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BANACH CONTRACTION PRINCIPLE AND SOME RESULTS IN SUPER 2-METRIC SPACES

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Abstract. In this paper, we introduce the general form of 2-metric spaces known as super 2-metric spaces. First we prove an analogous of Banach contraction principle in setting of super 2-metric and then present some interesting result in the setting of super 2-metric spaces.

Keywords: 2-metric spaces; super 2-metric spaces; Banach contraction principle.

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1. Introduction

The concept of 2-metric spaces was developed by Gähler in a series of papers [19] [20], [21] and many more papers dealing with 2-metric spaces are [1, 20, 21, 14, 10, 13, 24, 25, 23] and [26].

In metric spaces, one deals with the properties of length function and 2-metric spaces deal with the property of the area function for Euclidean triangles.

Gähler [19] introduced the notion of 2-metric spaces as follows:

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A 2-metric space is a space X together with a non-negative real valued function d on X^3 satisfying the following conditions:

(1.1) to each pair of distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$;

(1.2) $d(x,y,z) = 0$, when at least two of x, y, z are equal;

(1.3) $d(x,y,z) = d(x,z,y) = d(y,z,x) \ldots$;

(1.4) $d(x,y,z) \leq d(u,y,z) + d(x,u,z) + d(x,y,u)$ $\forall x,y,z,u \in X$.

A sequence $\langle x_n \rangle$ in X is called a Cauchy sequence if $\lim_{n\to\infty} d(x_n, x_m, a) = 0$ for all $a \in X$.
A sequence $\langle x_n \rangle$ in X is convergent to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for each $a \in X$ and x is called the limit point of the sequence.
A complete 2-metric space is one in which every Cauchy sequence converges.
In 2022, Karapinar *et al.* [7] introduced a more general form of distance function known as super metric spaces as follows:
Let X be a nonempty set and $m : X \times X \to [0,\infty)$ be a function such that

(1.1) if $m(x,y) = 0$, then $x = y$, for all $x, y \in X$;

(1.2) $m(x,y) = m(y,x)$, for all $x, y \in X$;

(1.3) there exists $s \geq 1$ such that for all $y \in X$ there exist distinct sequences $\langle x_n \rangle, \langle y_n \rangle \subset X$, with $m(x_n, y_n) \to 0$, whenever $n \to \infty$, such that

$$\lim_{n\to\infty} \sup m(y_n, y) \leq s \lim_{n\to\infty} \sup m(x_n, y).$$

Then we call (X,m) as a super metric space.

Motivated by the concepts of metric and super metric spaces, we introduce the concept of super 2-metric spaces as a general form of 2-metric spaces as follows :

Definition 1.1. Let X be a non-empty set. A function $m : X \times X \times X \to [0,\infty)$ is said to be a super 2-metric if

(1.1) for every $x, y \in X$ with $x \neq y$, there exists $a \in X$ such that $m(x,y,a) \neq 0$;

(1.2) $m(x,y,a) = 0$, then $x = y$, for all $a \in X$;

(1.3) $m(x,y,a) = m(x,a,y) = m(y,a,x) = m(y,x,a) = m(a,x,y) = m(a,y,x)$;

(1.4) there exists $s \geq 1$ such that for all $y \in X$, there exist two distinct sequences $\langle x_n \rangle, \langle y_n \rangle \subset X$ with $m(x_n, y_n, a) \to 0$ as n tends to infinity such that

$$\lim_{n\to\infty} \sup m(y_n, y, a) \leq s \lim_{n\to\infty} \sup m(x_n, y, a) \textit{ for all } a \in X.$$

Then (X,m) is called a super 2-metric space.

The topological structure of super 2-metric spaces is analogous to topological structure of 2-metric spaces.

Definition 1.2. Let (X,m) be a super 2-metic space and $a,b \in X$,$r \geq 0$. The set

$$B(a,b,r) = \{x \in X : m(a,b,x) < r\}$$

is called a super 2-ball centered at (a,b) with radius r. The topology generated by the collection of all super 2-balls acts as a basis of super 2-metric spaces.

Definition 1.3. Let (X,m) be a super 2-metric space. A sequence $\langle x_n \rangle$ in X is said to be

(i) convergent to a point x, if for every $a \in X$, $\lim_{n\to\infty} m(x_n, x, a) = 0$.
(ii) Cauchy sequence, if for every $a \in X$, $\lim_{n,m\to\infty} \sup m(x_n, x_m, a) = 0$, that is, for each $\varepsilon > 0$ and $a \in X$, there exists n_0 such that $m(x_n, x_m, a) < \varepsilon$ for all $n, m \geq n_0$.

A super 2-metric space is said to be complete if every Cauchy sequence in X converges to a point in X.

2. Analogue of Banach's Contraction Principle for Super 2-Metric Spaces

In 1922, Polish mathematician Banach proved a result which states that every contraction mapping on a complete metric space has a unique fixed point, popularly known as Banach contraction principle. After hundred years of journey, this principle has been proved in the setting of various types of metric spaces. Karapinar [7] gave the analogue of Banach contraction principle in the setting of super metric spaces as follows :
Let (X,m) be a complete super metric space and T be a self mapping on X satisfying

$$m(Tx,Ty) \leq \alpha m(x,y), 0 \leq \alpha < 1,$$

for all $x, y \in X$. Then T has a unique fixed point in X. Now we prove analogue of Banach contraction principle in the setting of super 2-metric spaces.

Before proving our main result we need the following Lemma:

Lemma 2.1. *Let T be a self mapping on a super 2-metric space X satisfying*

$$m(Tx, Ty, a) \leq \alpha\, m(x, y, a),\ 0 \leq \alpha < 1. \tag{1}$$

Then $\langle x_n \rangle$ is a Cauchy sequence in X.

Proof. Let $x_0 \in X$ and set $x_1 = Tx_0$.

For this x_1 , there exists x_2 such that $x_2 = Tx_1$. Continuing this way, we can define

$$x_{n+1} = Tx_n\ \forall n = 0, 1, 2, \dots\ . \tag{2}$$

From (1), we have

$$\begin{aligned} m(x_{n+1}, x_n, a) &= m(Tx_n, Tx_{n-1}, a) \\ &\leq \alpha\, m(x_n, x_{n-1}, a) \\ &\leq \alpha^2\, m(x_{n-1}, x_{n-2}, a) \\ &\ \vdots \\ &\leq \alpha^n\, m(x_1, x_0, a). \end{aligned} \tag{3}$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} m(x_n, x_{n+1}, a) = 0$, since $0 \leq \alpha < 1$.

Now by definition of super 2-metric spaces, for $s \geq 1$ and for all $x_{n+2} \in X$ there exist distinct sequences $\langle x_n \rangle, \langle x_{n+1} \rangle$ with $\lim_{n\to\infty} m(x_n, x_{n+1}, a) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+2}, a) \leq s \lim_{n\to\infty} \sup m(x_{n+1}, x_{n+2}, a)\ \textit{for all}\ a \in X.$$

Since $\lim_{n\to\infty} m(x_n, x_{n+1}, a) = 0$. Therefore, $\lim_{n\to\infty} \sup m(x_n, x_{n+2}, a) = 0$. Continuing in this way, for $s \geq 1$ and for all $x_{n+3} \in X$ there exist distinct sequences $\langle x_n \rangle, \langle x_{n+2} \rangle$ with $\lim_{n\to\infty} m(x_n, x_{n+2}, a) \to 0$ such that

$$\lim_{n\to\infty} \sup m(x_n, x_{n+3}, a) \leq s \lim_{n\longrightarrow\infty} \sup m(x_{n+2}, x_{n+3}, a)\ \textit{for all}\ a \in X.$$

i.e., $\lim_{n\to\infty} \sup m(x_n, x_{n+3}, a) = 0.$
Inductively, we can conclude

$$\lim_{n\to\infty} \sup m(x_n, x_m, a) = 0 \text{ for all } a \in X, m > n \, and \, m, n \in \mathbb{N}.$$

Thus $\langle x_n \rangle$ is Cauchy sequence in X. □

Now, we prove Banach contraction principle in the setting of super 2-metric spaces as follows:

Theorem 2.2. *Let (X,m) be a complete super 2-metric space and T be a contraction mapping on (X,m) i.e,*

$$m(Tx, Ty, a) \leq \alpha m(x, y, a), \text{ for all } x, y, a \in X, \ 0 \leq \alpha < 1. \tag{4}$$

Then T has a unique fixed point.

Proof. By Lemma 2.1, the sequence $\langle x_n \rangle$ defined by (2) is a Cauchy sequence. Since (X,m) is a complete super 2-metric space, therefore, the sequence $\langle x_n \rangle$ converges to a point say $z \in X$.
We claim that z be a fixed point of T.
Note that for all $a \in X$,

$$m(x_{n+1}, Tz, a) = m(Tx_n, Tz, a) \leq \alpha m(x_n, z, a).$$

Proceeding as limit $n \to \infty$ i.e., $Tz = z$.
This implies z is a fixed point for T.
Uniqueness:
Let $w (\neq z)$ be another fixed point.
$m(w, z, a) = m(Tw, Tz, a) \leq \alpha m(w, z, a)$, a contradiction, since $0 \leq \alpha < 1$.
This implies $z = w$.
Hence T has a unique fixed point. □

Now we present an example in support of the Theorem 2.2.

Example 2.3. Let $X=[0,\infty)$. Define $m : X\times X\times X \to [0,\infty)$ by

$$m(x,y,a)=\begin{cases} \frac{xya}{x+y+a} & x\neq y\neq a\neq 0\\ 0 & \text{when any two of x,y,a are equal}\\ max\{\frac{x}{2},\frac{y}{2},\frac{a}{2}\} & otherwise. \end{cases}$$

Proof. Suppose that $y\in X$. Consider two distinct non-zero sequences $\langle x_n\rangle$, $\langle y_n\rangle$ in X (different from a) such that $m(x_n,y_n,a)\to 0$ as $n\to\infty$, which implies that $m(x_n,y_n,a)=\frac{x_ny_na}{x_n+y_n+a}\to 0$ as $n\to\infty$ and we can choose $y_n\to 0$ and $x_n\to u$ as $n\to\infty$, where $u\in X$. Moreover, for any $y\in X$ and for all $a\in X$,

$$\limsup_{n\to\infty} m(y_n,y,a)=\limsup_{n\to\infty}\frac{y_nya}{y_n+y+a}=0\leq s\limsup_{n\to\infty} m(x_n,y,a)=\frac{uya}{u+y+a},$$

In case of $y=0$ the proof is straight forward, and hence it follows that (X,m) is a super 2-metric space. □

In 1969, Boyd and Wong [6] gave idea of $\phi-$contraction in setting of metric space: "there exists an upper semi-continuous function $\phi:[0,\infty)\longrightarrow[0,\infty)$ and $\phi(t)<t$ for each $t>0$ such that

$$d(fp,fq)\leq\phi(d(p,q))."$$

Now using Boyd and Wong $\phi-$contraction in setting of super 2-metric spaces we present a generalisation of Theorem 2.1.

Theorem 2.4. *Let (X,m) be a complete super 2-metric space, and f be a self map, $f:X\longrightarrow X$. Assume there exists a right continuous function $\phi:[0,\infty)\longrightarrow[0,\infty)$ such that $\phi(r)<r$, if $r>0$ and $\phi(t)=0$ iff $t=0$ satisfying,*

$$m(fx,fy,a)\leq\phi(m(x,y,a)), \tag{5}$$

for all $x,y,a\in X$. Then f has a unique fixed point in X.

Proof. Let $x_0\in X$. Define sequence $\langle x_n\rangle$ in X by

$$x_{n+1}=fx_n,\ for\ all\ n=0,1,2,...,\ where\ x_0\in X.$$

Consider $\alpha_n=m(x_n,x_{n-1},a)$. From (5), we have

$$\alpha_n=m(x_n,x_{n-1},a)=m(fx_{n-1},fx_{n-2},a)\leq\phi(m(x_{n-1},x_{n-2},a)). \tag{6}$$

on simplification, $\alpha_n \leq \phi(\alpha_{n-1}) < \alpha_{n-1}$, thus $\langle \alpha_n \rangle$ is a decreasing sequence of reals, so converges to some point say $L \geq 0$ in $\mathbf{R}^+$.
i.e., $\lim_{n\to\infty} m(x_n, x_{n-1}, a) = L$.
We claim $L = 0$, if $L \neq 0$ then $L \leq \phi(L) < L$, a contradiction, hence $L = 0$,
i.e.,$\lim_{n\to\infty} m(x_n, x_{n-1}, a) = 0$.
Since $\lim_{n\to\infty} m(x_n, x_{n-1}, a) = 0$, therefore from Lemma 2.1 we get that $\langle x_n \rangle$ is a Cauchy sequence in X. Since (X, m) is complete super 2-metric space, so $\langle x_n \rangle$ converges to a point, say z in X. From (5), we have,

$$m(x_n, fz, a) \leq \phi(m(x_n, z, a) \text{ for all } a \in X.$$

Proceeding limit $n \to \infty$ we have $fz = z$. Hence z is a fixed point of f.

Uniqueness:
Let $w(\neq z)$ be another fixed point of f. From (5) for all $a \in X$ we have,

$$m(z, w, a) = m(fz, fw, a) \leq \phi(m(z, w, a)) < m(z, w, a),$$

a contradiction, thus $w = z$. Hence f has a unique fixed point in X. □

3. Weakly Compatible Maps and Property (E.A.)

In 1998, Jungck and Rhoades [8] introduced the notion of weakly compatible as follows:

Definition 3.1. Two maps f and g are said to be weakly compatible if the maps commute at their coincidence points.

Example 3.2. Let $X = [0, 3]$. Define self maps f and g on X as $fx = \frac{x}{2}$ and $gx = x$, then $f(0) = g(0)$ and $fg(0) = gf(0)$. Hence f and g are weakly compatible.

In 2002, Amari and Moutawakil ([15]) introduced the notion of property(E.A.) as follows:

Definition 3.3. Let f and g be two self-maps of a metric space (X, d). The pair (f,g) is said to satisfy property(E.A.), if there exists a sequence $\langle x_n \rangle$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Example 3.4. Let $X = [0, 1]$. Define $f, g : X \longrightarrow X$ by $fx = \frac{x}{2}$ and $gx = \frac{3x}{4}$, for all $x \in X$. Consider sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$. Then f and g satisfy property(E.A.).

The importance of property(E.A.) is that it relaxes the continuity requirement of maps completely, weakens the completeness requirement of space and E.A. buys containment of ranges without any continuity requirement to the points of coincidence.

Now we generalise analogue of Banach contraction in the setting of super 2-metric space for a pair of maps.

Theorem 3.5. *Let (X,m) be a complete super 2-metric space and f and g be self maps of X satisfying conditions:*

$$m(fx,fy,a) \leq \alpha m(gx,gy,a), \tag{7}$$

where $0 \leq \alpha < 1$ and $g(X) \subseteq f(X)$. If one of the spaces $f(X)$ or $g(X)$ is a complete subspace in X, then f and g have a unique common fixed point, provided f and g are weakly compatible.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \text{ for all } n = 0,1,2,..., \text{ where } x_0 \in X.$$

Therefore, from (7),we have,

$$\begin{aligned} m(y_n,y_{n+1},a) = m(fx,fy,a) &\leq \alpha m(gx_n,gx_{n+1},a) \\ &= \alpha m(y_{n-1},y_n,a) \\ &\vdots \\ &\leq \alpha^n m(y_0,y_1,a). \end{aligned} \tag{8}$$

Taking limit as $n \to \infty$, we have,

$$m(y_n,y_{n+1},a) = 0.$$

From Lemma 2.1, sequence $\langle y_n \rangle$ is Cauchy sequence in X and (X,m) is complete super 2-metric space, so it converges to a point say z in X i.e.,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Let $g(X)$ be complete subspace of X, there exists $p \in X$ such that, $gp = z$.

From (7), we have,

$$m(fp,fx_n,a) \leq \alpha m(gp,gx_n,a) = \alpha m(z,y_{n-1},a).$$

Taking limit as $n \to \infty$, we get $fp = z$.

Since f and g are weakly compatible $fgp = gfp$, implies $fz = gz$. Now to show that z is a common fixed point of f and g. Consider,

$$m(fz, fx_n, a) \leq \alpha\, m(gz, gx_n, a)$$

$$m(fz, z, a) \leq \alpha\, m(fz, z, a).$$

This implies, $fz = z$ as $\alpha < 1$. Therefore, $fz = gz = z$. Hence z is a common fixed point of f and g.

Uniqueness follows easily from Theorem 2.1. □

Now we prove this result using closed subset instead of complete subspace.

Theorem 3.6. *Let (X,m) be a complete super 2-metric space and f and g be self maps of X such that*

$$m(fx, fy, a) \leq \alpha(m(gx, gy, a)) \text{ for all } x, y, a \in X \;, 0 \leq \alpha < 1, \tag{9}$$

and $g(X) \subseteq f(X)$, such that one of $f(X)$ and $g(X)$ is closed subset of X. Then f and g have a unique common fixed point in X, provided f and g are weakly compatible .

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \text{ for all } n = 0, 1, 2, \ldots, \text{ where } x_0 \in X.$$

From the proof of Theorem 3.5 we conclude that $\langle y_n \rangle$ is a Cauchy sequence in X and since either $f(x)$ or $g(X)$ is closed, for definiteness assume that $g(X)$ is closed subset of X. Note that $\langle y_n \rangle$ is contained in $g(X)$ and it has a limit point in $g(X)$, call it z. Let $p \in g^{-1}z$. Then $gp = z$. Now we show that $fp = z$. From (9), we have,

$$m(fx_n, fp, a) \leq \alpha\, m(gx_n, gp, a)$$

$$m(fx_n, fp, a) \leq \alpha\, m(y_{n-1}, z, a).$$

Letting limit $n \to \infty$, we get $fp = z$. Rest of the proof follows from Theorem 3.5. □

Now we generalise Banach contraction principle for a pair of mappings using $\phi-$contraction in setting of super 2-metric spaces.

Theorem 3.7. *Let (X,m) be a complete super 2-metric space and f and g be self maps of X. Assume there exists a right continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(r) < r$, if $r > 0$ and $\phi(t) = 0$ if and only if $t = 0$ satisfying following conditions:*

$$m(fx, fy, a) \leq \phi(m(gx, gy, a)) \text{ for all } x, y, a \in X, \tag{10}$$

and $g(X) \subseteq f(X)$, such that one of $f(X)$ and $g(X)$ is complete subspace of X. Then f and g have a unique common fixed point in X, provide f and g are weakly compatible.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, therefore choose $x_1 \in X$ such that $gx_1 = fx_0$. In general define x_{n+1} such that,

$$y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \ldots\ .$$

Consider $\alpha_n = m(y_n, y_{n-1}, a)$.
From (10), for all $a \in X$, we have

$$m(y_n, y_{n-1}, a) = m(fx_n, fx_{n-1}, a) \leq \phi(m(gx_n, gx_{n-1}, a)) = \phi(m(y_{n-1}, y_{n-2}, a)), \tag{11}$$

i.e., $\alpha_n \leq \phi(\alpha_{n-1}) < \alpha_{n-1}$. Thus $\langle \alpha_n \rangle$ is a decreasing sequence of reals , so converges in $\mathbb{R}^+$, say to L. i.e., $\lim_{n\to\infty} m(y_n, y_{n-1}, a) = L$. We claim $L = 0$ as if $L \neq 0$, $L \leq \phi(L) < L$, a contradiction, therefore, $L = 0$. Hence $\lim_{n\to\infty} m(y_n, y_{n-1}, a) = 0$. From Lemma 2.1, we get that $\langle y_n \rangle$ is Cauchy sequence. Since (X,m) is a complete super 2-metric space, so it converges to a point say $z \in X$. Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$.
Let $g(X)$ be complete subspace of X, therefore, there exists $p \in X$ such that $gp = z$.
Using (10), we have

$$m(fp, fx_n, a) \leq \phi(m(gp, gx_n, a)) = \phi(m(z, y_{n-1}, a)).$$

Proceeding limit $n \to \infty$,we have,
$fp = z, fp = gp = z$.
Since f and g are weakly compatible we have $fgp = gfp$, which implies $fz = gz$.
Now again using (10), we have

$$m(fz, fx_n, a) \leq \phi(m(gz, gx_n, a)) = \phi(m(fz, gx_n, a)).$$

Taking limit $n \to \infty$, we get,

$$m(fz,z,a) \leq \phi(m(fz,z,a)),$$

this implies $fz = gz = z$.

Hence z is a common fixed point.

Uniqueness:

Let $w(\neq z)$ be another common fixed point of f and g.

For all $a \in X$ and using (9), we have,

$$m(z,w,a) = m(fz,fw,a) \leq \phi(m(gz,gw,a)) = \phi(m(z,w,a)),$$

a contradiction, thus $w = z$. Hence f and g have a unique common fixed point. □

Now we prove analogue of Banach contraction principle for a pair of weakly compatible maps satisfying property(E.A.) in the setting of super 2-metric space.

Theorem 3.8. *Let (X,m) be a super 2-metric space and f and g be self maps of X satisfying conditions:*

$$m(gx,gy,a) \leq \alpha m(fx,fy,a), \tag{12}$$

where $0 \leq \alpha < 1$ and f and g satisfying E.A. property further, $f(X)$ is a closed subspace of X. Then f and g have a unique common fixed point, provided f and g are weakly compatible.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \textit{ for all } n = 0,1,2,..., \textit{ where } x_0 \in X.$$

Since f and g satisfy property(E.A.), there exists a sequence $\langle x_n \rangle$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$=$z \in X$. Since $f(X)$ is a closed subspace of X, $\lim_{n\to\infty} fx_n = z = fp = \lim_{n\to\infty} gx_n$ for some $p \in X$. This implies $z = fp \in f(X)$. Now we show that $fp = gp$. From (12), we have,

$$m(gp,gx_n,a) \leq \alpha m(fp,fx_n,a)$$

$$m(gp,gx_n,a) \leq \alpha m(z,y_n,a) \textit{ for all } a \in X.$$

Proceeding limit $n \to \infty$, we have, $fp = gp = z$. And rest of the proof follows from Theorem 3.5. □

4. Expansive Mappings

In 1984, Wang, Li, Gao and Iséki [22] and Rhoades [3] proved some fixed point theorems for expansive mappings, which corresponds to some contractive mappings in metric spaces.

Now we prove expansive mappings in the setting of super 2-metric spaces, that corresponds to some contractive mappings in metric and 2-metric spaces.

Let f be a mapping of a super 2-metric space (X,m) into itself. Then f is said to be expansive mapping "if there exists a constant $\alpha > 1$ such that for all $x,y,a \in X$, we have

$$m(fx,fy,a) \geq \alpha m(x,y,a)\text{''}.$$

Theorem 4.1. *Let (X,m) be a complete super 2-metric space and $T : X \longrightarrow X$ be a surjective mapping. Suppose that $\alpha > 1$ such that,*

$$m(Tx,Ty,a) \geq \alpha m(x,y,a), \tag{13}$$

for all x,y,a in X. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and define,

$$x_n = Tx_{n+1},\ n = 0,1,2\ldots\ .$$

If $x_0 = x_1$, then x_1 is fixed point of X and proof is completed. Suppose $x_0 \neq x_1$.
Thus $m(x_1,x_0,a) \geq 0$, without loss of generality we assume, $x_n \neq x_{n+1}$. So $m(x_{n+1},x_n,a) > 0$ for all $n = 0,1,2\ldots$.
Therefore from (13),

$$\begin{aligned}
m(x_{n+1},x_n,a) = m(Tx_{n+2},Tx_{n+1},a) &\geq \alpha m(x_{n+2},x_{n+1},a)\\
\frac{1}{\alpha} m(Tx_{n+2},Tx_{n+1},a) &\geq m(x_{n+2},x_{n+1},a)\\
\frac{1}{\alpha} m(x_{n+1},x_n,a) &\geq m(x_{n+2},x_{n+1},a)\\
m(x_{n+2},x_{n+1},a) &\leq \frac{1}{\alpha} m(x_{n+1},x_n,a)\\
&\leq \frac{1}{\alpha^2} m(x_n,x_{n-1},a)\\
&\leq \frac{1}{\alpha^3} m(x_{n-1},x_{n-2},a)
\end{aligned} \tag{14}$$

$$\vdots$$

$$\leq \frac{1}{\alpha^{n+1}} m(x_1, x_0, a)$$

By Lemma 2.1 , the sequence $\langle x_n \rangle$ is Cauchy sequence. Since (X,m) is a complete super 2-metric space, therefore , the sequence $\langle x_n \rangle$ converges to a point say $z \in X$.

Suppose $m(z,Tz,a) > 0$ for all $a \in X$. Note that for all $a \in X$,

$$m(x_{n-1}, Tz, a) = m(Tx_n, Tz, a) \leq \frac{1}{\alpha} m(x_n, z, a).$$

Taking limit as $n \to \infty$, we have,

$$m(z, Tz, a) = 0,$$

which implies $Tz = z$.

Hence z is the fixed point of T.

Uniqueness:

Let $w(\neq z)$ be another fixed point for T.

Since $m(w,z,a) \leq \frac{1}{\alpha} m(Tw,Tz,a) = \frac{1}{\alpha} m(w,z,a)$, a contradiction since $\alpha > 1$.

This implies $z = w$,

Hence T has a unique fixed point in X. □

We further generalise Theorem 4.1 for a pair of weakly compatible mappings in super 2-metric spaces.

Theorem 4.2. *Let (X,m) be a complete super 2-metric space and f and g be self maps on X satisfying the following conditions:*

$$m(fx, fy, a) \geq q\, m(gx, gy, a), \tag{15}$$

where $q > 1$ and $g(X) \subseteq f(X)$ for all $x,y,a \in X$. If one of the sub-spaces $f(X)$ or $g(X)$ is complete, then f and g have a unique common fixed point, provided f, g are weakly compatible.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$, choose $x_1 \in X$ such that $fx_1 = gx_0$. In general choose x_{n+1} such that,

$$y_n = fx_{n+1} = gx_n, \; n = 0, 1, 2... \; .$$

Then from (15),

$$m(y_n, y_{n+1}, a) = m(gx_n, gx_{n+1}, a) \leq \frac{1}{q} m(fx_n, fx_{n+1}, a) = \frac{1}{q} m(y_{n-1}, y_n, a). \tag{16}$$

From Lemma 2.1, the sequence $\langle y_n \rangle$ is Cauchy sequence. Since (X, m) is complete super 2-metric space, $\langle y_n \rangle$ converges to a point say $z \in X$.

Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} gx_n = z$.

Without loss of generality, assume $f(X)$ is complete subspace of X, so there exists a point p such that $fp = z$.

Now from (16), we have,

$$m(gp, gx_n, a) \leq \frac{1}{q} m(fp, fx_n, a) = \frac{1}{q} m(z, y_{n-1}, a).$$

Taking limit $n \to \infty$, we get $gp = z$. Since f and g are weakly compatible,therefore, $fgp = gfp$, i.e., $fz = gz$.

Now we show that z is a common fixed point of f and g from (16)

$$m(gz, gx_n, a) \leq \frac{1}{q} m(fz, fx_n, a).$$

Proceeding limit $n \to \infty$, we get, $gz = z$. Hence, z is a common fixed point of f and g.

Uniqueness:

Let $w(\neq z)$ be another fixed point of f and g.

$$m(z, w, a) = m(gz, gw, a) \leq \frac{1}{q} m(fz, fw, a) = \frac{1}{q} m(z, w, z),$$

a contradiction, as $q > 1$, hence, $z = w$. □

5. Φ−WEAK CONTRACTION

In 1997, Alber and Guerre-Delabriere [27] presented the notion of "$\phi-$weak contraction" in "Hilbert spaces". In 2001 Rhoades [4], extended this notion in setting of complete metric spaces as follows: "there exists a function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(fx, fy) \leq d(x, y) - \phi(x, y)\text{''}.$$

Now we generalise it in super 2-metric spaces.

Theorem 5.1. *Let (X,m) be a complete super 2-metric space and $T : X \longrightarrow X$ be a mapping such that*

$$m(Tx,Ty,a) \leq m(x,y,a) - \phi(m(x,y,a)) \tag{17}$$

for all $x,y,a \in X$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is continuous function with $\phi(t) = 0$ if and only if $t = 0$, $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Proof. Let us define a sequence $\langle x_n \rangle$ in X by

$$x_n = Tx_{n+1},\ for\ n = 0,1,2...\ where\ x_0 \in X.$$

If $x_0 = x_1$, then x_1 is a fixed point and proof is completed. Suppose $x_0 \neq x_1$. Thus $m(x_0,x_1,a) \geq 0$, without loss of generality we assume, $x_n \neq x_{n+1}$. So $m(x_n,x_{n+1},a) > 0$ for all $n = 0,1,2...$.
Let $\alpha_n = m(x_n,x_{n+1},a)$.
Therefore, from (17),we have,

$$m(x_n,x_{n+1},a) = m(Tx_{n-1},Tx_n,a) \leq m(x_{n-1},x_n,a) - \phi(m(x_{n-1},x_n,a)). \tag{18}$$

Which implies $m(x_n,x_{n+1},a) \leq m(x_{n-1},x_n,a)$, *i.e.*, $\alpha_n \leq \alpha_{n-1}$.
Sequence α_n is non-increasing sequence of reals, so it converges in $\mathbb{R}^+$.
Consequently, there exists $L \geq 0$ such that $\lim_{n\to\infty} \alpha_n = L$, i.e., $\lim_{n\to\infty} m(x_n,x_{n+1},a) = L$. We claim that $L = 0$ if $L \neq 0$. By using Continuity of ϕ and inequality (18), we get,

$$L \leq L - \phi(L),\ since,\ \phi(t) > 0\ for\ all\ t > 0,$$

implies , $\phi(L) = 0$, i.e., $L = 0$. Hence $\lim_{n\to\infty} m(x_n,x_{n+1},a) = 0$. By Lemma 2.1, $\langle x_n \rangle$ is Cauchy sequence. Since (X,m) is a complete super 2-metric space, $\langle x_n \rangle$ converges to a point say $z \in X$. we now show that z is fixed point of T.
From (17), for all $a \in X$, we have,

$$m(Tz,x_{n+1},a) = m(Tz,Tx_n,a) \leq m(z,x_n,a) - \phi(m(z,x_n,a)),$$

Proceeding limit $n \to \infty$, implies, $Tz = z$. Hence z is a fixed point of T.

Uniqueness:

Let $w(\neq z)$ be another fixed point.

$$m(z,w,a) = m(Tz,Tw,a) \leq m(z,w,a) - \phi(m(z,w,a)),$$

implies, $w = z$, as $\phi(t) > 0$ for all $t > 0$. □

Now we generalise Theorem 5.1 for a pair of weakly compatible mappings in super 2-metric space.

Theorem 5.2. *Let (X,m) be a complete super 2-metric space. Let f and g be self mappings, satisfying:*

$$m(fx,fy,a) \leq m(gx,gy,a) - \phi(m(gx,gy,a)) \tag{19}$$

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function with $\phi(t) > 0$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. Further with $f(X) \subseteq g(X)$ and $g(X)$ or $f(X)$ is complete subspace of X. Then f and g have a unique common fixed point, provided f, g are weakly compatible.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$. Choose $x_0 \in X$ such that $fx_1 = gx_0$. In general choose x_{n+1} such that,

$$y_n = fx_n = gx_{n+1} \ ,n = 0,1,2... \ .$$

Consider $\alpha_n = m(y_n,y_{n+1},a)$
From (19), we have,

$$\begin{aligned} m(y_n,y_{n+1},a) = m(fx_n,fx_{n+1},a) &\leq m(gx_n,gx_{n+1},a) - \phi(m(gx_n,gx_{n+1},a)) \\ &= m(y_{n-1},y_n,a) - \phi((m(y_{n-1},y_n,a)). \end{aligned} \tag{20}$$

i.e., $\alpha_n \leq \alpha_{n-1} - \phi(\alpha_{n-1})$. This implies α_n is non-increasing sequence, so it converges in $\mathbb{R}^+$. Consequently, there exists $L \geq 0$ such that $\lim_{n\to\infty} m(y_n,y_{n+1},a) = L$. By using definition of ϕ and inequality (20), we get,

$$L \leq L - \phi(L) \ \textit{since}, \ \phi(t) > 0 \textit{ for all } t > 0,$$

implies, $\phi(L) = 0$, i.e., $L = 0$. Hence $\lim_{n\to\infty} m(y_n,y_{n+1},a) = 0$. By Lemma 2.1, $\langle y_n \rangle$ is a Cauchy sequence in X and (X,m) is complete super 2-metric space, it converges to a point say $z \in X$.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Since $g(X)$ is complete subspace of X, there exists $p \in g(x)$ such that $gp = z$.

From (19), for all $a \in X$, we have,

$$m(fp, fx_n, a) \leq m(gp, gx_n, a) - \phi(m(gp, gx_n, a)) = m(z, y_{n-1}, a) - \phi(m(z, y_{n-1}, a)$$

Proceeding limit $n \to \infty$, we get,

$\lim_{n\to\infty} m(fp, z, a) = 0$, i.e., $fp = z$.

Thus $fp = gp$, since f and g are weakly compatible, so $fgp = gfp$, which implies $fz = gz$.

From (19), we have,

$$m(fz, fx_n, a) \leq m(gz, gx_n, a) - \phi(m(gz, gx_n, a) = m(fz, gx_n, a) - \phi(m(fz, gx_n, a).$$

Taking limit $n \to \infty$, we have,

$$m(fz, z, a) \leq m(fz, z, a) - \phi(m(fz, z, a)),$$

implies $fz = gz = z$. Hence z is the common fixed point of f and g.

Uniqueness:

Let $w(\neq z)$ be another fixed point. From (19), we have,

$$m(z, w, a) = m(fz, fw, a) \leq m(gz, gw, a) - \phi(m(gz, gw, a)) = m(z, w, a) - \phi(m(z, w, a)).$$

Thus $w = z$.

Hence z is the unique common fixed point of f and g. □

6. Integral Type Contraction Condition

In 2002 A. Branciari [2] gave existence of fixed point theorem for mappings satisfying general contraction inequality of integral type in complete metric space. Now we prove Branciari's result in the setting of super 2-metric space.

Theorem 6.1. *Let (X, m) be a complete super 2-metric space, and let $T : X \longrightarrow X$ be a mapping such that for each $x, y, a \in X$,*

$$\int_0^{m(Tx,Ty,a)} \phi(t)\,dt \leq k \int_0^{m(x,y,a)} \phi(t)\,dt \tag{21}$$

for $0 \leq k < 1$ *, where* $\phi : [0,\infty) \longrightarrow [0,\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0,\infty)$ *and non-negative, such that for each* $\varepsilon > 0$*,*

$$\int_0^{\varepsilon} \phi(t)\,dt > 0, \tag{22}$$

then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define $x_{n+1} = Tx_n$, $n = 0,1,2...$ For each integer $n \geq 1$, from (21), we get,

$$\begin{aligned} \int_0^{m(x_n,x_{n+1},a)} \phi(t)\,dt = \int_0^{m(Tx_{n-1},Tx_n,a)} \phi(t)\,dt &\leq k \int_0^{m(x_{n-1},x_n,a)} \phi(t)\,dt \\ &\leq k^2 \int_0^{m(x_{n-2},x_{n-1},a)} \phi(t)\,dt \\ &\vdots \\ &\leq k^n \int_0^{m(x_0,x_1,a)} \phi(t)\,dt. \end{aligned} \tag{23}$$

Taking limit $n \to \infty$, we get $\lim\limits_{n\to\infty} \int_0^{m(x_n,x_{n+1},a)} \phi(t)\,dt = 0$. From (22), we get ,

$$\lim_{n\to\infty} m(x_n,x_{n+1},a) = 0.$$

From definition of super 2-metric space, there exists $x_{n+2} \in X$,

$$\lim_{n\to\infty} sup \int_0^{m(x_n,x_{n+2},a)} \phi(t)\,dt \leq s \lim_{n\to\infty} sup \int_0^{m(x_n,x_{n+1},a)} \phi(t)\,dt,$$

implies $\lim\limits_{n\to\infty} m(x_n,x_{n+2},a) = 0$. By Lemma 2.1, we get $\langle x_n \rangle$ is Cauchy sequence and (X,m) is complete super 2-metric space, so $\langle x_n \rangle$ converges to a point say z in X. We now show that z is a fixed point of T.

From (21), we have,

$$\int_0^{m(x_{n+1},Tz,a)} \phi(t)\,dt = \int_0^{m(Tx_n,Tz,a)} \phi(t)\,dt \leq k \int_0^{m(x_n,z,a)} \phi(t)\,dt, \text{ for all } a \in X$$

Taking limit $n \to \infty$,

$$\int_0^{m(z,Tz,a)} \phi(t)\,dt \leq 0.$$

This implies $Tz = z$. Hence T has a fixed point, uniqueness follows easily from Theorem 2.1. □

Theorem 6.2. *Let (X,m) be a complete super 2-metric space and f and g be self mappings on X such that for each $x,y,a \in X$,*

$$\int_0^{m(fx,fy,a)} \phi(t)\,dt \leq k \int_0^{m(gx,gy,a)} \phi(t)\,dt \tag{24}$$

for $0 \leq k < 1$ and $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, non-negative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(t)\,dt > 0, \tag{25}$$

where $f(X) \subseteq g(X)$ and one of $g(X)$ or $f(X)$ is complete subspace of X. Then f and g has a unique common fixed point, provided f, g are weakly compatible.

Proof. Proof follows easily from Theorems 3.5 and 6.1 □

7. IMPLICIT RELATIONS

There were many kinds of fixed point theorems in fixed point theory literature. To unify these results authors have used certain types of implicit relations. In 2008 Akram et al. [16], introduced one such class of implicit relations as follows:

Definition 7.1. Let a non-empty set A consisting of all functions $\alpha : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ satisfying the following:
(i) α is continuous on the set $\mathbf{R}_+^3$ of all triplets of non-negative reals;
(ii) $a \leq kb$ for some $k \in [0,1)$, whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$ for all a,b.

Using this implicit relation we generalise our result in super 2-metric space.

Theorem 7.2. *Let (X,m) be a complete super 2-metric space and T be a self map on X, satisfying condition:*

$$m(Tx,Ty,a) \leq \alpha[m(x,y,a), m(x,Tx,a), m(y,Ty,a)] \tag{26}$$

for each $x,y,a \in X$ with some $\alpha \in A$. Then T has a fixed point.

Proof. Let x_0 be arbitrary point in X and define a sequence $\langle x_n \rangle$ in X by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2\ldots\ .$$

From (26), we have,

$$m(x_{n+1}, x_n, a) = m(Tx_n, Tx_{n-1}, a) \leq \alpha[m(x_n, x_{n-1}, a), m(x_n, Tx_n, a), m(x_{n-1}, Tx_{n-1}, a)]$$

$$= \alpha[m(x_n, x_{n-1}, a), m(x_n, x_{n+1}, a), m(x_{n-1}, x_n, a)]$$

$$= \alpha[m(x_n, x_{n-1}, a), m(x_n, x_{n+1}, a), m(x_n, x_{n-1}, a)]$$

(27) *which implies by Definition 7.1*

$$m(x_{n+1}, x_n, a) \leq k\, m(x_n, x_{n-1}, a)$$

Continuing in this way after n^{th} stage, we have,

$$m(x_{n+1}, x_n, a) \leq k^n m(x_1, x_0, a).$$

Taking limit $n \to \infty$, we have,

$\lim_{n\to\infty} m(x_{n+1}, x_n, a) = 0.$

From Lemma 2.1, the sequence $\langle x_n \rangle$ is Cauchy sequence. Since (X, m) is complete super 2-metric space, the sequence converges to a point say $z \in X$. we show that z is a fixed point, Consider,

$$m(x_{n+1}, Tz, a) = m(Tx_n, Tz, a) \leq \alpha[m(x_n, z, a), m(x_n, Tx_n, a), m(z, Tz, a)].$$

Taking limit $n \to \infty$, we have,

$$m(z, Tz, a) \leq \alpha[0, m(z, Tz, a), m(z, Tz, a)]$$

$$m(z, Tz, a) \leq k\, m(z, Tz, a).$$

i.e., $Tz = z$, as $k < 1$. Hence z is a fixed point. □

8. Any Kind of Weakly Compatible Maps

In 2010 Murthy et al. [17] introduced the notion of any kind of weakly compatible maps as follows:

Definition 8.1. A pair of self mappings (f,g) of a metric space (X,m) is said to be any kind of weakly compatible maps if and only if there is a sequence $\langle x_n \rangle$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, and $fgt = gft$ at this point.

Example 8.2. Define $f,g : [0,2] \longrightarrow [0,2]$ by $fx = 2$ if $x \in [0,1]$ and $fx = \frac{x}{2}$ if $x \in (1,2]$ and $gx = 2$ if $x \in [0,1]$ and $\frac{x+3}{5}$ if $x \in (1,2]$. Consider the sequence $\langle x_n \rangle = (2 - \frac{1}{2n})$. Clearly

$$fx_n = (1 - \frac{1}{4n}), gx_n = (1 - \frac{1}{10n}).$$

Therefore,

$$fx_n \to 1, gx_n \to 1$$

and $fg(1) = gf(1) = 1$, hence they are any kind of weakly compatible.

Now we prove a theorem for weakly compatible maps along with notion of any kind of weakly compatible.

Theorem 8.3. *Let (X,m) be a complete super 2-metric space and f and g be self maps of X satisfying conditions:*

$$m(fx, fy, a) \leq \alpha m(gx, gy, a), \tag{28}$$

where $0 \leq \alpha < 1$ and $f(X) \subset g(X)$. If one of the spaces $f(X)$ or $g(X)$ is a closed subset of X, f and g are any kind of weakly compatible maps. Then f and g have a unique common fixed point, provided f and g are weakly compatible maps.

Proof. Let us define a sequence $\langle y_n \rangle$ in X by

$$y_n = gx_{n+1} = fx_n, \ \textit{for all} \ n = 0,1,2,..., \ \textit{where} \ x_0 \in X.$$

Since f and g are any kind of weakly compatible maps, therefore, there exists a sequence $\langle x_n \rangle \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \in X$. Let $g(X)$ be a closet subset of X, then for sequence $\langle x_n \rangle$ in $g(X)$, there is a limit in $g(X)$. Call it be z such that $z = gp$. Therefore, $\lim_{n\to\infty} fx_n = z = gp = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies $z = gp \in g(X)$. Now we have to show $z = fp = gp$. Using (28), we have,

$$m(fp, fx_n, a) \leq \alpha m(gp, gx_n, a) = m(z, y_{n-1}, a)$$

Proceeding limit $n \to \infty$, gives $fp = z$. Rest of the proof follows from Theorem 3.5. $\square$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Constantin, Common fixed points of weakly commuting mappings in 2-metric spaces, Math. Japon. 36 (1991), 507–514.

[2] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002), 531–536. https://doi.org/10.1155/s0161171202007524.

[3] B.E. Rhoades , An expansion mapping theorem, Jnanabha, 23 (1993), 151–152.

[4] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal.: Theory Methods Appl. 47 (2001), 2683–2693. https://doi.org/10.1016/s0362-546x(01)00388-1.

[5] R. Chugh, S. Kumar, Common fixed points for weakly compatible maps, Proc. Math. Sci. 111 (2001), 241–247. https://doi.org/10.1007/bf02829594.

[6] D.W. Boyd, J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464. https://www.jstor.org/stable/2035677.

[7] E. Karapınar, F. Khojasteh, Super metric spaces, Filomat 36 (2022), 3545–3549. https://doi.org/10.2298/fil2210545k.

[8] G. Jungck, B.E. Rhoades , Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math. 29 (1998), 227–238.

[9] K. Iséki, Fixed point theorem in 2-metric spaces, Math. Sem. Notes 3 (1975), 133–136. https://doi.org/10.24546/E0001250.

[10] K. Iséki, P.L. Sharma, B.K. Sharma, Contraction type mapping on 2-metric space, Math. Japon. 21(1976), 67–70.

[11] K. Sanjay, A Note on Jungck's fixed point theorem, Fasc. Math. 45 (2010), 59–69.

[12] M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Cric. Mat. Palermo 22 (1906), 1–74.

[13] M.S. Khan, M. Swaleh, Results concering fixed points in 2-metric spaces, Math. Japon. 29 (1984), 519–525.

[14] M. Imdad, M.S. Khan, M.D. Khan, A common fixed point theorem in 2-metric spaces, Math. Japon. 36 (1991), 907–914.

[15] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002), 181–188. https://doi.org/10.1016/s0022-247x(02)00059-8.

[16] M. Akram, A.A. Zafar, A.A. Siddiqui, A general class of contractions: A-contractions, Novi Sad J. Math. 38 (2008), 25–33.

[17] P.P. Murthy, K. Sanjay, T. Kenan, Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces, Math. Commun. 15 (2010), 521–537.

[18] S. Banach, Surles operations dans les ensembles abstraites et leursapplications, Fund. Math. 3 (1922), 133–181.

[19] S. Gahler, 2-metrische Raume und ither topologische structure, Math. Narchr. 26 (1963), 115–148.

[20] S. Gahler, Linear 2-nonmeirte Raume, Math. Narchr. 27 (1964), 1–43.

[21] S. Ghaler, Uber die unformesior barkat 2-metriche Raume, Math. Nachr. 28 (1965), 235–244.

[22] S.Z. Wang , B.Y. Li , Z.M. Gao, K. Iséki , Some fixed point theorems on expansion mappings, Math. Japon. 29 (1984), 631–636.

[23] S.V.R. Naidu and J.R. Prasad, Fixed point theorems in 2-metric spaces, Indian J. Pure Appl. Math. 17 (1986), 974–993.

[24] S.V.R. Naidu, Fixed point theorems for self-maps on a 2-metric space, Pure Appl. Math. Sci. 41 (1995), 73–77.

[25] S.V.R. Naidu, Some fixed point theorems in metric and 2-metric spaces, Int. J. Math. Math. Sci. 28 (2001), 625–636. https://doi.org/10.1155/s016117120101064x.

[26] T. Van An, N. Van Dung, V.T. Le Hang, General fixed point theorems on metric spaces and 2-metric spaces, Filomat 28 (2014), 2037–2045. https://www.jstor.org/stable/24890043.

[27] Y.I. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich (Eds.), New Results in Operator Theory and Its Applications, Birkhäuser Basel, Basel, 1997: pp. 7–22. https://doi.org/10.1007/978-3-0348-8910-0_2.