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CONVERGENCE STUDY OF COMMON FIXED POINTS FOR PAIR OF MAPPINGS IN PARTIALLY ORDERED BANACH SPACES

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Abstract. This paper aims to delve into the weak and strong convergence theorems concerning a pair of monotone mappings that meet condition (E) and converge to a common fixed point. This investigation is conducted within the context of uniformly convex ordered Banach spaces, employing the Ishikawa iteration technique. Furthermore, an illustrative example is presented to demonstrate the implications of the theoretical results we have derived.

Keywords: monotone mapping; condition (E); common fixed points; Banach space.

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1. Introduction

Consider a real Banach space $(X, \|\cdot\|)$, where $\mathscr{C}$ is a nonempty subset of X. We designate a mapping $\Phi : \mathscr{C} \to \mathscr{C}$ defined within a subset of the Banach space X as a contraction map if there exists a real constant $0 \leq r < 1$ satisfying the following condition:

$$\|\Phi(\xi) - \Phi(v)\| \leq r\|\xi - v\| \quad \text{for all } \xi, v \in X. \tag{1.1}$$

An element $p \in X$ is termed as a fixed point of Φ if $p = \Phi(p)$. We denote by $F(\Phi)$ the set of all fixed points of Φ. The Banach-Caccioppoli fixed point theorem, established in references such

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as [5] and [2], asserts that any contraction mapping in the context of complete metric spaces possesses a unique fixed point p. This p is, indeed, the limit of all sequences $\{\xi_n\}$ derived from the Picard iterates $\xi_{k+1} = \Phi(\xi_k)$, see [14].

Notably, the class of nonexpansive mappings is significant within the realm of fixed point theory. Specifically, Φ is categorized as a nonexpansive mapping when (1.1) holds with $r = 1$. The exploration of fixed point existence for nonexpansive mappings originated in 1965, initiated independently by Browder [4], Göhde [9], and Kirk [11]. Browder [4] and Göhde [9] established an existence theorem for nonexpansive mappings in uniformly convex Banach spaces, while Kirk [11] attained a similar result in reflexive Banach spaces using the normal structure property.

While the Picard iteration has proven effective in approximating fixed points of contraction mappings and their variations, its success has not extended to nonexpansive mappings like Φ, even when the existence of a fixed point of Φ is known. For instance, consider the scenario where $\mathscr{C} = [0,1]$ and $\Phi(\xi) = 1-\xi$; here, Φ is a self-nonexpansive mapping on $\mathscr{C}$ with a unique fixed point at $1/2$. However, when starting with $\xi_1 = \xi \neq 1/2$, the sequence of Picard iterates alternates between ξ and $1-\xi$, resulting in oscillation.

To enhance convergence speed and overcome such challenges, various iterative methods have been proposed by different researchers. Notably, some prominent iterative processes are outlined below, with the Mann iteration process [12] being one such example:

$$
\begin{cases}
\xi_1 \in \mathscr{C} \\
\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Phi(\xi_n); n \in \mathbb{N},
\end{cases}
\tag{1.2}
$$

where α_n is a sequence in $[0,1]$.

In 1974, Ishikawa [10] generalized Mann iteration process from one step to two step as follows:

$$
\begin{cases}
\xi_1 \in \mathscr{C} \\
v_n = (1-\beta_n)\xi_n + \beta_n \Phi(\xi_n) \\
\xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Phi(v_n); n \in \mathbb{N},
\end{cases}
\tag{1.3}
$$

where α_n and β_n are sequence in $[0,1]$.

In the usual scenario, the nonexpansive conditions are typically expected to be met for all points within the mappings domain. Naturally, there is an inclination to explore cases where this requirement can be considerably relaxed without compromising the theorem's outcome. In order to address this problem, Suzuki [20] defined a class of mappings called Suzuki generalized nonexpansive mappings. These mappings satisfy the condition (C), which states that if the distance between a point ξ and its image $\Phi(\xi)$ is less than half the distance between ξ and another point v, then the distance between $\Phi(\xi)$ and $\Phi(v)$ is less than or equal to the distance between ξ and v. Following definition is due to Suzuki [20].

Definition 1.1. [20]. Let $\Phi : \mathscr{C} \to \mathscr{C}$ be a mapping on a subset $\mathscr{C}$ of a Banach space X. Then Φ is said to satisfy condition (C), if for all $\xi, v \in \mathscr{C}$

$$\frac{1}{2}\|\xi - \Phi(\xi)\| \leq \|\xi - v\| \text{ implies } \|\Phi(\xi) - \Phi(v)\| \leq \|\xi - v\|. \tag{1.4}$$

The condition (C) is weaker than nonexpansiveness, there are mappings that satisfy condition (C) but are not nonexpansive.

In 2011, García-Falset et al. [7] introduced a generalization of condition (C) called condition (E_μ). This condition states that there exists a constant $\mu \geq 1$ such that the distance between $\Phi(\xi)$ and $\Phi(v)$ is less than or equal to μ times the distance between ξ and $\Phi(\xi)$ plus the distance between ξ and v. García-Falset et al. [7] defined the condition (E_μ) as follows:

Definition 1.2. [7]. Let $\Phi : \mathscr{C} \to \mathscr{C}$ be a mapping on a subset $\mathscr{C}$ of a Banach space X. Then Φ is said to satisfy condition (E_μ) on $\mathscr{C}$ if there exists $\mu \geq 1$ such that for all $\xi, v \in \mathscr{C}$,

$$\|\xi - \Phi(v)\| \leq \mu\|\xi - \Phi(\xi)\| + \|\xi - v\|. \tag{1.5}$$

We say that a mapping satisfies condition (E) if it satisfies condition (E_μ) for some $\mu \geq 1$. Interestingly, any mapping satisfying condition (C) also satisfies condition (E), but the converse is not necessarily true.

Currently, it is noteworthy that the realm of fixed point theory for monotone nonexpansive mappings is undergoing a significant surge in interest and expansion amongst researchers, as evidenced by references [16, 17, 18, 19]. This expansion, however, extends beyond traditional boundaries. An intriguing augmentation of the Banach contraction principle applies to metric spaces endowed with partial orders, as demonstrated by Ran and Reurings [15]. This expansion continued to unfold, encompassing not only nonexpansive mappings but also their generalizations.

Very recently, Chen et al. [6] discussed weak and strong convergence theorems concerning fixed points of monotone generalized α-nonexpansive mappings within a uniformly convex Banach space (U.C.B.S.) endowed with a partial order. They achieved this through the utilization of an iteration method introduced in their paper.

In a similar vein, Muangchoo-in et al. [13] and Buthinah et al. [3] established weak and strong convergence results for pairs of α-nonexpansive mappings sharing common fixed point(s) in the setting of a uniformly convex ordered Banach space (U.C.O.B.S.). Their approach incorporated the following Ishikawa iteration:

$$\begin{cases} \xi_1 \in \mathscr{C} \\ v_n = (1-\beta_n)\xi_n + \beta_n \Phi(\xi_n) \\ \xi_{n+1} = (1-\alpha_n)\xi_n + \alpha_n \Psi(v_n); \quad n \in \mathbb{N}. \end{cases} \tag{1.6}$$

Motivated by the work of García-Falset et al. [7], as well as the contributions by Muangchoo-in et al. [13] and Buthinah et al. [3], our objective is to extend weak and strong convergence theorems to encompass common fixed points of two monotone mappings satisfying condition (E).

2. Preliminaries

Definition 2.1. [23] Let $\mathscr{C}$ be a nonempty subset of a normed space X. A mapping $\Phi : \mathscr{C} \to \mathscr{C}$ is said to satisfy Condition (I) if there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ satisfying $g(0) = 0$ and $g(r) > 0$ for all $r \in (0,\infty)$ such that $\|\xi - \Phi(\xi)\| \geq g(\inf_{v \in F(\Phi)} \|\xi - v\|)$ for all $\xi \in \mathscr{C}$.

Definition 2.2. [1] A Banach space X is said to have the monotone weak-Opial property, if for every monotone weakly convergent sequence $\{\xi_n\}$ in X with weak limit ξ,

$$\liminf_{n\to\infty} \|\xi_n - \xi)\| < \liminf_{n\to\infty} \|\xi_n - v\|$$

for all $v \in X$ with $v \neq \xi$ and $\xi_n \preceq v$ for all $n \in \mathbb{N}$.

Remark 2.3. All Hilbert spaces, finite dimensional Banach spaces and $l^p(1<p<\infty)$ have the Opial property. On the other hand $L_p[0,2\pi]$ $(p\neq 2)$ do not satisfy the Opial property [8].

Definition 2.4. Let $(X,\|.\|,\preceq)$ be an ordered Banach space and $\mathscr{C}$ be a nonempty subset of X. A mapping $\Phi:\mathscr{C}\to\mathscr{C}$ is said to be monotone if

$$\xi\preceq v \text{ implies } \Phi(\xi)\preceq\Phi(v),$$

where $\xi,v\in\mathscr{C}$.

Definition 2.5. Let $(X,\|.\|,\preceq)$ be an ordered Banach space and $\mathscr{C}$ be a nonempty subset of X. A mapping $\Phi:\mathscr{C}\to\mathscr{C}$ is said to be monotone mapping satisfying condition (E) if Φ is monotone and there exists $\mu\geq 1$ such that

$$\|\xi-\Phi(v)\|\leq\mu\|\xi-\Phi(\xi)\|+\|\xi-v\|$$

for all $\xi,v\in\mathscr{C}$ with ξ and v are comparable.

Proposition 2.6. [7] *Let $\Phi:\mathscr{C}\to X$ be a mapping which satisfies the condition (E) on $\mathscr{C}$ with some fixed point, then Φ is quassinonexpansive.*

Lemma 2.7. [22] *Suppose X is uniformly convex Banach space. Suppose $0<a<b<1$, and $\{\alpha_n\}$ is a sequence in $[a,b]$. Suppose $\{w_n\}$, $\{p_n\}$ are sequences in X such that $\|w_n\|\leqq 1, \|p_n\|\leqq 1$ for all n. Define $\{z_n\}$ in X by $\{z_n\}=(1-\alpha_n)w_n+\alpha_n p_n$. If $\lim_{n\to\infty}\|z_n\|=1$, then $\lim_{n\to\infty}\|w_n-p_n\|=0$.*

3. Main Results

In this section, we will begin by establishing the crucial lemmas that underpin our main outcomes. Subsequently, we delve into an exploration of weak and strong convergence theorems pertaining to a common fixed point of two monotone mappings that adhere to condition (E).

Lemma 3.1. *Let $\mathscr{C}$ be a convex closed subset of U.C.O.B.S. $(X,\preceq)$. Let $\Phi,\Psi:\mathscr{C}\to\mathscr{C}$ be two monotone mappings satisfying the condition (E). Let $\zeta\in Fix(\Phi)\cap Fix(\Psi)$. Let $\{\xi_n\}$ be a sequence defined by Ishikawa iteration (1.6) with $\alpha,\beta\in[a,b]\subset[0,1]$. Suppose $\xi_1\preceq\Phi(\xi_1)$ and $\xi_1\preceq\Psi(\xi_1)$ and ξ_1 with ζ are comparable. Then following holds:*

(i) $\lim_{n\to\infty} \parallel \xi_n - \zeta \parallel$ *exists.*

(ii) *If* ξ_1 *with* ζ *are comparable for all* $\zeta \in Fix(\Phi)\cap Fix(\Psi)$*, then* $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi))$ *exists.*

Proof. First we want to show that

$$\xi_n \preceq \Phi(\xi_n),\ \forall\ n\ \geq 1. \tag{3.1}$$

Given the hypothesis $\xi_1 \preceq \Phi(\xi_1)$, we can confirm that inequality (3.1) is satisfied for $n = 1$. Now, leveraging the convexity of order intervals and the monotonicity property of the mapping Φ,

$$\begin{aligned} \xi_n \preceq (1-\beta_n)\xi_n + \beta_n\Phi(\xi_n) &= v_n \preceq (1-\beta_n)\Phi(\xi_n) + \beta_n\Phi(\xi_n) \\ &= \Phi(\xi_n). \end{aligned} \tag{3.2}$$

Thus,

$$\xi_n \preceq v_n \preceq \Phi(\xi_n) \text{ and } \xi_n \preceq v_n \preceq \Phi(\xi_n) \preceq \Phi(v_n)\ \forall\ n\ \geq 1. \tag{3.3}$$

Since $\xi_n \preceq \Phi(v_n)\ \forall\ n\ \geq 1$ and by convexity of order intervals, we have $\xi_n \preceq \xi_{n+1}\ \forall\ n\ \geq 1$ and the sequence $\{\xi_n\}$ is monotone.

Let $\zeta \in\ Fix(\Phi)\ \cap\ Fix(\Psi)$. Since $\xi_1 \preceq \zeta$, then by monotonicity of Φ, we have

$$\Phi(\xi_1) \preceq \Phi(\zeta) = \zeta.$$

Thus from (3.3)

$$v_1 \preceq \Phi(\xi_1) \preceq \zeta.$$

In view of mathematical induction, one can easily show that

$$\xi_n \preceq v_n \preceq \Phi(\xi_n) \preceq \zeta. \tag{3.4}$$

From conditions on mapping Φ and using Eq. (3.4), we have

$$\|\Phi(\xi_n) - \zeta\| \leq \|\xi_n - \zeta\|.$$

Further,

$$\begin{aligned}
\|\xi_{n+1}-\zeta\| &= \|(1-\alpha_n)\xi_n+\alpha_n\Psi(v_n)-\zeta\| \\
&\le (1-\alpha_n)\|\xi_n-\zeta\|+\alpha_n\|\Psi(v_n)-\zeta\| \\
&\le (1-\alpha_n)\|\xi_n-\zeta\|+\alpha_n(1-\beta_n)\|\xi_n-\zeta\|+\alpha_n\beta_n\|\Phi(\xi_n)-\zeta\| \\
&\le (1-\alpha_n)\|\xi_n-\zeta\|+\alpha_n(1-\beta_n)\|\xi_n-\zeta\|+\alpha_n\beta_n\|\xi_n-\zeta\| \\
&= \|\xi_n-\zeta\|.
\end{aligned}$$

Thus $\|\xi_{n+1}-\zeta\|$ is nonincreasing, bounded, and $\lim_{n\to\infty}\|\xi_n-\zeta\|$ exists. Hence $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi))$ exists. □

Lemma 3.2. *Let $\mathscr{C}$, X, Φ, Ψ be same as in Lemma 3.1. Let $\zeta\in Fix(\Phi)\cap Fix(\Psi)$. Assume that $\exists\ \xi_1\in\mathscr{C}$ s.t. $\xi_1\preceq\Phi(\xi_1)$, $\xi_1\preceq\Psi(\xi_1)$ and ξ_1 and ζ are comparable. Let $\{\xi_n\}$ be a sequence same as in Lemma 3.1. Then $\lim_{n\to\infty}\|\Phi(\xi_n)-\xi_n\|=0$ and $\lim_{n\to\infty}\|\Psi(\xi_n)-\xi_n\|=0$.*

Proof. Let $\{\xi_n\}$ be same defined same as in By Lemma 3.1, and let $\zeta\in Fix(\Phi)\cap Fix(\Psi)$. Then by using Lemma 3.1, we have

$$\lim_{n\to\infty}\|\xi_n-\zeta\| \text{ exists.}$$

i.e. we can find a real number $r\ge 0$ such that

$$\lim_{n\to\infty}\|\xi_n-\zeta\|=r. \tag{3.5}$$

On using the condition of the mapping Φ, we have

$$\|\Phi(\xi_n)-\zeta)\|\le\|\xi_n-\zeta\|,\ \forall n\in\mathbb{N}.$$

Then

$$\limsup_{n\to\infty}\|\Phi(\xi_n)-\zeta\|\le r.$$

Moreover;

$$\begin{aligned}
\|v_n-\zeta\| &= \|(1-\beta_n)\xi_n+\beta_n\Phi(\xi_n)-\zeta\| \\
&\le \|(1-\beta_n)(\xi_n-\zeta)\|+\|\beta_n(\Phi(\xi_n)-\zeta)\| \\
&\le (1-\beta_n)\|\xi_n-\zeta\|+\beta_n\|\xi_n-\zeta\|
\end{aligned}$$

$$= \|\xi_n - \zeta\|.$$

Hence, we get

$$\limsup_{n\to\infty} \|v_n - \zeta\| \leq r. \tag{3.6}$$

By Property of Ψ and by Proposition (2.6), we deduce

$$\limsup_{n\to\infty} \|\Psi(v_n) - \zeta\| \leq r.$$

From Eq. (3.5), we have

$$\lim_{n\to\infty} \|(1-\alpha_n)(\xi_n - \zeta) + \alpha_n(\Psi(v_n) - \zeta)\| = r.$$

Hence by Lemma 2.7,

$$\lim_{n\to\infty} \|\Psi(v_n) - \xi_n\| = 0. \tag{3.7}$$

By the condition on mapping Ψ

$$\begin{aligned} \|\xi_n - \zeta\| &\leq \|\xi_n - \Psi(v_n)\| + \|\Psi(v_n) - \zeta\| \\ &\leq \|\xi_n - \Psi(v_n)\| + \|v_n - \zeta\|. \end{aligned}$$

Thus

$$r \leq \liminf_{n\to\infty} \|v_n - \zeta\|. \tag{3.8}$$

By Eq. (3.6) and Eq. (3.8), we have

$$\lim_{n\to\infty} \|v_n - \zeta\| = r. \tag{3.9}$$

This means

$$\lim_{n\to\infty} \|(1-\beta_n)(\xi_n - \zeta)\| + \|\beta_n(\Phi(\xi_n) - \zeta\| = r,$$

and by Lemma 2.7,

$$\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0. \tag{3.10}$$

In view of Eq. (1.6)

$$\begin{aligned}\|\Phi(\xi_n)-v_n\| &= \|\Phi(\xi_n)-(1-\beta_n)\xi_n-\beta_n\Phi(\xi_n)\| \\ &= (1-\beta_n)\|\Phi(\xi_n)-\xi_n\|.\end{aligned}$$

On taking limit $n\to\infty$, we have

$$\lim_{n\to\infty}\|\Phi(\xi_n)-v_n\|=0. \tag{3.11}$$

Let us suppose that

$$p_n=\frac{\Psi(\xi_n)-\zeta}{\|\xi_n-\zeta\|} \text{ and } w_n=\frac{\Phi(v_n)-\zeta}{\|\xi_n-\zeta\|}.$$

Thus, $\|p_n\|\leq 1$ and $\|w_n\|\leq 1$ for all $n\in\mathbb{N}$. Therefore, sequences $\{p_n\}$ and $\{w_n\}$ contained in a unit ball of X. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\|(1-\alpha_n)\,p_n+\alpha_n w_n\| &= \lim_{n\to\infty}\frac{\|(1-\alpha_n)\,(\Psi(\xi_n)-\zeta)+\alpha_n\,(\Phi(v_n)-\zeta)\|}{\|\xi_n-\zeta\|} \\ &\leq \lim_{n\to\infty}\frac{(1-\alpha_n)\,\|\Psi(\xi_n)-\zeta\|+\alpha_n\,\|\Phi(v_n)-\zeta\|}{\|\xi_n-\zeta\|} \\ &\leq \lim_{n\to\infty}\frac{(1-\alpha_n)\,\|\xi_n-\zeta\|+\alpha_n\,\|v_n-\zeta\|}{\|\xi_n-\zeta\|} \\ &\leq \lim_{n\to\infty}\frac{(1-\alpha_n)\,\|\xi_n-\zeta\|+\alpha_n\,\|\xi_n-\zeta\|}{\|\xi_n-\zeta\|} \\ &\leq \lim_{n\to\infty}\frac{\|\xi_n-\zeta\|}{\|\xi_n-\zeta\|}=\frac{r}{r}=1.\end{aligned}$$

In view of Lemma 2.7, we have $\lim_{n\to\infty}\|p_n-w_n\|=0$. However,

$$\|p_n-w_n\|=\frac{\|\Phi(v_n)-\Psi(\xi_n)\|}{\|\xi_n-\zeta\|}.$$

Thus,

$$\lim_{n\to\infty}\|\Phi(v_n)-\Psi(\xi_n)\|=\lim_{n\to\infty}\|p_n-w_n\|\lim_{n\to\infty}\|\xi_n-\zeta\|=0. \tag{3.12}$$

By the triangle inequality, and using that Φ satisfies condition (E)

$$\begin{aligned}\|\xi_n-\Psi(\xi_n)\| &\leq \|\xi_n-\Phi(v_n)\|+\|\Phi(v_n)-\Psi(\xi_n)\| \\ &= \|\xi_n-v_n\|+\mu\|\xi_n-\Phi(\xi_n)\|+\|\Phi(v_n)-\Psi(\xi_n)\| \\ &= \|\xi_n-\Phi(\xi_n)\|+\|\Phi(\xi_n)-v_n\|+\mu\|\xi_n-\Phi(\xi_n)\|+\|\Phi(v_n)-\Psi(\xi_n)\|\end{aligned}$$

$$= \quad (1+\mu)\|\xi_n - \Phi(\xi_n)\| + \|\Phi(\xi_n) - v_n\| + \|\Phi(v_n) - \Psi(\xi_n)\|.$$

From Eq. (3.10), Eq. (3.11) and Eq. (3.12),

$$\lim_{n\to\infty} \|\xi_n - \Psi(\xi_n)\| = 0.$$

□

Theorem 3.3. *(Weak Convergent theorem) Let $\mathscr{C}$, X, Φ, Ψ be same as in Lemma 3.1. Let X satisfies the monotone weak-Opial property and $\zeta \in Fix(\Phi) \cap Fix(\Psi)$. Assume there exists $\xi_1 \in \mathscr{C}$ such that $\xi_1 \preceq \Phi(\xi_1)$, $\xi_1 \preceq \Psi(\xi_1)$ and ξ_1 and ζ are comparable. Let $\{\xi_n\}$ be a sequence same as in Lemma 3.1. Then $\{\xi_n\}$ weak converges to a point in $Fix(\Phi) \cap Fix(\Psi)$.*

Proof. In view of Lemma 3.1 and Lemma 3.2, we have $\lim_{n\to\infty} \|\xi_n - \zeta\|$ exists for any $\zeta \in Fix(\Phi) \cap Fix(\Psi)$ and

$$\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0 = \lim_{n\to\infty} \|\Psi(\xi_n) - \xi_n\|. \tag{3.13}$$

By the boundedness of $\{\xi_n\}$ in U.C.O.B.S., $\{\xi_n\}$ has a weak subsequential limit. In order to prove that $\{\xi_n\}$ has a unique weak subsequential limit in $Fix(\Phi) \cap Fix(\Psi)$. Suppose ζ_1 and ζ_2 are two weak limits of the subsequential $\{\xi_{n_i}\}$ and $\{\xi_{n_j}\}$ of $\{\xi_n\}$, respectively and $\zeta_1 \neq \zeta_2$ from (3.13), demiclosedness of Φ at zero and $\xi_{n_i} \rightharpoonup \zeta_1$ as $n_i \to \infty$ it follows that $\Phi(\zeta_1) = \zeta_1$. Similarly $\Psi(\zeta_1) = \zeta_1$. Again it can easily prove that $\Phi(\zeta_2) = \zeta_2$ and $\Psi(\zeta_2) = \zeta_2$. Thus $\zeta_1, \zeta_2 \in Fix(\Phi) \cap Fix(\Psi)$. By the standard application of the monotone weak-Opial's property and on simplifying the expression,

$$\begin{aligned}
\lim_{n\to\infty} \|\xi_n - \zeta_1\| &= \lim_{i\to\infty} \|\xi_{n_i} - \zeta_1\| < \lim_{i\to\infty} \|\xi_{n_i} - \zeta_2\| \\
&= \lim_{n\to\infty} \|\xi_n - \zeta_2\| = \lim_{j\to\infty} \|\xi_{n_j} - \zeta_2\| \\
&< \lim_{j\to\infty} \|\xi_{n_j} - \zeta_1\| = \lim_{n\to\infty} \|\xi_n - \zeta_1\|,
\end{aligned}$$

we arrive at a contradiction. Therefore $\zeta_1 = \zeta_2$ and $\{\xi_n\}$ converges weakly to a point in $Fix(\Phi) \cap Fix(\Psi)$. □

Theorem 3.4. *(Strong Convergent theorem). Let $\mathscr{C}$, X, Φ, Ψ be same as in Lemma 3.1. Assume $\exists\ \xi_1 \in \mathscr{C}$ s.t. $\xi_1 \preceq \Phi(\xi_1)$, $\xi_1 \preceq \Psi(\xi_1)$ and $\xi_1 \preceq \zeta$ for all $\zeta \in Fix(\Phi) \cap Fix(\Psi)$. Let $\{\xi_n\}$ be a* *sequence same as in Lemma 3.1. Then* $\{\xi_n\}$ *converges strongly to a point in* $Fix(\Phi)\cap Fix(\Psi)$ *if and only if* $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi)) = 0$*. where* $d(\xi_n, Fix(\Phi)\cap Fix(\Psi))$ *is the distance from* ξ *to* $Fix(\Phi)\cap Fix(\Psi)$*.*

Proof. Suppose $\{\xi_n\}$ converges strongly to a point in $Fix(\Phi)\cap Fix(\Psi)$. Hence it is clear that $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi)) = 0$. Conversely, let $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi)) = 0$.
Since by Lemma 3.1 $\lim_{n\to\infty} d(\xi_n, Fix(\Phi)\cap Fix(\Psi))$ exists. Therefore, by the standard argument it can be prove that the sequence $\{\xi_n\}$ is Cauchy. Since $\mathscr{C}$ is a closed subset of X, then we can find a $z\in\mathscr{C}$ such that $\xi_n \to z$, and

$$\xi_n \preceq z \text{ for all } n\in\mathbb{N}.$$

On using the definition of mapping Ψ, we have

$$\begin{aligned}
\|z-\Psi(z)\| &\leq \|z-\xi_n\| + \|\xi_n - \Psi(z)\| \\
&\leq \|z-\xi_n\| + \|\xi_n - z\| + \mu\|\xi_n - \Psi(\xi_n)\| \\
&= 2\|\xi_n - z\| + \mu\|\xi_n - \Psi(\xi_n)\| \to 0.
\end{aligned}$$

As $n\to\infty$. Thus $\Psi(z) = z$.
Similarly, since Φ satisfies condition (E), then

$$\begin{aligned}
\|z-\Phi(z)\| &\leq \|z-\xi_n\| + \|\xi_n - \Phi(z)\| \\
&\leq \|z-\xi_n\| + \|\xi_n - z\| + \mu\|\xi_n - \Phi(\xi_n)\| \\
&= 2\|\xi_n - z\| + \mu\|\xi_n - \Psi(\xi_n)\|
\end{aligned}$$

on taking limit $n\to\infty$ and using the fact that $\Psi(z) = z$, we set $\|z-\phi(z)\|\to 0$ given that $\phi(z) = z$. Then we conclude that the sequence $\{\xi_n\}$ converges strongly to a point in $Fix(\Phi)\cap Fix(\Psi)$. □

In conclusion, we hereby present a strong convergence theorem applicable to mappings that fulfill both condition (E) and condition (I).

Theorem 3.5. *Let* $\mathscr{C}$*,* X*,* Φ*,* Ψ *be same as in Lemma 3.1. Let* Φ*,* Ψ *are two mapping defined on* X *satisfying the condition* (I) *with* $Fix(\Phi)\cap Fix(\Psi)\neq\emptyset$*. Let* $\{\xi_n\}$ *be a sequence same as in Lemma 3.1. Then* $\{\xi_n\}$ *converges strongly to a fixed point of* Φ *and* Ψ*.*

Proof. From Lemma 3.1 and Lemma 3.2, it follows that

$$\lim_{n\to\infty} \|\Phi(\xi_n) - \xi_n\| = 0 \text{ and } \lim_{n\to\infty} \|\Psi(\xi_n) - \xi_n\| = 0. \tag{3.14}$$

Since, Φ and Ψ satisfy condition (I), we have

$$\|\xi_n - \Phi(\xi_n)\| \geq g(d(\xi_n, Fix(\Phi) \cap Fix(\Psi)) \text{ and } \|\xi_n - \Psi(\xi_n)\| \geq g(d(\xi_n, Fix(\Phi) \cap Fix(\Psi)). \tag{3.15}$$

From Eq. (3.14), we get

$$\liminf_{n\to\infty} g(d((\xi_n, Fix(\Phi) \cap Fix(\Psi)) = 0.$$

Since $g : [0,\infty) \to [0,\infty)$ is a non decreasing function with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0,\infty)$, we have

$$\liminf_{n\to\infty}(d((\xi_n, Fix(\Phi) \cap Fix(\Psi)) = 0.$$

Therefore all the assumptions of Theorem 3.5 are satisfied and hence $\{\xi_n\}$ converges strongly to a point in $Fix(\Phi) \cap Fix(\Psi)$. □

Theorem 3.6. *Let $\mathscr{C}$, X, Φ,Ψ be same as in Lemma 3.1. Suppose that $\mathscr{C}$ is compact subset of X. Assume there exists $\xi_1 \in \mathscr{C}$ such that $\xi_1 \preceq \Phi(\xi_1)$, $\xi_1 \preceq \Psi(\xi_1)$ and $\xi_1 \preceq z$ for all $z \in Fix(\Phi) \cap Fix(\Psi)$. Let $\{\xi_n\}$ be a sequence same as in Lemma 3.1. Then $\{\xi_n\}$ converges strongly to a point in $Fix(\Phi) \cap Fix(\Psi)$.*

Proof. Since $\mathscr{C}$ is compact set, there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ that strongly converges to $p^\dagger \in \mathscr{K}$. By the triangle inequality and condition on mapping Φ, we get

$$\|\xi_{n_j} - \Phi(p^\dagger)\| \leq \mu\|\xi_{n_j} - \Phi(\xi_{n_j})\| + \|\xi_{n_j} - p^\dagger\|.$$

Taking $j \to \infty$, implies

$$\limsup_{j\to\infty} \|\xi_{n_j} - \Phi(p^\dagger)\| \leq \mu \lim_{j\to\infty} \|\xi_{n_j} - \Phi(\xi_{n_j})\| + \limsup_{j\to\infty} \|\xi_{n_j} - p^\dagger\|,$$

and, from Lemma 3.2, we have $\Phi(p^\dagger) = p^\dagger$. Lemma 3.1 ensures that $\lim_{n\to\infty} \|\xi_n - p^\dagger\|$ exists. Therefore, $p^\dagger$ is a strong limit of the sequence $\{\xi_n\}$. Similarly $\Psi(p^\dagger) = p^\dagger$ and this completes the proof.

□

4. Example

Let $X=\mathbb{R}$ and $\mathscr{C}=[-1,1]$ with the usual norm $\|.\|$.

Let $\Phi,\Psi:\mathscr{C}\to\mathscr{C}$ be defined as

$$\Phi(\xi)=\begin{cases}\frac{-\xi}{2}, & \text{if } \xi\in[-1,0)\\ -\xi, & \text{if } \xi\in[0,1]\end{cases} \quad \Psi(\xi)=\begin{cases}\frac{|-\xi|}{3}, & \text{if } \xi\in[-1,1)\\ \frac{-1}{2} & \text{if } \xi=1\end{cases}$$

Now we show that Φ satisfy the condition (E). We consider different cases as follows;

(1) Let $\xi,v\in[-1,0)$,we have

$$\begin{aligned}\|\xi-\Phi(v)\| &\leq \|\xi-\Phi(\xi)\|+\|\Phi(\xi)-\Phi(v)\|\\ &\leq \|\xi-\Phi(\xi)\|+\frac{1}{2}\|v-\xi\|\\ &\leq \|\xi-\Phi(\xi)\|+\|\xi-v\|.\end{aligned}$$

(2) Let $\xi,v\in[0,1]$,

$$\begin{aligned}\|\xi-\Phi(v)\| &\leq \|\xi-\Phi(\xi)\|+\|\Phi(\xi)-\Phi(v)\|\\ &= \|\xi-\Phi(\xi)\|+\|\xi-v\|.\end{aligned}$$

(3) Let $\xi\in[-1,0)$ and $v\in[0,1]$,

$$\begin{aligned}\|\xi-\Phi(v)\| &= |\xi+v|\leq|\xi|+|v|\\ &\leq \frac{3}{2}|\xi|+|\xi-v| \ as\ (\xi<0\ and\ v\geq 0)\\ &= \|\xi-\Phi(\xi)\|+\|\xi-v\|.\end{aligned}$$

(4) Let $\xi\in[0,1]$ and $v\in[-1,0]$,

$$\begin{aligned}\|\xi-\Phi(v)\| &= |\xi+\frac{v}{2}|\\ &\leq \frac{3}{2}|\xi|+|\frac{\xi}{2}-\frac{v}{2}|\\ &\leq \|2\xi\|+\|\xi-v\|\\ &= \|\xi-\Phi(\xi)\|+\|\xi-v\|.\end{aligned}$$

Now we show that Ψ satisfy the condition (E).

(1) if Let $\xi \in [-1,0), v \in [-1,1]$.Then $|\xi - \Psi(\xi)| = \frac{4}{3}|\xi|$

$$\begin{aligned} |\xi - \Psi(v)| &\leq |\xi| + \frac{1}{3}|v| \\ &\leq \frac{4}{3}|\xi| + \frac{1}{3}|\xi - v| \\ &\leq |\xi - \Psi(\xi)| + |\xi - v|. \end{aligned}$$

(2) if Let $\xi \in [0,1), v \in [-1,1]$.Then $|\xi - \Psi(\xi)| = \frac{2}{3}|\xi|$

$$\begin{aligned} |\xi - \Psi(v)| &\leq |\xi| + \frac{1}{3}|v| \\ &\leq \frac{4}{3}|\xi| + \frac{1}{3}|\xi - v| \\ &\leq 2|\xi - \Psi(\xi)| + |\xi - v|. \end{aligned}$$

(3) if Let $\xi = 1, v \in [-1,1]$.Then $|\xi - \Psi(\xi)| = \frac{4}{3}$

$$\begin{aligned} |1 - \Psi(v)| &\leq 1 - \frac{|v|}{3} \\ &= \frac{2}{3} + \frac{1 - |v|}{3} \\ &\leq \frac{1}{2}|1 - \Psi(1)| + \frac{1}{3}|1 - v| \\ &\leq |1 - \Psi(1)| + |1 - v|. \end{aligned}$$

Therefore, Φ, Ψ satisfies the condition (E) with $\mu \geq 2$ and both have common fixed point that is 0.

Conflict of Interests

The authors declare that there is no conflict of interests.

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