Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:57
https://doi.org/10.28919/afpt/8836
ISSN: 1927-6303

FIXED POINTS OF OPERATORS WITH MULTIPLICATIVE CLOSED GRAPHS ON BIPOLAR b-MULTIPLICATIVE METRIC SPACES

J. MERCY AROKIA RANI[1,*], A. LEEMA MARIA PRAKASAM[2]

[1]Department of Mathematics, St. Joseph's College (Autonomous), (Affiliated to Bharathidasan University), Tiruchirappalli-620002, Tamilnadu, India

[2] PG and Research Department of Mathematics, Holy Cross College (Autonomous), (Affiliated to Bharathidasan University), Tiruchirappalli-620002, Tamilnadu, India

Abstract: This work focusses on fixed point iterations of a multiplicative contraction mapping on the bipolar b-metric space domain. We develop a map that maps points from one set in the series to the next and fulfils the multiplicative contraction condition with regard to an expanding sequence of subsets in a bipolar b-multiplicative metric space. In this manner, we provide many fixed-point theorems for different multiplicative contraction mappings with multiplicative closed graph. This comprehensive study greatly contributes to the identification of fixed-point theory inside the multiplicative metric spaces framework and generates significant predictions.

Keywords: b-multiplicative metric space; bipolar multiplicative metric space; multiplicative closed graph; multiplicative contraction; exponential transformation.

2020 AMS Subject Classification: 47H10.

1. INTRODUCTION

Metric spaces were first established by Frechet [1] in 1906. Since then, metric spaces have been extensively generalized by concept abstraction, metric function modification, or the removal or relaxation of certain axioms. In recent years, fixed point research has focused increasingly on these

*Corresponding author
E-mail addresses: jmercyarokiarani@gmail.com
Received August 14, 2024

structures, and several insightful discoveries have been produced in this field [2–9]. Multiplicative metric spaces were first presented by A. E. Bashirov and colleagues in [10]. M. Ozavsar as well as A. C. Cevikel demonstrated fixed point findings of multiplicative contraction mappings as well as deduced topological structures of multiplicative metric spaces (MMS) in [11]. Numerous works on fixed point theory in MMS have been published; they include [12-16]. Czerwik [17] established b-Metric space, that is a simplification of a metric space. M. U. Ali et al. invented b-MMS in [18]. In b-MMSs, there are certain fixed-point outcomes and topological characteristics.

A bipolar metric space is one of the most recent generalizations, as proposed by Mutlu and Gurdal [19]. A. Mutlu and U. Gurdal proposed the idea of bipolar metric space, providing a new concept for measuring the distance between members of two distinct sets. A generalization of metric space is called bipolar metric space. The reason for this is because distances occur more often between components of two distinct sets in real-world applications than midway points of a single set. Thus, bipolar metrics were created to ratify different kinds of distances. Basic examples include the affinity a session of scholars has for a certain set of events, the period mean distances among individuals as well as places, and several more. Other instances include the separation between sets and points in metric spaces and the separation among lines as well as points in a Euclidean space. A number of articles are being published for fixed point (FP) theory in bipolar metric spaces (BMS); for example, check [20-23] and the references therein.

Here we were motivated by Bipolar MMS [24] and the multiplicative closed graph operators on b-MMS in a FP theorem [25] for introducing and analyzing the Bipolar b-MMS (BBMMS) and including other fixed-point theorems with more multiplicative closed graphs for different kinds of multiplicative contraction mapping.

2. PRELIMINARY

In this part, let us provide some known preliminary findings. Refer to [24] and [25] for further details.

Definition 2.1: [24] Assume two non-empty sets, S and T. The following conditions must be met for a mapping $p : Q \times T \to [1, \infty)$ to be considered a bipolar multiplicative metric.

(i) $p(q,d) = 1 \Rightarrow q = d$, whenever $(q,d) \in Q \times D$,

(ii) $q = d \Rightarrow p(q,d) = 1$, whenever $(q,d) \in Q \times D$,

(iii)$p(q,d) = p(d,q), \forall\, q,d \in Q \cap D$,

(iv)$p(q_1,d_2) \leq p(q_1,d_1)p(q_2,d_1)p(s_2,t_2), \forall s_1,s_2 \in S, and\ t_1,t_2 \in T$.

It is known as a bipolar multiplicative metric space (or, BMMS) for the triple (Q, D, p).

Remark 2.2: [24] Let a BMMS be (Q, D, p). A set (Q, D, p) is said to be disjoint if $Q \cap D = \emptyset$. If $Q \cap D \neq \emptyset$, then the space (Q, D, p) is a joint. The terms "right pole" and "left pole" of (Q, D, p) refer to the sets T and S, respectively.

Example 2.3: [24] Let $Q = (1,\infty), D = (0,1]$. Define $p: Q \times D \to [0,\infty)$ as p(q, d) = $\left|\frac{q^2}{d^2}\right| *$, whenever (q, d) $\in Q \times D$, where $|.| * : N^+ \to N^+$is defined on a set of positive real numbers N^+ as follows: $|z| = z\ if\ k \geq 1\ and\ |k| = \frac{1}{k}\ if\ k < 1$. Then (Q, D, p) is a disjoint BMMS.

Remark 2.4: [24] (Q, Q, p) is a BMMS if (Q, p) is an MMS. On the other hand, (Q, p) is an MMS if (Q, D, p) is a BMMS such that Q = D.

Definition 2.5: [25] Presuming that F $\neq \emptyset$ is a set as well as $h \in R$ with $h \geq 1$. A multiplicative metric is a mapping $q: F \times F \to R+= [0,\infty)$ fulfilling the following four principles

(i) $q(u,t) \geq 1, \forall\ u,t \in f$,

(ii) $q(u,t) = 1$ if and only if(or, iff) $u = t$ in F,

(iii) $q(u,t) = q(t,u), \forall\ u,t \in F$,

(iv) $q(u,t) \leq [q(u,\rho)q(\rho,\iota)]h, \forall\ u,\iota,\rho \in F$.

Then, we call the triple (F, q, h) a b-MMS.

Definition 2.6: [25] Presuming that $(F, q\ h)$ is a b-MMS, $\{u_n\}$ is a sequence in F, as well as $u \in F$. In that case $\{u_n\}$ is known multiplicative converging to u, if for each multiplicative open ball $B_e(u) = \{\iota: q(u,t) < e\}, e > 1$, there exists $N \in \mathbb{N}$ so that $u_n \in B_e(u), \forall\, n > N$. This is represented by $u_n \to u(n \to \infty)$.

Lemma 2.7: [25] Given a b-MMS $(F, q\ h)$, $\{u_n\}$ is a sequence in F, as well as $u \in F$. If $q(u_n,u) \to 1$ (n → ∞), then $u_n \to u(n \to \infty)$.

Lemma 2.8: [25] Considering that $\{u_n\}$ is a sequence in F as well as $(F, q\ h)$ is a b-MMS. There is a distinct multiplicative limit point for each multiplicative convergent sequence $\{u_n\}$.

Definition 2.9: [25] Assuming that a b-MMS is $(F, q\ h)$. If there exists $N \in \mathbb{N}$ so that $q(u_n, u_m) < e, \forall m, n \geq N$, then the sequence $\{u_n\} \in F$ is referred to as a multiplicative Cauchy sequence (or, MCS) for all $e > 1$.

Lemma 2.10: [25] Considering that $\{u_n\}$ is a sequence in F as well as $(F, q\ h)$ is a b-MMS. When $q(u_n, u_m) \rightarrow 1(m, n \rightarrow \infty)$, $\{u_n\}$ is an MCS.

3. BIPOLAR B-MULTIPLICATIVE METRIC SPACES

Definition 3.1: Presuming that $F \neq \Phi$ is a set as well as $w \in \mathbb{R}$ with $w \geq 1$, a BBMMS is a structure (F, P_1, P_2, w), where $P_1: F \times F \to \mathbb{R}^+ = [0, \infty)$ and $P_2: F \times F \to \mathbb{R}^+ = [0, \infty)$ are mappings satisfying the following axioms for all $v, l, d \in F$:

1. $P_1(v, l) \geq 1$ and $P_1(v, l) \geq 1$
2. $P_1(v, l) = 1$ if and only if $v = l$ and $P_2(v, l) = 1$ if and only if $v = l$,
3. $P_1(v, l) = P_1(l, v)$ as well as $P_2(v, l) = P_2(l, v)$
4. $P_1(v, l) \leq [P_1(v, d) P_1(d, v)]^w$ and $P_2(v, l) \leq [P_2(v, d) P_2(d, v)]^w$

This paradigm extends the multiplicative metric qualities to two distinct metrics, P_1 and P_2, therefore encapsulating the essence of a BBMMS.

Definition 3.2: Given a BBMMS (F, P_1, P_2, w), a sequence $\{v_n\}$ in F, and $v \in F$, this statement is true. If for each multiplicative open ball $O_{e1,e2}(v) = \{l: P_1(v, l) < e1 \quad and\ P_1(v, l) < e1\}, e1, e2 > 1$, there exists $w \in \mathbb{N}$ such that $v_n \in O_{e1,e2}(v)$ for all $n > N$. It is obtained by $v_n \to v$ as $n \to \infty$.

Lemma 3.3: Given that (F, P_1, P_2, w) is a BBMMS, $\{v_n\}$ is a sequence in F and $v \in F$. Then, if and only if $P_1(v_n, v) \to 1$ and if $P_2(v_n, v) \to 1$ as $n \to \infty$.

Lemma 3.4: Given that (F, P_1, P_2, w) is a BBMMS and $\{v_n\}$ is a sequence in F, we may assume the following. Then, there is a distinct multiplicative limit point for each multiplicative convergent sequence $\{v_n\}$.

Definition 3.5: Assuming that (F, P_1, P_2, w) is a BBMMS. The sequence $\{v_n\} \in F$ is referred to as MCS if, for each $e > 1$, there exists $N \in \mathbb{N}$ so that $P_1(v_n, v_m) < e$ and $P_2(v_n, v_m) < e$ for all $m, n \geq N$.

Lemma 3.6: Assume that (F, P_1, P_2, w) is a BBMMS and that $\{v_n\}$ is a sequence in F. Then, if and only if $P_1(v_n, v_m) \to 1$ and $P_2(v_n, v_m) \to 1$ as $m, n \to \infty$, then $\{v_n\}$ is a multiplicative Cauchy sequence (MCS).

Proof

Assume $\{v_n\}$ is an MCS. By definition 2.5, for each $e > 1$, there exists $N \in \mathbb{N}$ so that $P_1(v_n, v_m) < e$ and $P_2(v_n, v_m) < e$ for all $m, n \geq N$.

Given $e > 1$, we can choose $e = 1 + \varphi$ where $\varphi > 0$. Since P_1 and P_2 are multiplicative metrics, $P_1(v_n, v_m) < 1 + \varphi$ and $P_2(v_n, v_m) < 1 + \varphi$ for all $m, n \geq N$.

As $\varphi \to 0$, $e \to 1$. Therefore, $P_1(v_n, v_m) \to 1$ and $P_2(v_n, v_m) \to 1$ as $m, n \to \infty$.

Assume $P_1(v_n, v_m) \to 1$ and $P_2(v_n, v_m) \to 1$ as $m, n \to \infty$.

Given $e > 1$, there exists $N_1 N_2 \in \mathbb{N}$ such that for all $m, n \in N_1$, $P_1(v_n, v_m) < e$ and for all $, n \in N_2$, $P_2(v_n, v_m) < e$.

Let $N = \max(N_1, N_2)$. Then for all $m, n \in N$, $P_1(v_n, v_m) < e$ and $P_2(v_n, v_m) < e$.

Therefore, $\{v_n\}$ is an MCS.

The evidence is now complete.

4. MAIN RESULTS

In this section, let's demonstrate a few FP theorems for different multiplicative contractions on BBMMS.

Theorem 4.1: Assuming that (F, P_1, P_2, w) be a complete BBMMS and let multiplicative contractive generalized (MCG) mapping be associated with $G: F \to F$. Let $\{F_i\}$ be an increasing series of subsets of F such that $\sum = \bigcup_{j=1}^{\infty} F_i$, $G(F_i) \subseteq F_{i+1}$, $\forall_i$, and for each i, $P_1(Gd, Gy) \leq P_1(d, y)\beta_i$, $P_2(Gd, Gy) \leq P_2(d, y)\eta_i$, $\forall_{d,y} \in F_i$, where β_i and η_i are positive fixed values so that: $\sum_{n=1}^{\infty} w^n \beta_1 \beta_{2\ldots} \beta_n < \infty$, $\sum_{n=1}^{\infty} w^n \eta_1 \eta_{2\ldots} \eta_n < \infty$. Then, for any fixed point $d_1 \in F\{G_n d_1\}$

multiplicatively converges to a FP in both P_1 and P_2. Moreover, if $\beta_i \in (0,1)$ & $\eta_i \in (0,1)$ for all i, then G has a unique FP (UFP) in F.

Proof:

Let $d_1 \in F$ be arbitrarily chosen. Define the sequence $\{d_n\}$ using the formula $d_{n+1} = G(d_n)$. We aim to demonstrate the Cauchy nature of this sequence in terms of both P_1 and P_2 metrics. Firstly, we demonstrate that $\{d_n\}$ is a Cauchy sequence (CS) in (F, P_1). For $n \geq m \geq 1$,

$$P_1(d_{n+1}, d_n) = P_1(G(d_n), G(d_{n+1})) \leq P_1(d_n, d_{n-1})\beta_n$$

By induction, for any $k \leq n$,

$$P_1(d_{n+1}, d_k) \leq P_1(d_k, d_{k+1})\beta_{k+1} \dots \beta_n$$

Hence, we have

$$P_1(d_{n+1}, d_1) \leq P_1(d_1, d_2)\beta_2 \dots \beta_n$$

The series $\sum_{n=1}^{\infty} w^n \beta_1\beta_{2\dots}\beta_n < \infty$ ensures that the product $\beta_1\beta_{2\dots}\beta_n$ tends to zero as $n \to \infty$. Therefore, $\{d_n\}$ is a CS in (F, P_1).

Similarly, we indicate that $\{d_n\}$ is a CS in (F, P_2). For $n \geq m \geq 1$,

$$P_2(d_{n+1}, d_n) = P_2(G(d_n), G(d_{n+1})) \leq P_2(d_n, d_{n-1})\eta_n$$

By induction, for any $k \leq n$,

$$P_2(d_{n+1}, d_k) \leq P_2(d_k, d_{k+1})\eta_{k+1} \dots \eta_n$$

Hence, we have

$$P_2(d_{n+1}, d_1) \leq P_2(d_1, d_2)\eta_2 \dots \eta_n$$

The series $\sum_{n=1}^{\infty} w^n \eta_1\eta_{2\dots}\eta_n < \infty.$ ensures that the product $\eta_1\eta_{2\dots}\eta_n$ tends to zero as $n \to \infty$. Therefore, $\{d_n\}$ is a CS in (F, P_2).

Since (F, P_1) and (F, P_2) are complete, the sequences $\{d_n\}$ converges to a point $d^* \in F$ in both metrices.

We need to show that d^* is a fixed point for G. Since G is continuous in both metrics,

$$d^* = \lim_{n\to\infty} d_n = \lim_{n\to\infty} G(d_{n-1}) = G(d^*)$$

Suppose $\beta_i \in (0,1)$ & $\eta_i \in (0,1)$ for all i. Let d^* and d^{**} be two FP of G. Then,

$$P_1(d^*, d^{**}) = P_1(G(d^*), G(d^{**})) \leq P_1(d^*, d^{**})\beta_i,$$

where $\beta_i < 1$. This implies that $P_1(d^*, d^{**}) = 0$, so $d^* = d^{**}$ in P_1.

Similarly,

$$P_2(d^*, d^{**}) = P_2(G(d^*), G(d^{**})) \leq P_2(d^*, d^{**})\eta_i,$$

where $\eta_i < 1$. This implies that $P_1(d^*, d^{**}) = 0$, so $d^* = d^{**}$ in P_2.

Thus d^* is the unique FP of G.

Therefore, G has a unique FP in F. This completes the proof.

Corollary 4.2: Assuming that $(F, \widehat{P_1}, \widehat{P_2})$ is a complete bipolar MMS, and $G: F \to F$ has a closed graph. Let $\{F_i\}$ be an expanding sequence of subsets of F so that: $\sum = \bigcup_{j=1}^{\infty} F_i$, $G(F_i) \subseteq F_{i+1}$, $\forall_i$, and for each i, $\widehat{P_1}(Gd, Gy) \leq \widehat{P_1}(d, y)\beta_i$, $\widehat{P_2}(Gd, Gy) \leq \widehat{P_2}(d, y)\eta_i$, $\forall_{d,y} \in F_i$, where β_i and η_i are positive fixed values so that: $\sum_{n=1}^{\infty} w^n \beta_1 \beta_{2\dots} \beta_n < \infty$, $\sum_{n=1}^{\infty} w^n \eta_1 \eta_{2\dots} \eta_n < \infty$. Then, for any FP $d_1 \in F\{G_n d_1\}$ multiplicatively converges (MC) to a FP in both $\widehat{P_1}$ and $\widehat{P_2}$. Moreover, if $\beta_i \in (0,1)$ & $\eta_i \in (0,1)$ for all i, then G has a unique FP in F.

Proof

Let $P_1(d, y) = \exp(\widehat{P_1}(d, y))$ and $P_2(d, y) = \exp(\widehat{P_2}(d, y))$ for all $d, y \in F$. Note that (F, P_1, P_2) is now a complete BBMMS with $w = 1$.

Given that $\widehat{P_1}(Gd, Gy) \leq \widehat{P_1}(d, y)\beta_i$ for all $d, y \in F_i$ and $\beta_i \in (0, \infty)$, we have:

$$P_1(Gd, Gy) = \exp(\widehat{P_1}(Gd, Gy)) \leq \exp(\beta_i \left(\widehat{P_1}(d, y)\right)$$

$$= (exp\ (P_1(Gd, Gy)))^{\beta_i}$$

$$= (P_1(d, y))^{\beta_i}$$

Similarly,

$$P_2(Gd, Gy) = (P_2(d, y))^{\eta_i}$$

Since $P_1(d, y) = \exp(\widehat{P_1}(d, y))$ and $P_2(d, y) = \exp(\widehat{P_2}(d, y))$, convergence in P_1 and P_2 implies convergence in $\widehat{P_1}$ and $\widehat{P_2}$. Thus $\{G_n d_1\}$ converges to a FP in both $\widehat{P_1}$ and $\widehat{P_2}$.

If $\beta_i \in (0,1)$ & $\eta_i \in (0,1)$ for all i, then any two FP d^* and d^{**} must satisfy:

$\widehat{P_1}(d^*, d^{**}) = \widehat{P_1}(G(d^*), G(d^{**})) \leq \exp(\beta_i \left(\widehat{P_1}(d^*, d^{**})\right)$, where $\beta_i < 1$. This implies $\widehat{P_1}(d^*, d^{**}) = 0$, so $d^* = d^{**}$ in $\widehat{P_1}$.

Similarly,

$\widehat{P_2}(d^*, d^{**}) = \widehat{P_2}(G(d^*), G(d^{**})) \leq \exp\left(\eta_i \left(\widehat{P_1}(d^*, d^{**})\right)\right.$, where $\eta_i < 1$. This implies $\widehat{P_2}(d^*, d^{**}) = 0$, so $d^* = d^{**}$ in $\widehat{P_2}$.

Therefore, d^* is the unique FP of G.

This concludes the evidence of corollary 4.2 in the context of a BBMMS.

Example 4.3: Let $F = [1/4, \infty)$. Assume the metrics P_1 and P_2 on F are defined as: $P_1(d, y) = max\left\{\left(d/y\right)^{1/4}, \left(y/d\right)^{1/4}\right\}$, $P_2(d, y) = max\left\{\left(d/y\right)^{1/2}, \left(y/d\right)^{1/2}\right\}$ for all $d, y \in F$. Then (F, P_1, P_2, w) is a complete BBMMS with $w = 1$. Assume that $F_n = [1/4, n]$, and let $\beta_n = \frac{n^2}{(n+1)^2} \in [1/4, 1)$ for $n = 1,2,3$ .... Then, $\sum_{n=1}^{\infty} w^n \beta_1 \beta_{2\ldots} \beta_n < \infty$.

Define $G: F \to F$ by $G(d) = d^{1/4}$ if $d \in F_n$, for $n \in \mathbb{N}$.

For $d, y \in F_n$, we get:

$$P_1(G(d), G(y)) = max\left\{\left(d^{1/4}/y^{1/4}\right)^{1/4}, \left(y^{1/4}/d^{1/4}\right)^{1/4}\right\}$$

$$= max\left\{\left(d/y\right)^{1/16}, \left(y/d\right)^{1/16}\right\} \leq (P_1(d, y))^{1/4} \leq P_1(d, y)\beta_n$$

$$P_2(G(d), G(y)) = max\left\{\left(d^{1/4}/y^{1/4}\right)^{1/2}, \left(y^{1/4}/d^{1/4}\right)^{1/2}\right\}$$

$= max\left\{\left(d/y\right)^{1/8}, \left(y/d\right)^{1/8}\right\} \leq (P_2(d, y))^{1/4} \leq P_2(d, y)\beta_n$, for all for $n \in \mathbb{N}$.

Thus, the hypotheses of Theorem 4.1 are fulfilled. Moreover, the unique FP is 1.

Theorem 4.4: Assume that (F, P_1, P_2, w) is a complete BBMMS, and $G: F \to F$ is an MCG mapping. Let $F_1 \subseteq F_2 \subseteq \cdots$ be subset of F such that $F = \bigcup_{j=1}^{\infty} F_i, G(F_i) \subseteq F_{i+1}$ for all i, and $P_1(Gd, Gy) \leq \left(P_1(Gd, d)P_1(Gy, y)\right)^{\gamma_i}$, $P_2(Gd, Gy) \leq \left(P_2(Gd, d)P_2(Gy, y)\right)^{\gamma_i}$ for all $d, y \in F_i$ and for all i, where $\gamma_i \in (0,1)$ are actual positive fixed values such that $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = {}^{\gamma_i}/{1-\gamma_i}, \forall_i$. Thus, G has a UPF in F. Furthermore, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to the unique FP.

Proof

Fix $d_1 \in F_1$ as well as $d_{n+1} = Gd_n = G_n d_1$ for all n= 1,2, ... Then we have $P_1(G_{n+1}d_1, G_n d_1) \leq (P_1(G_{n+1}d_1, G_n d_1)P_1(G_n d_1, G_{n-1}d_1))^{\gamma_{n+1}} = P_1(G_{n+1}d_1, G_n d_1)^{\gamma_{n+1}} P_1(G_n d_1, G_{n-1}d_1)^{\gamma_{n+1}}$, and similarly,

$$P_2(G_{n+1}d_1, G_n d_1) \leq (P_2(G_{n+1}d_1, G_n d_1)P_2(G_n d_1, G_{n-1}d_1))^{\gamma_{n+1}}$$
$$= P_2(G_{n+1}d_1, G_n d_1)^{\gamma_{n+1}} P_2(G_n d_1, G_{n-1}d_1)^{\gamma_{n+1}}$$

Now we get,

$$P_1(G_{n+1}d_1, G_n d_1) \leq (P_1(G_n d_1, G_{n-1}d_1)^{\frac{\gamma_{n+1}}{1-\gamma_{n+1}}} = P_1(G_n d_1, G_{n-1}d_1)^{a_{n+1}},$$

$$P_2(G_{n+1}d_1, G_n d_1) \leq (P_2(G_n d_1, G_{n-1}d_1)^{\frac{\gamma_{n+1}}{1-\gamma_{n+1}}} = P_2(G_n d_1, G_{n-1}d_1)^{a_{n+1}}.$$

Further, for $1 \leq n < m$, we have

$$P_1(G_m d_1, G_m d_1) \leq P_1(G_m d_1)$$
$$\leq P_1(G_m d_1, G_{m-1}d_1)^{w^{m-1}} P_1(G_{m-1}d_1, G_{m-2}d_1)^{w^{m-2}} \ldots P_1(G_{m+1}d_1, G_m d_1)^{w^n}$$
$$\leq P_1(Gd_1, d_1)^{\sum_{i=n}^{m-1} w^i a_2 a_{3\ldots} a_{i+1}}$$

$$P_2(G_m d_1, G_m d_1) \leq P_2(G_m d_1)$$
$$\leq P_2(G_m d_1, G_{m-1}d_1)^{w^{m-1}} P_2(G_{m-1}d_1, G_{m-2}d_1)^{w^{m-2}} \ldots P_2(G_{m+1}d_1, G_m d_1)^{w^n}$$
$$\leq P_2(Gd_1, d_1)^{\sum_{i=n}^{m-1} w^i a_2 a_{3\ldots} a_{i+1}}$$

Therefore, $P_1(G_m d_1, G_m d_1) \to 1$ and $P_2(G_m d_1, G_m d_1) \to 1$ as $m, n \to \infty$. By lemma 3.6, $\{G_m d_1\}_{m=1}^{\infty}$ is a MCS in F. Let $\{G_m d_1\}_{m=1}^{\infty}$ MC to s^* in F, that is multiplicatively complete. Evoke that $\{G_{m+1} d_1\}_{m=1}^{\infty}$ is too a MCS and its multiplicatively converges to d^* in F. Also, the MCG property of G gives $Gd^* = d^*$. Hence, we gained a FP d^* of G. This process can be prolonged to the overall case: $d_1 \in F_n$, for some n.

If $Gd^* = d^*$ and $Gy^* = y^*$ in G, then let d^* and y^* be in F_n for some n, so we have

$$1 \leq P_1(d^*, y^*) = P_1(Gd^*, Gy^*) \leq (P_1(Gd^*, d^*)P_1(Gy^*, y^*))^{\gamma_n} = 1,$$

$$1 \leq P_2(d^*, y^*) = P_2(Gd^*, Gy^*) \leq (P_2(Gd^*, d^*)P_2(Gy^*, y^*))^{\gamma_n} = 1.$$

Therefore, $d^* = y^*$. Thus, G has a UFP.

Corollary 4.5: Let $(F, \widehat{P_1}, \widehat{P_2})$ is a complete BMS, and $G: F \to F$ has a closed graph. Let $\{F_i\}$ be an increasing sequence of subsets of F such that: $\sum = \bigcup_{j=1}^{\infty} F_i$, $G(F_i) \subseteq F_{i+1}$, $\forall_i$, and for each i, $\widehat{P_1}(Gd, Gy) \leq \gamma_i(\widehat{P_1}(Gd, d)\ ,\ \widehat{P_1}(Gy, y))$, $\widehat{P_2}(Gd, Gy) \leq \gamma_i(\widehat{P_2}(Gd, d)\ ,\ \widehat{P_2}(Gy, y))$, $\forall_{d,y} \in F_i$, where $\gamma_i \in (0,1)$ are actual positive fixed values so that: $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = {\gamma_i}/{1-\gamma_i}$, $\forall_i$. Thus, G has a UPF in F. Furthermore, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to the UFP.

Proof

Let $P_1(d, y) = \exp(\widehat{P_1}(d, y))$ and $P_2(d, y) = \exp(\widehat{P_2}(d, y))$ for all $d, y \in F$. Then (F, P_1, P_2, w) is now a complete BBMMS with $w = 1$. Also, $P_1(Gd, Gy) \leq \left(P_1(Gd, d)P_1(Gy, y)\right)^{\gamma_i}$, $P_2(Gd, Gy) \leq \left(P_2(Gd, d)P_2(Gy, y)\right)^{\gamma_i}$ for all $d, y \in F_i$ and for all i, where $\gamma_i \in (0, \infty)$ are actual positive fixed values so that $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = {\gamma_i}/{1-\gamma_i}$, $\forall_i$. By theorem 4.4 adopted for BBMMS, G has a UPF in F. Furthermore, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to d^*.

Thus, corollary 4.5 follows from theorem 4.4 in the context of BBMMS.

Theorem 4.6: Assume that (F, P_1, P_2, w) is a complete BBMMS, and $G: F \to F$ is an MCG mapping. Let $F_1 \subseteq F_2 \subseteq \cdots$ be subset of F such that $F = \bigcup_{j=1}^{\infty} F_i, G(F_i) \subseteq F_{i+1}$ for all i, and $P_1(Gd, Gy) \leq P_1(d, y)^{\gamma_i} P_2(y, Gd)^{\delta_i}$, $P_2(Gd, Gy) \leq P_2(d, y)^{\gamma_i} P_1(y, Gd)^{\delta_i}$, for all $d, y \in F_i$ and for all i, where $\gamma_i, \delta_i \in (0,1)$ are actual positive fixed values such that $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = \frac{\gamma_i + w\delta_i}{1 - w\delta_i}$, $\forall_i$. Then G has a UPF in F. Moreover, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to the UFP.

Proof

Fix $d_1 \in F_1$ as well as set $d_{n+1} = Gd_n = G_n d_1$ for all n=1,2,3… Then we have $P_1(G_{n+1}d_1, G_n d_1) \leq P_1(G_n d_1, G_{n-1}d_1)^{\gamma_{n+1}} P_2(G_n d_1, G_{n-1}d_1)^{\delta_{n+1}}$ and similarly,

$$P_2(G_{n+1}d_1, G_n d_1) \leq P_2(G_n d_1, G_{n-1}d_1)^{\gamma_{n+1}} P_1(G_n d_1, G_{n-1}d_1)^{\delta_{n+1}}$$

Since $P_1(G_n d_1, G_n d_1) = 1$ and $P_2(G_n d_1, G_n d_1) = 1$ we get,

$P_1(G_{n+1}d_1, G_n d_1) \leq (P_1(G_n d_1, G_{n-1}d_1)^{\gamma_{n+1}} P_2(G_n d_1, G_{n+1}d_1)^{\delta_{n+1}}$,

$P_2(G_{n+1}d_1, G_nd_1) \leq (P_2(G_nd_1, G_{n-1}d_1)^{\gamma_{n+1}} P_1(G_nd_1, G_{n+1}d_1)^{\delta_{n+1}}.$

Now we get,

$P_1(G_{n+1}d_1, G_nd_1) \leq P_1(G_nd_1, G_{n-1}d_1)^{\gamma_{n+1}+w\delta_{n+1}},$

$P_2(G_{n+1}d_1, G_nd_1) \leq P_2(G_nd_1, G_{n-1}d_1)^{\gamma_{n+1}+w\delta_{n+1}}.$

Therefore,

$P_1(G_{n+1}d_1, G_nd_1) \leq P_1(G_nd_1, G_{n-1}d_1)^{a_{n+1}},$

$P_2(G_{n+1}d_1, G_nd_1) \leq P_2(G_nd_1, G_{n-1}d_1)^{a_{n+1}},$

where $a_{n+1} = \frac{\gamma_{n+1}+w\delta_{n+1}}{1-w\delta_{n+1}}$

Proceeding further,

$P_1(G_{n+1}d_1, G_nd_1) \leq P_1(G_1^d, d_1) \prod_{i=2}^{n+1} a_i,$

$P_2(G_{n+1}d_1, G_nd_1) \leq P_2(G_1^d, d_1) \prod_{i=2}^{n+1} a_i,$

Further, for $1 \leq n < m$, we have

$$P_1(G_md_1, G_nd_1) \leq P_1(G_1^d, d_1) \sum_{i=n}^{m-1} \prod_{i=2}^{n+1} a_j$$

$$P_2(G_md_1, G_nd_1) \leq P_2(G_1^d, d_1) \sum_{i=n}^{m-1} \prod_{i=2}^{n+1} a_j$$

Therefore, $P_1(G_md_1, G_nd_1) \to 1$ and $P_2(G_md_1, G_nd_1) \to 1$ as $m, n \to \infty$. By lemma 3.6, $\{G_md_1\}_{m=1}^{\infty}$ is a MCS in F. Let $\{G_md_1\}_{m=1}^{\infty}$ MC to d^* in F, that is multiplicatively complete. Evoke that $\{G_{m+1}d_1\}_{m=1}^{\infty}$ is too a MCS and its MC to d^* in F. Also, the MCG property of G gives $Gd^* = d^*$. Thus, we gained a FP d^* of G. This process can be expanded to the overall case: $d_1 \in F_n$, for some n.

If $Gd^* = d^*$ and $Gy^* = y^*$ in G, then let d^* and y^* be in F_n for some n, so we have

$$1 \leq P_1(d^*, y^*) = P_1(Gd^*, Gy^*) \leq P_1(d^*, y^*)^{\gamma_n} P_2(y^*, Gd^*)^{\delta_n} \leq P_1(d^*, y^*)^{\gamma_n+}\delta_n,$$

$1 \leq P_2(d^*, y^*) = P_2(Gd^*, Gy^*) \leq P_2(d^*, y^*)^{\gamma_n} P_1(y^*, Gd^*)^{\delta_n} \leq P_2(d^*, y^*)^{\gamma_n+}\delta_n.$

Then,

$P_1(d^*, y^*) \leq P_1(d^*, y^*)(\gamma_n + \delta_n)^m,$

$P_2(d^*, y^*) \leq P_2(d^*, y^*)(\gamma_n + \delta_n)^m$, $\forall_i \in N$.

Since $(\gamma_n + \delta_n)^m \to 0$ as $m \to \infty$ because $\gamma_n + \delta_n < 1$, it follows that $P_1(d^*, y^*) = 1$ and $P_2(d^*, y^*) = 1$.

Thus, $d^* = y^*$. Hence, G has a unique FP.

Therefore, the proof is complete and we have established that G has a UFP in the BBMMS (F, P_1, P_2, w), and for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to this UFP.

Remark 4.7: In theorem 4.6, replacing the condition $P(Gd, Gy) \leq P(d, y)^{\gamma_i} P(y, Gd)^{\delta_i}$, for all $d, y \in F_i$ and for all i, where $\gamma_i, \delta_i \in (0,1)$ are actual positive fixed values so that $\gamma_n + \delta_n < 1$, $\forall_i$ as well as $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = \frac{\gamma_i + w\delta_i}{1 - w\delta_i}$, $\forall_i$, with the condition $P(Gd, Gy) \leq P(Gd, Gy)^{\gamma_i} P(y, Gd)^{\delta_i}$, for all $d, y \in F_i$ and for all i, where $\gamma_i, \delta_i \in (0,1)$ are actual positive fixed values so that $\gamma_n + \delta_n < 1$, $\forall_i$ as well as $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = \frac{w\delta_i + 1}{1 - w\delta_i + 1 - \gamma_i + 1}$, $\forall_i$, also guarantees the existence of a unique FP in the BBMMS (F, P_1, P_2, w).

Theorem 4.8: Assume that (F, P_1, P_2, w) is a complete BBMMS, and $G: F \to F$ is an MCG mapping. Let $F_1 \subseteq F_2 \subseteq \cdots$ be subset of F such that $F = \bigcup_{j=1}^{\infty} F_i, G(F_i) \subseteq F_{i+1}$ for all i, and $P_1(Gd, Gy) \leq \left(P_1(Gd, y) P_1(Gy, d)\right)^{\gamma_i}$, $P_2(Gd, Gy) \leq \left(P_2(Gd, y) P_2(Gy, d)\right)^{\gamma_i}$ for all $d, y \in F_i$ and for all i, where $\gamma_i \in (0, {}^1\!/_2)$ are actual positive fixed values so that $\sum_{n=1}^{\infty} w^n a_1 a_{2\ldots} a_n < \infty$, where $a_i = {}^{w\gamma_i}\!/_{1 - w\gamma_i}$, $\forall_i$. Thus, G has a UPF in F. Furthermore, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to the UFP.

Proof

Fix $d_1 \in F_1$ and $d_{n+1} = Gd_n = G_n d_1$ for all n = 1,2, ... Then we have $P_1(G_{n+1} d_1, G_n d_1) \leq (P_1(G_{n+1} d_1, G_n d_1) P_1(G_n d_1, G_{n-1} d_1))^{\gamma_{n+1}}$

Since $P_1(G_{n+1} d_1, G_n d_1) = 1$, we get

$$P_1(G_{n+1} d_1, G_n d_1) \leq (P_1(G_{n+1} d_1, G_n d_1) P_1(G_n d_1, G_{n-1} d_1))^{\gamma_{n+1}}$$
$$= P_1(G_{n+1} d_1, G_n d_1)^{\gamma_{n+1}} P_1(G_n d_1, G_{n-1} d_1))^{\gamma_{n+1}}$$

Now we get,

$$P_1(G_{n+1}d_1, G_nd_1) \le (P_1(G_nd_1, G_{n-1}d_1)^{\frac{\gamma_{n+1}}{1-\gamma_{n+1}}}$$
$$= P_1(G_nd_1, G_{n-1}d_1)^{a_{n+1}},$$
$$\le P_1(Gd_1, d_1)^{\sum_{i=n}^{m-1} w^i a_2 a_{3\dots} a_{i+1}}$$

Further, for $1 \le n < m$, we have

$$P_1(G_md_1, G_md_1) \le P_1(G_md_1)$$
$$\le P_1(G_md_1, G_{m-1}d_1)^{w^{m-1}} P_1(G_{m-1}d_1, G_{m-2}d_1)^{w^{m-2}} \dots P_1(G_{m+1}d_1, G_md_1)^{w^n}$$
$$\le P_1(Gd_1, d_1)^{\sum_{i=n}^{m-1} w^i a_2 a_{3\dots} a_{i+1}}$$

Therefore, $P_1(G_md_1, G_md_1) \to 1$ as $m, n \to \infty$. Similarly, $P_2(G_md_1, G_md_1) \to 1$ as $m, n \to \infty$. By lemma 3.6, $\{G_md_1\}_{m=1}^{\infty}$ is a MCS in F. Let $\{G_md_1\}_{m=1}^{\infty}$ MC to s^* in F, that is multiplicatively complete. Evoke that $\{G_{m+1}d_1\}_{m=1}^{\infty}$ is too a MCS as well as its multiplicatively converges to d^* in F. Also, the MCG property of G gives $Gd^* = d^*$. Thus, we gained a FP d^* of G. This process can be expanded to the overall case: $d_1 \in F_n$, for some n.

If $Gd^* = d^*$ and $Gy^* = y^*$ in G, then let d^* and y^* be in F_n for some n, so we have

$$1 \le P_1(d^*, y^*) = P_1(Gd^*, Gy^*) \le (P_1(Gd^*, y^*)P_1(Gy^*, d^*))^{\gamma_n} = 1,$$

$1 \le P_2(d^*, y^*) = P_2(Gd^*, Gy^*) \le (P_2(Gd^*, y^*)P_2(Gy^*, d^*))^{\gamma_n} = 1.$

Therefore, $d^* = y^*$. Hence, G has a unique FP.

Corollary 4.9: Let $(F, \widehat{P_1}, \widehat{P_2})$ is a complete bipolar metric space, as well as $G: F \to F$ has a closed graph. Let $\{F_i\}$ be an expanding sequence of subsets of F such that: $\sum = \bigcup_{j=1}^{\infty} F_i$, $G(F_i) \subseteq F_{i+1}$, $\forall_i$, and for each i , $\widehat{P_1}(Gd, Gy) \le \gamma_i(\widehat{P_1}(Gd, y)$, $\widehat{P_1}(Gy, d))$, $\widehat{P_2}(Gd, Gy) \le \gamma_i(\widehat{P_2}(Gd, y)$, $\widehat{P_2}(Gy, d))$, $\forall_{d,y} \in F_i$, where $\gamma_i \in (0, \infty)$ are positive fixed values so that: $\sum_{n=1}^{\infty} w^n a_1 a_{2\dots} a_n < \infty$, where $a_i = {}^{\gamma_i}/{1 - \gamma_i}$, $\forall_i$. Then G has a UPF in F. Furthermore, for any fixed $d_1 \in F$, $\{G_nd_1\}$ MC to the UFP.

Proof

Let $\hat{P} = \exp(\widehat{P_1})$ and where $\widehat{P_1}(d, y) = \exp(\widehat{P_1}(d, y)$ for all $d, y \in F$. Then $(F, \hat{P}, w)$ is now a complete BBMMS with $w = 1$. Also, $P_1(Gd, Gy) \le \left(\hat{P}(Gd, y)P_1(Gy, d)\right)^{\gamma_i}$ for all $d, y \in F_i$ and for all i, where $\gamma_i \in (0, \infty)$ are actual positive fixed values so that $\sum_{n=1}^{\infty} w^n a_1 a_{2\dots} a_n < \infty$, where $a_i = {}^{\gamma_i}/_{1-\gamma_i}, \forall_i$. By theorem 4.8 adopted for BBMMS, G has a UPF in F. Moreover, for any fixed $d_1 \in F$, $\{G_n d_1\}$ MC to d^*.

Thus, corollary 4.9 follows from theorem 4.8 in the context of BBMMS.

5. Conclusion

The sufficient and necessary condition for a bipolar b-multiplicative metric space (F, P_1, P_2, w) to be a multiplicative metric space is $w = 1$. By applying the exponential transformation, all the fixed-point outcomes are transformable from bipolar b-multiplicative metric spaces into metric spaces. Three examples it is given in Corollaries 4. 2. 4. 5 and 4. 9. Thus, it is very important to analyze FP of multiplicative contractions in BBMMS.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] M.M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Matem. Palermo 22 (1906), 1–72. https://doi.org/10.1007/bf03018603.

[2] H. Aydi, W. Shatanawi, C. Vetro, On generalized weakly G-metric spaces, Computers Math. Appl. 62 (2011), 4222–4229. https://doi.org/10.1016/j.camwa.2011.10.007.

[3] Z. Mustafa, B. Sims, A new approach to generalized metric spaces. J. Nonlinear Convex Anal. 7 (2006), 289–297.

[4] T. Rasham, P. Agarwal, L.S. Abbasi, et al. A study of some new multivalued fixed point results in a modular like metric space with graph, J. Anal. 30 (2022), 833–844. https://doi.org/10.1007/s41478-021-00372-z.

[5] T. Rasham, M. Nazam, H. Aydi, et al. Hybrid pair of multivalued mappings in modular-like metric spaces and applications, AIMS Math. 7 (2022) 10582–10595. https://doi.org/10.3934/math.2022590.

[6] T. Rasham, A. Shoaib, S. Alshoraify, et al. Study of multivalued fixed point problems for generalized contractions in double controlled dislocated quasi metric type spaces, AIMS Math. 7 (2021), 1058–1073. https://doi.org/10.3934/math.2022063.

[7] M. Gamal, T. Rasham, W. Cholamjiak, et al. New iterative scheme for fixed point results of weakly compatible maps in multiplicative G_M-metric space via various contractions with application, AIMS Math. 7 (2022) 13681–13703. https://doi.org/10.3934/math.2022754.

[8] T. Rasham, M. De La Sen, A novel study for hybrid pair of multivalued dominated mappings in b-multiplicative metric space with applications, J. Inequal. Appl. 2022 (2022) 107. https://doi.org/10.1186/s13660-022-02845-6.

[9] T. Rasham, M. Nazam, H. Aydi, et al. Existence of common fixed points of generalized Δ-implicit locally contractive mappings on closed ball in multiplicative G-metric spaces with applications, Mathematics 10 (2022), 3369. https://doi.org/10.3390/math10183369.

[10] A.E. Bashirov, E.M. Kurpınar, A. Özyapıcı, Multiplicative calculus and its applications, J. Math. Anal. Appl. 337 (2008), 36–48. https://doi.org/10.1016/j.jmaa.2007.03.081.

[11] M. Özavşar, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, J. Eng. Technol. Appl. Sci. 2 (2017), 65-79.

[12] A.A.N. Abdou, Fixed point theorems for generalized contraction mappings in multiplicative metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 2347-2363.

[13] R.P. Agarwal, E. Karapınar, B. Samet, An essential remark on fixed point results on multiplicative metric spaces, Fixed Point Theory Appl. 2016 (2016), 21. https://doi.org/10.1186/s13663-016-0506-7.

[14] F. Gu, Y.J. Cho, Common fixed point results for four maps satisfying ϕ-contractive condition in multiplicative metric spaces, Fixed Point Theory Appl. 2015 (2015), 165. https://doi.org/10.1186/s13663-015-0412-4.

[15] X. He, M. Song, D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, Fixed Point Theory Appl. 2014 (2014), 48. https://doi.org/10.1186/1687-1812-2014-48.

[16] S.M. Kang, P. Kumar, S. Kumar, et al. Common fixed points for compatible mappings and its variants in multiplicative metric spaces, Int. J. Pure Appl. Math. 102 (2015), 383-406. https://doi.org/10.12732/ijpam.v102i2.14.

[17] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostrav. 1 (1993), 5-11. http://dml.cz/dmlcz/120469.

[18] M.U. Ali, T. Kamran, A. Kurdi, Fixed point theorems in b-multiplicative metric spaces, U.P.B. Sci. Bull., Ser. A. 79 (2017), 107-116.

[19] A. Mutlu, U. Gürdal, Bipolar metric spaces and some fixed-point theorems. J. Nonlinear Sci. Appl. 9 (2016), 5362-5373.

[20] G.N.V. Kishore, K.P.R. Rao, H. IsIk, et al. Covarian mappings and coupled fixed point results in bipolar metric spaces, Int. J. Nonlinear Anal. Appl. 12 (2021), 1-15. https://doi.org/10.22075/ijnaa.2021.4650.

[21] G.N.V. Kishore, R.P. Agarwal, B.S. Rao, et al. Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications, Fixed Point Theory Appl 2018 (2018), 21. https://doi.org/10.1186/s13663-018-0646-z.

[22] A. Mutlu, K. Ozkan, U. Gurdal, Coupled fixed point theorems on bipolar metric spaces, Eur. J. Pure Appl. Math. 10 (2017), 655-667.

[23] B.S. Rao, G.N.V. Kishore, Common fixed point theorems in bipolar metric spaces with applications to integral equations, Int. J. Eng. Technol. 7 (2018), 1022-1026.

[24] G. C Moorthy, G. Siva, Bipolar multiplicative metric spaces and fixed point theorems of covariant and contravariant mappings, Math. Anal. Convex Optim. 2 (2021), 39–49. https://doi.org/10.52547/maco.2.1.4.

[25] G. Siva, Fixed points of multiplicative closed graph operators on b-multiplicative metric spaces, Malaya J. Mat. 12 (2024), 21–30. https://doi.org/10.26637/mjm1201/002.