $\mathscr{B}$ and $\mathscr{G}_1,\mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition. Then, for all $u,v \in \mathscr{K}$ and for $\theta \in [0,1]$,*

(i) $\max\{\|\mathscr{G}_1u-\mathscr{G}_1^2u\|, \|\mathscr{G}_2u-\mathscr{G}_2^2u\|\} \leq \min\{\|u-\mathscr{G}_1u\|, \|u-\mathscr{G}_2u\|\}$,

(ii) at least one of the following ((a) and (b)) holds:

(a) $\frac{\theta}{2}\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}\leq\|u-v\|$,

(b) $\frac{\theta}{2}\min\{\|\mathscr{G}_1u-\mathscr{G}_1^2u\|,\|\mathscr{G}_2u-\mathscr{G}_2^2u\|\}\leq\max\{\|\mathscr{G}_1u-v\|,\|\mathscr{G}_2u-v\|\}$.

The condition (a) implies $\max\{\|\mathscr{G}_1u-\mathscr{G}_1v\|,\|\mathscr{G}_2u-\mathscr{G}_2v\|\}\leq(1-\frac{\theta}{2})\|u-v\|+\mu\min\{\|u-\mathscr{G}_1v\|+\|v-\mathscr{G}_1u\|,\|u-\mathscr{G}_2v\|+\|v-\mathscr{G}_2u\|\}$ *and* *the condition (b) implies* $\max\{\|\mathscr{G}_1^2u-\mathscr{G}_1v\|,\|\mathscr{G}_2^2u-\mathscr{G}_2v\|\}\leq(1-\frac{\theta}{2})\min\{\|\mathscr{G}_1u-v\|,\|\mathscr{G}_2u-v\|\}+\mu\min\{\|\mathscr{G}_1u-\mathscr{G}_1v\|+\|v-\mathscr{G}_1^2u\|,\|\mathscr{G}_2u-\mathscr{G}_2v\|+\|v-\mathscr{G}_2^2u\|\}$.

(iii)

$$\begin{aligned}\max\{\|u-\mathscr{G}_1v\|,\|u-\mathscr{G}_2v\|\}\leq(3-\theta)\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}+(1-\frac{\theta}{2})\|u-v\|\\+\mu\min\{2\|u-\mathscr{G}_1u\|+\|u-\mathscr{G}_1v\|+\|v-\mathscr{G}_1u\|\\+2\|\mathscr{G}_1u-\mathscr{G}_1^2u\|,2\|u-\mathscr{G}_2u\|+\|u-\mathscr{G}_2v\|+\|v-\mathscr{G}_2u\|\\+2\|\mathscr{G}_2u-\mathscr{G}_2^2u\|\}.\end{aligned}$$

Proof. (i) For all $u\in\mathscr{K}$ and $\gamma=\frac{\theta}{2},\theta\in[0,1]$, we have

$$\begin{aligned}&\gamma\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}\\&\leq\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}\\&\leq\max\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}\\&\leq\max\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}+\mu\max\{\|\mathscr{G}_1u-\mathscr{G}_1^2u\|,\|\mathscr{G}_2u-\mathscr{G}_2^2u\|\}.\end{aligned}$$

So, by Definition 3.1 (substitute v by $\mathscr{G}_1u$), we get

$$\begin{aligned}&\max\{\|\mathscr{G}_1u-\mathscr{G}_1^2u\|,\|\mathscr{G}_2u-\mathscr{G}_2^2u\|\}\\&\leq(1-\gamma)\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}+\mu\min\{\|u-\mathscr{G}_1^2u\|,\|u-\mathscr{G}_2^2u\|\}\\&\leq(1-\gamma)\min\{\|u-\mathscr{G}_1u\|,\|u-\mathscr{G}_2u\|\}+\mu\min\{\|u-\mathscr{G}_1u\|+\\&\|\mathscr{G}_1u-\mathscr{G}_1^2u\|,\|u-\mathscr{G}_2u\|+\|\mathscr{G}_2u-\mathscr{G}_2^2u\|\}.\end{aligned}\tag{3.23}$$

Now consider the following cases:

Case 1: Let $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ and $\|u-\mathscr{G}_2u\|$ and $\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq (1-\gamma)\|u-\mathscr{G}_1u\| + \mu(\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|), \tag{3.24}$$

which implies that, because $\dfrac{(1-\gamma+\mu)}{1-\mu} \leq 1$

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \frac{(1-\gamma+\mu)}{1-\mu}\|u-\mathscr{G}_1u\| \leq \|u-\mathscr{G}_1u\|, \tag{3.25}$$

hence,

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \|u-\mathscr{G}_1u\|. \tag{3.26}$$

Case 2: Let $\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ and $\|u-\mathscr{G}_2u\|$ and $\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| &\leq (1-\gamma)\|u-\mathscr{G}_1u\| + \mu(\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|)\\ &= (1-\gamma+\mu)\|u-\mathscr{G}_1u\| + \mu\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|,\end{aligned} \tag{3.27}$$

from (3.27) we obtain

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| - \mu\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq (1-\gamma+\mu)\|u-\mathscr{G}_1u\|, \tag{3.28}$$

because $\dfrac{(1-\gamma+\mu)}{1-\mu} \leq 1$ and $\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$, then we have that

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \frac{(1-\gamma+\mu)}{1-\mu}\|u-\mathscr{G}_1u\| \leq \|u-\mathscr{G}_1u\|, \tag{3.29}$$

hence,

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \|u-\mathscr{G}_1u\|. \tag{3.30}$$

Thus, from (3.26) and (3.30) we have

$$\max\{\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|, \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|\} \leq \|u-\mathscr{G}_1u\|. \tag{3.31}$$

Case 3: Let $\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ and $\|u-\mathscr{G}_1u\|$ and $\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq (1-\gamma)\|u-\mathscr{G}_2u\| + \mu(\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|), \tag{3.32}$$

which implies that

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \frac{(1-\gamma+\mu)}{1-\mu}\left\|u-\mathscr{G}_2u\right\| \leq \left\|u-\mathscr{G}_2u\right\|, \tag{3.33}$$

hence,

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|u-\mathscr{G}_2u\right\|. \tag{3.34}$$

Case 4: Let $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ and $\left\|u-\mathscr{G}_1u\right\|$ and $\left\|u-\mathscr{G}_1u\right\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| &\leq (1-\gamma)\left\|u-\mathscr{G}_2u\right\| + \mu(\left\|u-\mathscr{G}_2u\right\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|)\\ &= (1-\gamma+\mu)\left\|u-\mathscr{G}_2u\right\| + \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|,\end{aligned}$$

which implies that

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| - \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq (1-\gamma+\mu)\left\|u-\mathscr{G}_2u\right\|, \tag{3.35}$$

since $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ then, from the above inequality we get

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|u-\mathscr{G}_2u\right\|. \tag{3.36}$$

Thus, from (3.34) and (3.36), we conclude that

$$\max\{\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|, \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|\} \leq \left\|u-\mathscr{G}_2u\right\|. \tag{3.37}$$

Case 5: Let $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ and let $\left\|u-\mathscr{G}_2u\right\|$ and $\left\|u-\mathscr{G}_1u\right\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| &\leq (1-\gamma)\left\|u-\mathscr{G}_1u\right\| + \mu(\left\|u-\mathscr{G}_2u\right\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|)\\ &\leq (1-\gamma+\mu)\left\|u-\mathscr{G}_1u\right\| + \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|,\end{aligned}$$

from the above inequality we get

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| - \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq (1-\gamma+\mu)\left\|u-\mathscr{G}_1u\right\|, \tag{3.38}$$

sine $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ hence,

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|u-\mathscr{G}_1u\right\|. \tag{3.39}$$

Case 6: Let $\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ and let $\|u-\mathscr{G}_2u\|$ and $\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ be minimum. Then from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| &\leq (1-\gamma)\,\|u-\mathscr{G}_1u\| + \mu(\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|)\\ &\leq (1-\gamma+\mu)\,\|u-\mathscr{G}_1u\| + \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|,\end{aligned}$$

from the above inequality we get

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| - \mu\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq (1-\gamma+\mu)\,\|u-\mathscr{G}_1u\|, \tag{3.40}$$

hence,

$$\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \|u-\mathscr{G}_1u\|. \tag{3.41}$$

Hence, from (3.39) and (3.41) we get

$$\max\{\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|, \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|\} \leq \|u-\mathscr{G}_1u\|. \tag{3.42}$$

Case 7: Let $\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ and let $\|u-\mathscr{G}_1u\|$ be minimum, $\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ be minimum. Then, from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| &\leq (1-\gamma)\,\|u-\mathscr{G}_2u\| + \mu(\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|)\\ &\leq (1-\gamma+\mu)\,\|u-\mathscr{G}_2u\| + \mu\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|,\end{aligned}$$

from the above inequality we get

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| - \mu\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq (1-\gamma+\mu)\,\|u-\mathscr{G}_2u\|, \tag{3.43}$$

hence,

$$\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\| \leq \|u-\mathscr{G}_2u\|. \tag{3.44}$$

Case 8: Let $\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| \leq \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|$ and let $\|u-\mathscr{G}_1u\|$ be minimum, $\|u-\mathscr{G}_2u\| + \left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\|$ be minimum.. Then from (3.23) we obtain

$$\begin{aligned}\left\|\mathscr{G}_2u-\mathscr{G}_2^2u\right\| &\leq (1-\gamma)\,\|u-\mathscr{G}_2u\| + \mu(\|u-\mathscr{G}_1u\| + \left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|)\\ &\leq (1-\gamma+\mu)\,\|u-\mathscr{G}_2u\| + \mu\left\|\mathscr{G}_1u-\mathscr{G}_1^2u\right\|,\end{aligned}$$

from the above inequality we get

$$\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|-\mu\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\| \leq(1-\gamma+\mu)\left\|u-\mathscr{G}_2 u\right\|, \tag{3.45}$$

since $\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\| \leq\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|$ hence,

$$\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\| \leq\left\|u-\mathscr{G}_2 u\right\|. \tag{3.46}$$

Thus, from (3.44) and (3.46) we have

$$\max \left\{\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|,\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|\right\} \leq\left\|u-\mathscr{G}_2 u\right\|. \tag{3.47}$$

Therefore, from (3.31), (3.37), (3.42) and (3.47) we conclude that
$\max \left\{\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|,\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|\right\} \leq \min \left\{\left\|u-\mathscr{G}_1 u\right\|,\left\|u-\mathscr{G}_2 u\right\|\right\}$,

(ii) We can prove this by contradiction, assume that $\frac{\theta}{2} \min \left\{\left\|u-\mathscr{G}_1 u\right\|,\left\|u-\mathscr{G}_2 u\right\|\right\}>\|u-v\|$, and $\frac{\theta}{2} \min \left\{\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|,\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|\right\}>\max \left\{\left\|\mathscr{G}_1 u-v\right\|,\left\|\mathscr{G}_2 u-v\right\|\right\}$. Suppose $\left\|u-\mathscr{G}_2 u\right\|$ be minimum of $\left\|u-\mathscr{G}_1 u\right\|$, $\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|$ minimum of $\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|$ and $\left\|\mathscr{G}_2 u-v\right\|$ be maximum of $\left\|\mathscr{G}_1 u-v\right\|$. Thus, we have $\frac{\theta}{2}\left\|u-\mathscr{G}_1 u\right\|>\|u-v\|$ and $\frac{\theta}{2}\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|>\left\|\mathscr{G}_1 u-v\right\|$. Now by using (i) and $\theta \leq 1$, we have

$$\begin{aligned}
\left\|u-\mathscr{G}_1 u\right\| & \leq\|u-v\|+\left\|v-\mathscr{G}_1 u\right\| \\
& <\frac{\theta}{2}\left\|u-\mathscr{G}_1 u\right\|+\frac{\theta}{2}\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\| \\
& \leq \frac{\theta}{2}\left\|u-\mathscr{G}_1 u\right\|+\frac{\theta}{2}\left\|u-\mathscr{G}_1 u\right\| \\
& \leq\left\|u-\mathscr{G}_1 u\right\| \\
\Rightarrow\left\|u-\mathscr{G}_1 u\right\| & <\left\|u-\mathscr{G}_1 u\right\|,
\end{aligned}$$

which is a contradiction. And by using similar arguments, we obtain $\left\|u-\mathscr{G}_2 u\right\|<\left\|u-\mathscr{G}_2 u\right\|$, which is again a contradiction. Hence, our assumption is false. Therefore, at least one of (a) and (b) holds.

(iii) By using (ii), we get

$$\begin{aligned}
\max \left\{\left\|u-\mathscr{G}_1 v\right\|,\left\|u-\mathscr{G}_2 v\right\|\right\} & \leq \max \left\{\left\|u-\mathscr{G}_1 u\right\|,\left\|u-\mathscr{G}_2 u\right\|\right\}+\max \left\{\left\|\mathscr{G}_1 u-\mathscr{G}_1 v\right\|,\left\|\mathscr{G}_2 u-\mathscr{G}_2 v\right\|\right\} \\
& \leq(3-\theta) \min \left\{\left\|u-\mathscr{G}_1 u\right\|,\left\|u-\mathscr{G}_2 u\right\|\right\}+
\end{aligned}$$

$$\begin{aligned}
&\max\{\|\mathscr{G}_1 u-\mathscr{G}_1 v\|,\|\mathscr{G}_2 u-\mathscr{G}_2 v\|\}\\
&\leq (3-\theta)\min\{\|u-\mathscr{G}_1 u\|,\|u-\mathscr{G}_2 u\|\}+(1-\gamma)\|u-v\|+\\
&\mu\min\{\|u-\mathscr{G}_1 v\|+\|v-\mathscr{G}_1 u\|,\|u-\mathscr{G}_1 v\|+\|v-\mathscr{G}_1 u\|\}\\
&\leq (3-\theta)\min\{\|u-\mathscr{G}_1 u\|,\|u-\mathscr{G}_2 u\|\}+(1-\frac{\theta}{2})\|u-v\|\\
&+\mu\min\{2\|u-\mathscr{G}_1 u\|+\|u-\mathscr{G}_1 v\|+\|v-\mathscr{G}_1 u\|\\
&+2\left\|\mathscr{G}_1 u-\mathscr{G}_1^2 u\right\|,2\|u-\mathscr{G}_2 u\|+\|u-\mathscr{G}_2 v\|+\|v-\mathscr{G}_2 u\|\\
&+2\left\|\mathscr{G}_2 u-\mathscr{G}_2^2 u\right\|\}.
\end{aligned}$$

Which completes our proof. □

And, we can give the following lemma, which will play an important role in the sequel.

Lemma 3.6. *Let $\mathscr{B}$ be a Banach space and $\mathscr{K}$ be a nonempty convex and bounded subset of $\mathscr{B}$. Let $\mathscr{G}_1,\mathscr{G}_2:\mathscr{K}\longrightarrow\mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition with $Fix(\mathscr{G})=Fix(\mathscr{G}_1)\cap Fix(\mathscr{G}_2)\neq\emptyset$ on $\mathscr{K}$. Let $p\in Fix(\mathscr{G})$, $u_1\in\mathscr{K}$ and $\{u_n\}$ be sequence defined by (3.1) is in $\mathscr{K}$. Then, $\lim_{n\to\infty}\|u_n-p\|$ exists for all $p\in Fix(\mathscr{G})$.*

Proof. Let $Fix(\mathscr{G})=Fix(\mathscr{G}_1)\cap Fix(\mathscr{G}_2)\neq\emptyset$, $p\in Fix(\mathscr{G})$ and $\{u_n\}$ be a sequence in $\mathscr{K}$. Then, by Lemma 3.3, $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-nonexpansive mappings. Now, by using (3.1), (3.12) and 3.14, we have the following

$$\begin{aligned}
\|u_{n+1}-p\| &= \|(1-\alpha_n)\mathscr{G}_1 w_n+\alpha_n\mathscr{G}_2 v_n-p\|\\
&\leq (1-\alpha_n)\|\mathscr{G}_1 w_n-p\|+\alpha_n\|\mathscr{G}_2 v_n-p\|\\
&\leq (1-\alpha_n)\|w_n-p\|+\alpha_n\|v_n-p\|\\
&= (1-\alpha_n)\|(1-\beta_n)\mathscr{G}_1 u_n+\beta_n\mathscr{G}_2 v_n-p\|+\alpha_n\|(1-\theta_n)u_n+\theta_n\mathscr{G}_1 u_n-p\|\\
&\leq (1-\alpha_n)\left[(1-\beta_n)\|\mathscr{G}_1 u_n-p\|+\beta_n\|\mathscr{G}_2 v_n-p\|\right]+\\
&\alpha_n\left[(1-\theta_n)\|u_n|-p\|+\theta_n\|\mathscr{G}_1 u_n-p\|\right]\\
&\leq (1-\alpha_n)\left[(1-\beta_n)\|u_n-p\|+\beta_n\|v_n-p\|\right]+\\
&\alpha_n\left[(1-\theta_n)\|u_n-p\|+\theta_n\|u_n-p\|\right]
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\big[(1-\beta_n)\|u_n-p\| + \beta_n \|(1-\theta_n)u_n + \theta_n \mathscr{G}_1 u_n - p\|\big] + \\
&\alpha_n\big[(1-\theta_n)\|u_n-p\| + \theta_n\|u_n-p\|\big] \\
&\le (1-\alpha_n)\Big[(1-\beta_n)\|u_n-p\| + \beta_n\big[(1-\theta_n)\|u_n-p\| + \theta_n\|u_n-p\|\big]\Big] + \\
&\alpha_n\big[(1-\theta_n)\|u_n-p\| + \theta_n\|u_n-p\|\big] \\
&= (1-\alpha_n)\Big[(1-\beta_n)\|u_n-p\| + \beta_n\|u_n-p\|\Big] + \alpha_n\|u_n-p\| \\
&= (1-\alpha_n)\|u_n-p\| + \alpha_n\|u_n-p\| \\
&= \|u_n-p\|,
\end{aligned} \tag{3.48}
$$

for each $n \in \mathbb{N}$. Thus, from inequality (3.48) the sequence $\{\|u_n-p\|\}$ is a monotonically decreasing sequence and bounded below for all $p \in Fix(\mathscr{G})$. Therefore, $\lim_{n\to\infty}\|u_n-p\|$ exists. This completes the proof. □

Now using the above facts, we prove the following theorem which is useful for the next results.

Theorem 3.7. *Let $\mathscr{B}$ be a uniformly convex Banach space and $\mathscr{K}$ be a non-empty closed convex subset of $\mathscr{B}$. Let $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition. Let $\{u_n\}$ be a sequence in $\mathscr{K}$ as defined by (3.1). Then, $Fix(\mathscr{G}) = Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$ if and only if $\lim_{n\to\infty}\|u_n - \mathscr{G}_i u_n\| = 0,\ i = 1,2.$*

Proof. Let $p \in Fix(\mathscr{G})$. By Lemma 3.6 $\lim_{n\to\infty}\|u_n-p\|$ exists and assume that

$$\lim_{n\to\infty}\|u_n-p\| = d. \tag{3.49}$$

And also, from (3.12), we have that

$$\|\mathscr{G}_1 u_n - p\| \le \|u_n-p\|.$$

Which implies that

$$\limsup_{n\to\infty}\|\mathscr{G}_1 u_n - p\| \le \limsup_{n\to\infty}\|u_n-p\| = d. \tag{3.50}$$

Similarly, we have that

$$\limsup_{n\to\infty} \|\mathscr{G}_2 u_n - p\| \le \limsup_{n\to\infty} \|u_n - p\| = d. \tag{3.51}$$

Then, by using (3.48) and (3.49), we have that

$$\begin{aligned} d = \lim_{n\to\infty} \|u_{n+1} - p\| &= \lim_{n\to\infty} \|(1-\alpha_n)\mathscr{G}_1 w_n + \alpha_n \mathscr{G}_2 v_n - p\| \\ &\le \lim_{n\to\infty} \|u_n - p\| \\ &= d. \end{aligned} \tag{3.52}$$

Hence, from (3.52), we have that

$$\lim_{n\to\infty} \|(1-\alpha_n)\mathscr{G}_1 w_n + \alpha_n \mathscr{G}_2 v_n - p\| = d. \tag{3.53}$$

From (3.53), we have

$$\lim_{n\to\infty} \|(1-\alpha_n)(\mathscr{G}_1 w_n - p) + \alpha_n(\mathscr{G}_2 v_n - p)\| = d. \tag{3.54}$$

And from (3.54) we obtain

$$\limsup_{n\to\infty} \|(1-\alpha_n)(\mathscr{G}_1 w_n - p) + \alpha_n \mathscr{G}_2(v_n - p)\| = d. \tag{3.55}$$

Hence, from (3.50), (3.51), (3.55) and Lemma 2.3, we get the following

$$\lim_{n\to\infty} \|\mathscr{G}_1 w_n - \mathscr{G}_2 v_n\| = 0. \tag{3.56}$$

From (3.1) we have the following

$$\begin{aligned} \|u_{n+1} - \mathscr{G}_2 v_n\| &= \|\|(1-\alpha_n)\mathscr{G}_1 w_n + \alpha_n \mathscr{G}_2 v_n - \mathscr{G}_2 v_n\| \\ &= \|(1-\alpha_n)\mathscr{G}_1 w_n - (1-\alpha_n)\mathscr{G}_2 v_n\| \\ &= (1-\alpha_n)\|\mathscr{G}_1 w_n - \mathscr{G}_2 v_n\|. \end{aligned}$$

which implies that

$$\|u_{n+1} - \mathscr{G}_2 v_n\| = (1-\alpha_n)\|\mathscr{G}_1 w_n - \mathscr{G}_2 v_n\|. \tag{3.57}$$

Then taking limit as $n \longrightarrow \infty$ on both sides of (3.57) and using (3.56), we obtain the following

$$\lim_{n\to\infty} \|u_{n+1} - \mathscr{G}_2 v_n\| = 0. \tag{3.58}$$

Then, using triangle inequality we obtain the following

$$\|u_{n+1}-p\| \leq \|u_{n+1}-\mathscr{G}_2 v_n\| + \|\mathscr{G}_2 v_n - p\|$$
$$\leq \|u_{n+1}-\mathscr{G}_2 v_n\| + \|v_n - p\|. \tag{3.59}$$

By taking $\liminf$ as $n \longrightarrow \infty$ in both sides of (3.59) and using (3.49) and (3.58) we have the following

$$d \leq \liminf_{n\to\infty} \|v_n - p\|. \tag{3.60}$$

Now, by using Lemma 3.3, (3.1) and (3.12), we obtain

$$\|v_n - p\| = \|(1-\theta_n)u_n + \theta_n \mathscr{G}_1 u_n - p\|$$
$$\leq (1-\theta_n)\|u_n - p\| + \theta_n \|\mathscr{G}_1 u - p\|$$
$$\leq (1-\theta_n)\|u_n - p\| + \theta_n \|u_n - p\| \tag{3.61}$$
$$= \|u_n - p\|.$$

Then taking $\limsup$ as $n \longrightarrow \infty$ in both sides of (3.61), we obtain

$$\limsup_{n\to\infty} \|v_n - p\| \leq d. \tag{3.62}$$

Hence, by combining (3.60) and (3.62), we obtain the following

$$\lim_{n\to\infty} \|v_n - p\| = d. \tag{3.63}$$

From (3.1), (3.61) and (3.63), we obtain

$$d = \lim_{n\to\infty} \|v_n - p\| = \lim_{n\to\infty} \|(1-\theta_n)u_n + \theta_n \mathscr{G}_1 u_n - p\|$$
$$\leq \lim_{n\to\infty} \|u_n - p\| = d, \tag{3.64}$$

which implies that

$$\lim_{n\to\infty} \|(1-\theta_n)u_n + \theta_n \mathscr{G}_1 u_n - p\| = d. \tag{3.65}$$

Then, using (3.50), (3.65) and Lemma 2.3, we get the following

$$\lim_{n\to\infty} \|u_n - \mathscr{G}_1 u_n\| = 0. \tag{3.66}$$

Now, using (3.1) and (3.56), we obtain

$$\begin{aligned}\lim_{n\to\infty}\|u_{n+1}-\mathscr{G}_1w_n\| &= \lim_{n\to\infty}\|(1-\alpha_n)\mathscr{G}_1w_n+\alpha_n\mathscr{G}_2v_n-G_1w_n\|\\ &= \lim_{n\to\infty}\alpha_n\|\mathscr{G}_2v_n-\mathscr{G}_1w_n\|\\ &= 0, \qquad (3.67)\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|u_n-\mathscr{G}_1w_n\| = 0. \qquad (3.68)$$

Again, using (3.1) and (3.56), we obtain

$$\begin{aligned}\lim_{n\to\infty}\|u_{n+1}-\mathscr{G}_2v_n\| &= \lim_{n\to\infty}\|(1-\alpha_n)\mathscr{G}_1w_n+\alpha_n\mathscr{G}_2v_n-\mathscr{G}_2v_n\|\\ &= \lim_{n\to\infty}(1-\alpha_n)\|\mathscr{G}_1w_n-\mathscr{G}_2v_n\| \qquad (3.69)\\ &= 0,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|u_n-\mathscr{G}_2v_n\| = 0. \qquad (3.70)$$

Since, $\mathscr{G}_2$ satisfies the $B_{\gamma,\mu}$ condition, then we have that

$$\begin{aligned}\|\mathscr{G}_2v_n-\mathscr{G}_2u_n\| &\le (1-\gamma)\|v_n-u_n\|+\mu(\|v_n-\mathscr{G}_2u_n\|+\|u_n-\mathscr{G}_2v_n\|) \qquad (3.71)\\ &\le (1-\gamma)\|v_n-u_n\|+\mu(\|v_n-u_n\|+\|u_n-\mathscr{G}_2u_n\|+\|u_n-\mathscr{G}_2v_n\|)\end{aligned}$$

Then, by using (3.71) and triangle inequality, we obtain

$$\begin{aligned}\|\mathscr{G}_2u_n-u_n\| &\le \|\mathscr{G}_2v_n-\mathscr{G}_2u_n\|+\|\mathscr{G}_2v_n-\mathscr{G}_1w_n\|+\|\mathscr{G}_1w_n-u_n\|\\ &\le (1-\gamma)\|v_n-u_n\|+\mu(\|v_n-u_n\|+\|u_n-\mathscr{G}_2u_n\|\\ &\quad+\|u_n-\mathscr{G}_2v_n\|)+\|\mathscr{G}_2v_n-\mathscr{G}_1w_n\|+\|\mathscr{G}_1w_n-u_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}&\|\mathscr{G}_2u_n-u_n\|\\ &\le \frac{(1-\gamma+\mu)}{1-\mu}\|v_n-u_n\|+\frac{\mu}{1-\mu}\|u_n-\mathscr{G}_2v_n\|+\frac{1}{1-\mu}\|\mathscr{G}_2v_n-\mathscr{G}_1w_n\|+\frac{1}{1-\mu}\|\mathscr{G}_1w_n-u_n\|\end{aligned}$$

$$\leq \|v_n - u_n\| + \frac{\mu}{1-\mu}\|u_n - \mathscr{G}_2 v_n\| + \frac{1}{1-\mu}\|\mathscr{G}_2 v_n - \mathscr{G}_1 w_n\| + \frac{1}{1-\mu}\|\mathscr{G}_1 w_n - u_n\|.$$

Since, from (3.1) we have $\|v_n - u_n\| = \theta_n \|u_n - \mathscr{G}_1 u_n\|$. Thus

$$\|\mathscr{G}_2 u_n - u_n\| \leq \theta_n \|u_n - \mathscr{G}_1 u_n\| + \frac{\mu}{1-\mu}\|u_n - \mathscr{G}_2 v_n\| + \frac{1}{1-\mu}\|\mathscr{G}_2 v_n - \mathscr{G}_1 w_n\| + \frac{1}{1-\mu}\|\mathscr{G}_1 w_n - u_n\|. \tag{3.72}$$

Taking limit as $n \longrightarrow \infty$ on both direction of (3.72) and using (3.56), (3.66), (3.68) and (3.70), we get the following

$$\lim_{n\to\infty} \|u_n - \mathscr{G}_2 u_n\| = 0. \tag{3.73}$$

This complete the forward proof.

Now, we prove the converse. Let $\lim_{n\to\infty} \|u_n - \mathscr{G}_i u_n\| = 0,\ i = 1,2$. Then

$$0 = \gamma \min\{\|u_n - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 u_n\|\} \leq \|u_n - p\| + \mu \max\{\|p - \mathscr{G}_1 p\|, \|p - \mathscr{G}_2 p\|\}. \tag{3.74}$$

So, by Definition 3.1, we have

$$\max\{\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\|, \|\mathscr{G}_2 u_n - \mathscr{G}_2 p\|\}$$
$$\leq (1-\gamma)\|u_n - p\| + \mu \min\{\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 p\| + \|p - \mathscr{G}_2 u_n\|\}.$$

From the above inequality we obtain

$$\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\| \leq (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|) \tag{3.75}$$

implies that

$$\begin{aligned}
\|\mathscr{G}_1 p - u_n\| - \|u_n - \mathscr{G}_1 u_n\| &\leq (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|) \\
\Rightarrow \|\mathscr{G}_1 p - u_n\| &\leq \|u_n - \mathscr{G}_1 u_n\| + (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|) \\
&\leq \|u_n - \mathscr{G}_1 u_n\| + (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - u_n\| \\
&\quad + \|u_n - \mathscr{G}_1 u_n\|) \\
&= (1+\mu)\|u_n - \mathscr{G}_1 u_n\| + (1-\gamma+\mu)\|u_n - p\| + \mu\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\|
\end{aligned}$$

because, $\dfrac{(1-\gamma+\mu)}{(1-\mu)} \leq 1$ implies that

$$\begin{aligned}(1-\mu)\left\|\mathscr{G}_1 p-u_n\right\| &\leq (1+\mu)\left\|u_n-\mathscr{G}_1 u_n\right\|+(1-\gamma+\mu)\left\|u_n-p\right\| \\ \Rightarrow \left\|\mathscr{G}_1 p-u_n\right\| &\leq \frac{(1+\mu)}{(1-\mu)}\left\|u_n-\mathscr{G}_1 u_n\right\|+\frac{(1-\gamma+\mu)}{(1-\mu)}\left\|u_n-p\right\| \\ &\leq \frac{(1+\mu)}{(1-\mu)}\left\|u_n-\mathscr{G}_1 u_n\right\|+\left\|u_n-p\right\|\end{aligned}$$

which implies that

$$\left\|\mathscr{G}_1 p-u_n\right\| \leq \frac{(1+\mu)}{(1-\mu)}\left\|u_n-\mathscr{G}_1 u_n\right\|+\left\|u_n-p\right\|. \tag{3.76}$$

Then, taking $\limsup$ as $n \longrightarrow \infty$ on both sides of inequality (3.76), we get

$$\begin{aligned}\limsup_{n\to\infty}\left\|\mathscr{G}_1 p-u_n\right\| &\leq \frac{(1+\mu)}{(1-\mu)}\limsup_{n\to\infty}\left\|u_n-\mathscr{G}_1 u_n\right\|+\limsup_{n\to\infty}\left\|u_n-p\right\| \\ &= \limsup_{n\to\infty}\left\|u_n-p\right\|\end{aligned}$$

implies that

$$\limsup_{n\to\infty}\left\|\mathscr{G}_1 p-u_n\right\| \leq \limsup_{n\to\infty}\left\|u_n-p\right\|. \tag{3.77}$$

So, by Definition 2.2, we have the following result

$$r(\mathscr{G}_1 p,\{u_n\}) = \limsup_{n\to\infty}\left\|u_n-\mathscr{G}_1 p\right\| \leq \limsup_{n\to\infty}\left\|u_n-p\right\| = r(p,\{u_n\}). \tag{3.78}$$

This implies that $\mathscr{G}_1 p \in A(\mathscr{K},\{u_n\})$. Since, $\mathscr{B}$ is uniformly convex Banach space, $A(\mathscr{K},\{u_n\})$ is singleton, hence $\mathscr{G}_1 p = p$. In a similar way, one can show that $\mathscr{G}_2 p = p$. This completes the proof. □

The next lemma studies the demiclosedness principle of two mappings satisfying the $B_{\gamma,\mu}$ condition.

Lemma 3.8. *Let $\mathscr{K}$ be a nonempty closed convex subset of a Banach space $\mathscr{B}$ with the Opial?s condition. Let $\mathscr{G}_1,\mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition on $\mathscr{K}$. If $\{u_n\}$ is a sequence in $\mathscr{K}$ such that $\{u_n\}$ converges weakly to p and $\lim_{n\to\infty}\left\|\mathscr{G}_1 u_n-u_n\right\| = 0 = \lim_{n\to\infty}\left\|\mathscr{G}_2 u_n-u_n\right\|$, then $I-\mathscr{G}_1$ and $I-\mathscr{G}_2$ are demiclosed at zero.*

Proof. By Proposition 3.5 (for $\gamma = \frac{\theta}{2}, \theta \in [0,1]$)

(3.79)

$$0 = \gamma \min\{\|\mathscr{G}_1 u_n - u_n\|, \|\mathscr{G}_2 u_n - u_n\|\} \leq \|u_n - p\| \leq \|u_n - p\| + \mu \max\{\|p - \mathscr{G}_1 p\|, \|p - \mathscr{G}_2 p\|\}.$$

So, by Definition 3.1, we get

$$\begin{aligned} &\max\{\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\|, \|\mathscr{G}_2 u_n - \mathscr{G}_2 p\|\} \\ &\leq (1-\gamma)\|u_n - p\| + \mu \min\{\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 p\| + \|p - \mathscr{G}_2 u_n\|\}. \end{aligned} \tag{3.80}$$

Now, by using (3.80)

$$\begin{aligned} \max\{\|u_n - \mathscr{G}_1 p\|, \|u_n - \mathscr{G}_2 p\|\} &\leq \max\{\|u_n - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 u_n\|\} + \\ &\max\{\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\|, \|\mathscr{G}_2 u_n - \mathscr{G}_2 p\|\} \\ &\leq \max\{\|u_n - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 u_n\|\} + \\ &\max\{\|\mathscr{G}_1 u_n - \mathscr{G}_1 p\|, \|\mathscr{G}_2 u_n - \mathscr{G}_2 p\|\} \\ &\leq \max\{\|u_n - \mathscr{G}_1 u_n\|, \|u_n - \mathscr{G}_2 u_n\|\} + \\ &(1-\gamma)\|u_n - p\| + \mu \min\{\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|, \\ &\|u_n - \mathscr{G}_2 p\| + \|p - \mathscr{G}_2 u_n\|\}. \end{aligned}$$

Then from the above inequality one can obtain

$$\begin{aligned} \|u_n - \mathscr{G}_1 p\| &\leq \|u_n - \mathscr{G}_1 u_n\| + (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - \mathscr{G}_1 u_n\|) \\ &\leq \|u_n - \mathscr{G}_1 u_n\| + (1-\gamma)\|u_n - p\| + \mu(\|u_n - \mathscr{G}_1 p\| + \|p - u_n\| + \|u_n - \mathscr{G}_1 u_n\|), \end{aligned}$$

which implies that

$$\begin{aligned} \|u_n - \mathscr{G}_1 p\| &\leq \frac{(1+\mu)}{1-\mu}\|u_n - \mathscr{G}_1 u_n\| + \frac{(1-\gamma+\mu)}{1-\mu}\|u_n - p\| \\ &\leq \frac{(1+\mu)}{1-\mu}\|u_n - \mathscr{G}_1 u_n\| + \|u_n - p\|. \end{aligned} \tag{3.81}$$

Then taking $\liminf$ as $n \longrightarrow \infty$ on both side of (3.81), we obtain

$$\liminf_{n\to\infty}\|u_n - \mathscr{G}_1 p\| \leq \liminf_{n\to\infty}\|u_n - p\|.$$

Since $\mathscr{B}$ satisfies the Opial's condition, if $p \neq \mathscr{G}_1 p$, then we have

$$\liminf_{n\to\infty} \|u_n - p\| < \liminf_{n\to\infty} \|u_n - \mathscr{G}_1 p\|, \tag{3.82}$$

which is a contradiction. Hence $p = \mathscr{G}_1 p$. That is, $(I - \mathscr{G}_1)p = 0$. By similar arguments we obtain $(I - \mathscr{G}_2)p = 0$. Therefore, $I - \mathscr{G}_1$ and $I - \mathscr{G}_2$ are demiclosed at zero □

The following result is the weak convergence of iteration (3.1) to get the common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$.

Theorem 3.9. *Let $\mathscr{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{B}$. Assume that $\mathscr{B}$ satisfies the Opial condition. Let $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition with $Fix(\mathscr{G}) = Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$. Let a sequence $\{u_n\}$ be defined as the iteration scheme (3.1). Then, $\{u_n\}$ converges weakly to an element of $Fix(\mathscr{G})$.*

Proof. Since $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ are two mappings satisfying the $B_{\gamma,\mu}$ condition with $Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$, then by Theorem 3.7 we obtain

$$\lim_{n\to\infty} \|\mathscr{G}_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\mathscr{G}_2 u_n - u_n\|. \tag{3.83}$$

Since by Lemma 3.6 $\lim_{n\to\infty} \|u_n - p\|$ exists, hence $\{u_n\}$ is bounded. Consider that $\mathscr{B}$ satisfying Opial's conditions and let p_1 and p_2 be two weak sub-sequential limits of $\{u_n\}$. Assume that $\{u_{n_s}\}$ weakly converges to p_1 and $\{u_{n_t}\}$ weakly converges to p_2. We need to show that $p_1, p_2 \in Fix(\mathscr{G})$. Since by Lemma 3.8, we have that $I - \mathscr{G}_1$ is demiclosed at zero. Hence $(I - \mathscr{G}_1)p_1 = 0$. Then it follows that $p_1 = \mathscr{G}_1 p_1$. By similar arguments, we obtain that $p_1 = \mathscr{G}_2 p_1$. Therefore, p_1 is a common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$. Similarly, we obtain that p_2 is a common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$.

Next, we need to proof that $p_1 = p_2$. Let $p_1 \neq p_2$. Then we get that;

$$\begin{aligned}
\lim_{n\to\infty} \|u_n - p_1\| &= \liminf_{s\to\infty} \|u_{n_s} - p_1\| \\
&< \liminf_{s\to\infty} \|u_{n_s} - p_2\| \\
&= \lim_{n\to\infty} \|u_n - p_2\| \\
&= \liminf_{t\to\infty} \|u_{n_t} - p_2\|
\end{aligned}$$

$$< \liminf_{t\to\infty} \|u_{n_t} - p_1\|$$

$$= \lim_{n\to\infty} \|u_n - p_1\|,$$

which is a contradiction. Thus, $p_1 = p_2$ and this infers that $\{u_n\}$ weakly converges to the common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$. This completes the proof. □

Remark 3.10. *The Opial's property in some sub-classes of uniformly convex Banach spaces does not hold. So that, the above discussed result is not true for these some sub-classes of uniformly convex Banach spaces. Therefore, we use another way of proof, in the next result, where we consider the Frechet differentiable norm instead of Opial's property.*

Theorem 3.11. *Let $\mathscr{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{B}$ withe Fréchet differentiable norm. Let $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition, $I - \mathscr{G}_1$ and $I - \mathscr{G}_2$ be demiclosed at zero and $\lim_{k\to 0} \|ku_n + (1-k)p - q\|$ exists for all $p, q \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$. Then, $\{u_n\}$ converges weakly to a common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$.*

Proof. Since $Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$, from Theorem 3.7 $\lim_{n\to\infty} \|u_n - \mathscr{G}_i u_n\| = 0, i = 1, 2$. We need to show that u_n has a unique limit point. Assume $\{u_{n_i}\}$ weakly converges to z_1 and $\{u_{n_j}\}$ weakly converges to z_2. But also $I - \mathscr{G}_1$ and $I - \mathscr{G}_2$ are demiclosed at zero, this fact leads that $z_1, z_2 \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$. Putting $u = p - q$ and $v = k(u_n - p)$ in (2.8)

$$\begin{aligned}\frac{1}{2}\|p-q\|^2 + \langle k(u_n - p), J(p-q)\rangle &\le \frac{1}{2}\|ku_n + (1-k)p - q\|^2 \\ &\le \frac{1}{2}\|p-q\|^2 + \langle k(u_n - p), J(p-q)\rangle + h(k\|u_n - p\|).\end{aligned}$$

Using the given condition, we obtain

$$\begin{aligned}\frac{1}{2}\|p-q\|^2 + k\limsup_{n\to\infty}\langle u_n - p, J(p-q)\rangle &\le \frac{1}{2}\lim_{n\to\infty}\|ku_n + (1-k)p - q\|^2 \\ &\le \frac{1}{2}\|p-q\|^2 + k\liminf_{n\to\infty}\langle u_n - p, J(p-q)\rangle + \mathscr{O}(k).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\langle u_n - p, J(p-q)\rangle \le \liminf_{n\to\infty}\langle u_n - p, J(p-q)\rangle + \frac{\mathscr{O}(k)}{k}. \tag{3.84}$$

Then, taking $k \longrightarrow 0^+$, we get $\lim_{n\to\infty} \langle u_n - p, J(p-q)\rangle$ exists. Now, we have $\langle z_1 - p, J(p-q)\rangle = r$ (say) and also $\langle z_2 - p, J(p-q)\rangle = r$. So, $\langle z_1 - z_2, J(p-q)\rangle = 0$, for all $p, q \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$. From this we obtain

$$\|z_1 - z_2\|^2 = \langle z_1 - z_2, J(z_1 - z_2)\rangle = 0, \tag{3.85}$$

which is possible, if $z_1 = z_2$. Hence, $\{u_n\}$ converges weakly to a common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$. This completes the proof. □

We now state the following Lemma that enables us to prove the next result.

Lemma 3.12. [42] *Assume that the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$of non-negative real numbers such that $\alpha_{n+1} \leq \alpha_n + \beta_n$ for all $n \in \mathbb{N}$. If $\sum_n \beta_n$ converges, then $\lim_{n\to\infty} \alpha_n$ exists.*

Then, by using Lemma 3.12 and Definition 2.6, we want to prove the following result of strong convergence of the iterative scheme (3.1).

Theorem 3.13. *Let $\mathscr{K}$ be any non-empty closed and convex subset of a uniformly convex Banach space $\mathscr{B}$. Let $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ be two mappings satisfying the $B_{\gamma,\mu}$ condition with $Fix(\mathscr{G}) = Fix(\mathscr{G}_1 \cap Fix(\mathscr{G}_2) \neq \emptyset$. For any $u_1 \in \mathscr{K}$, we define the sequence $\{u_n\}$ as (3.1). Assume that $\mathscr{G}_1$ and $\mathscr{G}_2$ are satisfy the Condition (B). Then $\{u_n\}$ converges strongly to some common fixed point of $\mathscr{G}_1$ and $\mathscr{G}_2$.*

Proof. Let $p \in Fix(\mathscr{G})$. Then by Lemma 3.6, $\lim_{n\to\infty} \|u_n - p\|$ exists for all $p \in Fix(\mathscr{G})$. But also from (3.48), we obtain

$$\|u_{n+1} - p\| \leq \|u_n - p\|, \text{ for all } n \in \mathbb{N}. \tag{3.86}$$

Then

$$d(u_{n+1}, p) \leq d(u_n, p). \tag{3.87}$$

Therefore, by Lemma 3.12 $\lim_{n\to\infty} d(u_n, p)$ exists. But also from Theorem 3.7, we have that

$$\lim_{n\to\infty} \|\mathscr{G}_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\mathscr{G}_2 u_n - u_n\|. \tag{3.88}$$

Since $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy condition (B) and by Definition 2.6, we have that the following

$$\lim_{n\to\infty} \max(\|\mathscr{G}_1 u_n - u_n\|, \|\mathscr{G}_2 u_n - u_n\|) \geq \lim_{n\to\infty} h(d(u_n, Fix(\mathscr{G}))), \tag{3.89}$$

which implies that

$$\lim_{n\to\infty} h(d(u_n, Fix(\mathscr{G}))) = 0, \tag{3.90}$$

hence, $\lim_{n\to\infty} d(u_n, Fix(\mathscr{G})) = 0$.

Then, we can choose a sub-sequence $\{u_{n_j}\}$ of $\{u_n\}$, $\varepsilon > 0$ and some sequences $\{p_j\}$ in $Fix(\mathscr{G})$ such that $\|u_{n_j} - p_j\| \leq \frac{\varepsilon}{2}$ for all $j \in \mathbb{N}$. Then, we need to show that $\{u_n\}$ is a Cauchy sequence. Then, for all $m, n \geq j$, we have the following

$$\begin{aligned}
\|u_{m+n} - u_n\| &\leq \|u_{m+n} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-1} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-2} - p_j\| + \|u_n - p_j\| \\
&\leq \|u_{m+n-3} - p_j\| + \|u_n - p_j\| \\
&\vdots \\
&\leq 2\|u_n - p_j\| \\
&= 2\|u_{n_j} - p_j\| \\
&< \varepsilon,
\end{aligned}$$

which implies that $\|u_{m+n} - p_n\| < \varepsilon$. Hence, $\{u_n\}$ is a Cauchy sequence in $\mathscr{K}$. Since $\mathscr{K}$ is a closed convex subset of $\mathscr{B}$, $\lim_{n\to\infty} u_n = p$, for some $p \in \mathscr{K}$. Sine $Fix(\mathscr{G})$ is closed and $\liminf_{n\to\infty} d(u_n, Fix(\mathscr{G})) = 0$. Therefore, $p \in Fix(\mathscr{G})$. This completes the proof. □

Remark 3.14. *Our finding extend and unify those Patir et al.*[33] *and Suzuki* [39] *which deal with the existence and approximation of fixed point results concerning two mappings satisfying* $B_{\gamma,\mu}$ *condition.*

Example 3.15. *Let* $\mathscr{B} = \mathbb{R}$ *with the usual norm and* $\mathscr{K} = [0, \infty)$. *Let* $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{K} \longrightarrow \mathscr{K}$ *be defined by*

(3.91)
$$\mathscr{G}_1 x = \begin{cases} 0, \ if \ x \in [0,2) \\ 1, \ if \ x \in [2,\infty). \end{cases}$$

(3.92)
$$\mathscr{G}_2 x = \begin{cases} 0, \ if \ x \in [0,2) \\ 1.1, \ if \ x \in [2,\infty). \end{cases}$$

Proof. When $\gamma = 1, \mu = \frac{1}{2}$. Let us consider the following different cases;

Case i: If $x, y \in [0,2)$ and $y \geq \frac{4}{3}x$, then

(3.93)
$$\gamma \|x - \mathscr{G}_1 x\| = \|x - \mathscr{G}_1 x\| = \|x\| \leq \|y - x\| + \mu \|y\| = \|x - y\| + \frac{1}{2}\|y\| = \|x - y\| + \mu \|y - \mathscr{G}_1 y\|,$$

implies

(3.94)
$$\|\mathscr{G}_1 x - \mathscr{G}_1 y\| = 0 \leq (1-\gamma)\|x - y\| + \mu(\|x - \mathscr{G}_1 y\| + \|y - \mathscr{G}_1 x\|) = \frac{1}{2}(\|x\| + \|y\|).$$

Similarly it is holds for $\mathscr{G}_2$. Hence,

(3.95)
$$\gamma \min\{\|x - \mathscr{G}_1 x\|, \|x - \mathscr{G}_2 x\|\} \leq \|x - y\| + \mu \max\{\|y - \mathscr{G}_1 y\|, \|y - \mathscr{G}_2 y\|\}$$

implies

$$\max\{\|\mathscr{G}_1 x - \mathscr{G}_1 y\|, \|\mathscr{G}_2 x - \mathscr{G}_2 y\|\} \leq$$
$$(1-\gamma)\|x - y\| + \mu \min\{\|x - \mathscr{G}_1 y\| + \|y - \mathscr{G}_1 x\|, \|x - \mathscr{G}_2 y\| + \|y - \mathscr{G}_2 x\|\}$$

Case ii: If $x, y \in [2,\infty)$ and $y \geq \frac{4}{3}x - \frac{1}{3}$, then

(3.96)
$$\gamma \|x - \mathscr{G}_1 x\| = \|x - 1\| \leq \|x - y\| + \mu \|y - \mathscr{G}_1 y\| = \|x\| + \frac{1}{2}\|2x - 1\|,$$

implies

$$\|\mathscr{G}_1 x - \mathscr{G}_1 y\| = 0 \leq (1-\gamma)\|x - y\| + \mu(\|x - \mathscr{G}_1 y\| + \|y - \mathscr{G}_1 x\|)$$
$$= \frac{1}{2}(||x - 1|| + ||2x - 1||).$$

Similarly it is holds for $\mathscr{G}_2$. Hence,

$$\gamma \min\{\|x-\mathscr{G}_1 x\|, \|x-\mathscr{G}_2 x\|\} \leq \|x-y\| + \mu \max\{\|y-\mathscr{G}_1 y\|, \|y-\mathscr{G}_2 y\|\}, \tag{3.97}$$

implies

$$\begin{aligned}&\max\{\|\mathscr{G}_1 x-\mathscr{G}_1 y\|, \|\mathscr{G}_2 x-\mathscr{G}_2 y\|\}\\ &\leq (1-\gamma)\|x-y\| + \mu \min\{\|x-\mathscr{G}_1 y\|+\|y-\mathscr{G}_1 x\|, \|x-\mathscr{G}_2 y\|+\|y-\mathscr{G}_2 x\|\}\end{aligned}$$

Case iii: If $x \in [0,2)$ and $y \geq 3$, then

$$\begin{aligned}\gamma\|x-\mathscr{G}_1 x\| = \|x-\mathscr{G}_1 x\| &= \|x\|\\ &\leq \|x-y\| + \mu\|y-\mathscr{G}_1 y\|\\ &= \|x-y\| + \frac{1}{2}\|y-1\|,\end{aligned}$$

implies that

$$\begin{aligned}\|\mathscr{G}_1 x-\mathscr{G}_1 y\| = 1&\\ &\leq (1-\gamma)\|x-y\| + \mu(\|x-\mathscr{G}_1 y\|+\|y-\mathscr{G}_1 x\|)\\ &= \frac{1}{2}(\|x-1\|+\|y\|).\end{aligned}$$

Similarly holds for $\mathscr{G}_2$. Hence,

$$\gamma \min\{\|x-\mathscr{G}_1 x\|, \|x-\mathscr{G}_2 x\|\} \leq \|x-y\| + \mu \max\{\|y-\mathscr{G}_1 y\|, \|y-\mathscr{G}_2 y\|\}, \tag{3.98}$$

implies

$$\begin{aligned}&\max\{\|\mathscr{G}_1 x-\mathscr{G}_1 y\|, \|\mathscr{G}_2 x-\mathscr{G}_2 y\|\} \leq\\ &(1-\gamma)\|x-y\| + \mu \min\{\|x-\mathscr{G}_1 y\|+\|y-\mathscr{G}_1 x\|, \|x-\mathscr{G}_2 y\|+\|y-\mathscr{G}_2 x\|\}.\end{aligned}$$

Therefore, $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy the $B_{\gamma,\mu}$ condition on $\mathscr{K} = [0,\infty)$, when $\gamma = 1, \mu = \frac{1}{2}$ with $Fix(\mathscr{G}) = Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2) \neq \emptyset$, since $0 \in Fix(\mathscr{G}_1) \cap Fix(\mathscr{G}_2)$.

Now, we consider the following non-decreasing map $h(x) = \frac{x}{5}$, which satisfies $h(r) > 0$ if $r \in (0,\infty)$ and $h(0) = 0$. Then,

$$\begin{aligned} d(x_n, Fix(\mathscr{G})) &= \inf ||x - z||_{z \in Fix(\mathscr{G})} \\ &= \inf \|x - 0\| \\ &= \inf \|x\| \\ &= \begin{cases} 0, \ if\ x \in [0,2) \\ 2, \ if\ x \in [2,\infty). \end{cases} \end{aligned} \tag{3.99}$$

And from the above given facts, we can obtain the following

$$h(d(x_n, Fix(\mathscr{G}))) = \begin{cases} 0, \ if\ x \in [0,2) \\ \frac{2}{5}, \ if\ x \in [2,\infty). \end{cases} \tag{3.100}$$

So that we can consider the following cases

Case I: If $x \in [0,2)$, then we have

$$\|\mathscr{G}_1 x - x\| = \|0 - x\| = \|x\| \text{ and } \|\mathscr{G}_2 x - x\| = \|0 - x\| = \|x\|. \tag{3.101}$$

It follows that $\max\{\|\mathscr{G}_1 x - x\|, \|\mathscr{G}_2 x - x\| \geq h(d(x_n, Fix(\mathscr{G})))\}$.

Case II: If $x \in [2,\infty)$, we have

$$\|\mathscr{G}_1 x - x\| = \|1 - x\| = \|x - 1\| \text{ and } \|\mathscr{G}_2 x - x\| = \|1.1 - x\| = \|x - 1.1\|. \tag{3.102}$$

In this case $\max\{\|\mathscr{G}_1 x - x\|, \|\mathscr{G}_2 x - x\| \geq h(d(x_n, Fix(\mathscr{G})))\}$.

Thus, $\mathscr{G}_1$ and $\mathscr{G}_2$ satisfy the condition (B). Therefore, all the hypothesis of Theorem 3.13 satisfied. □

4. Conclusion

In this article, the existence and convergence of common fixed point have been studied for two mappings satisfying the $B_{\gamma,\mu}$ condition. In our study, we have shown that the weak and strong convergence of iterative approximations to common fixed points of pair of mappings satisfying the condition $B_{\gamma,\mu}$. Also an example which has been shown to be two mappings satisfying the $B_{\gamma,\mu}$ condition have been presented. Generally, our results mainly extend the results of

Patir et al. [33], approximate with the three step iteration process of Abbas and Nazir [1] and several other well known results in the literature. It is an open problem to prove existence and approximation of common fixed points of finite or infinite family of mappings satisfying the Condition $B_{\gamma,\mu}$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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