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COMPATIBLE MAPPINGS AND ITS VARIANTS IN GENERALIZED RECTANGULAR METRIC SPACES

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Abstract. In this paper, we present the notion of compatible maps, including type (A) and type (B) compatible maps in generalized rectangular metric spaces which is an extension of the rectangular metric spaces. This article includes illustrative examples from different contexts to further demonstrate our findings. Our results generalize many known result in fixed point theory.

Keywords: fixed point theorem; compatible maps; compatible maps of type(A); compatible maps of type (B); generalized rectangular metric spaces.

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1. Introduction

Fixed point theory, a prominant field in mathematics holds significant importance in the realms of science and mathematics. It has emerged as a fundamental framework in mathematical analysis, providing essential tools for solving a wide range of problems encountered in various scientific and engineering disciplines. The field has witnessed rapid advancements in the past twenty years, primarily driven by its extensive applications in diverse domains, including non-linear analysis, topology and engineering. Consequently, it has garnered substantial attention from researchers world wide. In many abstract metric spaces, the banach contraction

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principle is being thoroughly studied and refined, employing a variety of techniques, ever since its inception. Numerous authors have extended, generalized and enhanced Banach's fixed point theorem in a variety of ways.

In 1976, the fixed point theorem was established by Jungck [1] for commuting maps. However the results were contingent upon the continuity of one of the maps. In metric space, Sessa [2] introduced a weaker version of commutativity for the pair of self maps.

Later, in 1986, Jungck [3] introduced a more generalized commuting mappings known as compatible mappings. He demonstrated that weakly commuting maps are compatible, although the converse may not hold true. The concept of D-metric space was first introduced by Dhage [4] in the year 1992. Dhage presented several results on fixed points for self maps that satisfy contarction conditions for complete bounded D-metric space. Geometrically, D-metric space $D(x,y,z)$ represent the perimeter of the triangle with vertices x, y, $z \in \mathbb{R}^2$.

Subsequently, Mustafa and Sims [5] conducted a study demonstrating that majority of the results pertaining to Dhage's D-metric spaces are invalid and they introduced an enhanced form of the generalized metric space and called it as G-metric spaces.

In 2000, Branciari [6] introduced the concept of rectangular metric spaces, where a three term expression takes the place of the triangle inequality in a metric space. Building upon Branciari's work, Adewale, Olaleru, Olaoluwa and Akewe [7] extend the concept further in 2021 by presenting the idea of generalized rectangular metric spaces. This extension enriches the understanding and application of rectangular metrics in a broader context.

2. Preliminaries

In this section we provide preliminary information and definitions that are utilized throughout this paper.

The concept of G-metric space is defined by Mustafa and Sims [5] as follows:

Definition 2.1. ([5]) Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a mapping that satisfy the following conditions:

(G1.) $G(\xi,\eta,\tau) = 0$ if and only if $\xi = \eta = \tau$, for all $\xi,\eta,\tau \in X$.

(G2.) $G(\xi,\xi,\eta) > 0$, for all $\xi,\eta \in X$ with $\xi \neq \eta$.

(G3.) $G(\xi,\xi,\eta) < G(\xi,\eta,\tau)$, for all $\xi,\eta,\tau \in X$ with $\tau \neq \eta$.

(G4.) $G(\xi,\eta,\tau) = G(\xi,\tau,\eta) = G(\eta,\xi,\tau) = ...$

(G5.) $G(\xi,\eta,\tau) \leq G(\xi,\alpha,\alpha) + G(\alpha,\eta,\tau)$, for all $\alpha,\xi,\eta,\tau \in X$.

Then the function G is referred to as G-metric, and the pair (X,G) is termed a G-metric space.

Branciari [6] introduced the rectangular metric space, which is defined as follows:

Definition 2.2. ([6]) Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a mapping that satisfy the following conditions:

(1.) $d(\xi,\eta) = 0$ if and only if $\xi = \eta$, for all $\xi,\eta \in X$.

(2.) $d(\xi,\eta) = d(\eta,\xi)$, for all $\xi,\eta \in X$.

(3.) $d(\xi,\eta) \leq d(\xi,\alpha) + d(\alpha,\beta) + d(\beta,\eta)$, for all $\xi,\eta \in X$ and all distinct points $\alpha,\beta \in X - \{\xi,\eta\}$.

Then the function d is termed as rectangular metric, while the pair (X,d) is referred to as a rectangular metric space.

In 2021, Adewale, Olaleru, Olaoluwa and Akewe [7] introduced the generalized rectangular metric space, which is defined as follows:

Definition 2.3. ([7]) Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a mapping that satisfy the following conditions:

(1.) $G(\xi,\eta,\tau) = 0$ if and only if $\xi = \eta = \tau$, for all $\xi,\eta,\tau \in X$.

(2.) $G(\xi,\xi,\eta) > 0$, for all $\xi,\eta \in X$ with $\xi \neq \eta$.

(3.) $G(\xi,\eta,\tau) = G(\xi,\tau,\eta) = G(\eta,\xi,\tau) = ...$

(4.) $G(\xi,\eta,\tau) \leq G(\xi,\alpha,\alpha) + G(\alpha,\beta,\beta) + G(\beta,\eta,\eta) + G(\eta,\eta,\tau)$, for all $\xi,\eta,\tau \in X$ and all distinct points $\alpha,\beta \in X - \{\xi,\eta,\tau\}$.

Then the function G is referred to as generalized rectangular metric, and the pair (X,G) is termed a generalized rectangular metric space.

Definition 2.4. ([7]) Let X be a non-empty set and the pair (X, G) be a generalized rectangular metric space. Then the sphere with center $\xi \in X$ and radius $r > 0$ is

$$S_G(\xi,r) = \{\tau \in X : G(\xi,\tau,\tau) < r\}.$$

Definition 2.5. ([7]) Let (X,G) be a generalized rectangular metric space. The sequence $\{\xi_n\} \subset X$ is convergent to τ if it converges to τ in the generalized rectangular metric space.

Definition 2.6. ([7]) Let (X,G) be a generalized rectangular metric space. A sequence $\{\xi_n\}$ in X is considered to converge to a point in X if there exists $\xi \in X$ such that $\lim_{n\to\infty} G(\xi_n,\xi,\xi) = 0$.

Definition 2.7. ([7]) Let (X,G) be a generalized rectangular metric space. A sequence $\{\xi_n\}$ is said to be a cauchy sequence in X if, for each $\varepsilon > 0$ there exist a positive integer $\mathbb{N}$ such that $G(\xi_n,\xi_m,\xi_l) < \varepsilon$ for all $n,m,l \geq \mathbb{N}$ .i.e $G(\xi_n,\xi_m,\xi_l) \to 0$ *as* $n,m,l \to \infty$.

Remark 2.8. ([7]) If $\eta = \tau$ and we set $G(\xi,\eta,\eta) = d(\xi,\eta)$, definition 2.3 reduces to rectangular metric space in [6].

Proposition 2.9. *(*[3]*) Consider a metric space (X,d). Let f, g be two self-mappings on X that are compatible:*

(1) If $f(t) = g(t)$, then $fg(t) = gf(t)$.
(2) Suppose that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.
   (a) If f exhibit continuity at t, then $\lim_{n\to\infty} gf\xi_n = f(t)$.
   (b) If both f and g exhibit continuity at t , then $f(t) = g(t)$ and $fg(t) = gf(t)$.

3. Main Results

There has been significant interest in investigating common fixed points for pairs(or families) of mappings that satisfy the contractive conditions across different spaces. Numerous authors have made significant contributions in this field, resulting in several interesting and elegant results.

The introduction of commutativity by Jungck [1] was a significant break through in the field of fixed point theory. In his notable work, Jungck [1] utilized commutativity to establish common fixed point theorem. Later on, Rani and Ankit [8] introduced commuting maps in generalized rectangular metric spaces.

Jungck [3] established the concept of compatible maps in metric spaces in 1986, stated as follows:

Definition 3.1. ([3]) Two self-mappings f, g on a metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fg\xi_n, gf\xi_n) = 0$, whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

We are now going to present the notion of compatible maps in a generalized rectangular metric space as follows:

Definition 3.2. A pair of self-mappings f, g on a generalized rectangular metric space (X,G) is said to be compatible if $\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 0$ or $\lim_{n\to\infty} G(gf\xi_n, fg\xi_n, fg\xi_n) = 0$, whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

Theorem 3.3. *Let (X,G) be a complete generalized rectangular metric space . Suppose f and g are self-mappings of X that satisfy the following conditions:*

(3.3.1) $f(X) \subseteq g(X)$,

(3.3.2) f or g is continuous,

(3.3.3) $G(f\xi, f\eta, f\tau) \leq \alpha\ G(g\xi, g\eta, g\tau)$ for every $\xi, \eta, \tau \in X$ and $0 \leq \alpha < 1$.

If both f and g are compatible mappings in X, then under the given conditions, they possess a unique common fixed point in X.

Proof. Let ξ_0 be an arbitrary point in X. We can choose a point $\xi_1 \in X$ from (3.3.1) such that $f\xi_0 = g\xi_1$. In general one can choose ξ_{n+1} such that $\eta_n = f\xi_n = g\xi_{n+1}, n = 0, 1, 2, ...$ From (3.3.3), we have

$$G(f\xi_n, f\xi_{n+1}, f\xi_{n+1}) \leq \alpha G(f\xi_{n-1}, f\xi_n, f\xi_n).$$

Continuing in the same way, we have

$$G(f\xi_n, f\xi_{n+1}, f\xi_{n+1}) \leq \alpha^n G(f\xi_0, f\xi_1, f\xi_1).$$

and hence

$$G(\eta_n, \eta_{n+1}, \eta_{n+1}) \leq \alpha^n G(\eta_0, \eta_1, \eta_1). \tag{3.1}$$

Setting $P_n = G(\eta_n, \eta_{n+1}, \eta_{n+1})$ we have

$$P_n \leq \alpha^n P_0. \tag{3.2}$$

By repeated use of (3.2) in definition of generalized rectangular metric space and all distinct points $\eta_{n+1}, \eta_{n+2}, ..., \eta_{m-1}$ with $m > n$, we have the following for all odd $m - n$:

$$\begin{aligned} G(\eta_n, \eta_m, \eta_m) &\leq G(\eta_n, \eta_{n+1}, \eta_{n+1}) + G(\eta_{n+1}, \eta_{n+2}, \eta_{n+2}) + G(\eta_{n+2}, \eta_m, \eta_m), \\ &\leq P_n + P_{n+1} + G(\eta_{n+2}, \eta_m, \eta_m), \\ &\leq \sum_{i=n}^{n+3} P_i + G(\eta_{n+4}, \eta_m, \eta_m), \\ &\leq \sum_{i=n}^{m-1} P_i \leq \sum_{i=n}^{\infty} P_i. \end{aligned} \tag{3.3}$$

In a similarly manner, if m - n $\geq$ 4 is even, we obtain

$$G(\eta_n, \eta_m, \eta_m) \leq \sum_{i=n}^{m-3} P_i + G(\eta_{m-2}, \eta_m, \eta_m). \tag{3.4}$$

From (3.2) and (3.3), we have

$$\begin{aligned} G(\eta_n, \eta_m, \eta_m) &\leq \alpha^n P_0 + \alpha^{n+1} P_0 + \alpha^{n+2} P_0 + ... + \alpha^{m-2} P_0 + \alpha^{m-1} P_0, \\ &\leq \alpha^n [1 + \alpha + \alpha^2 + \alpha^3 + ... + \alpha^{m-n-1}] P_0, \\ &\leq \frac{\alpha^n}{(1-\alpha)} P_0. \end{aligned} \tag{3.5}$$

From (3.2) and (3.4), we have

$$\begin{aligned} G(\eta_n, \eta_m, \eta_m) &\leq \alpha^n (1-\alpha)^{-1} P_0 + G(\eta_{m-2}, \eta_m, \eta_m), \\ &\leq \alpha^n (1-\alpha)^{-1} P_0 + \alpha^{m-2} G(\eta_0, \eta_2, \eta_2). \end{aligned}$$

Taking the limit of $G(\eta_n, \eta_m, \eta_m)$ *as* $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} G(\eta_n, \eta_m, \eta_m) = 0. \tag{3.6}$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$G(\eta_n, \eta_m, \eta_l) \leq G(\eta_n, \eta_{n-1}, \eta_{n-1}) + G(\eta_{n-1}, \eta_{n-2}, \eta_{n-2}) + G(\eta_{n-2}, \eta_m, \eta_m) + G(\eta_m, \eta_m, \eta_l). \tag{3.7}$$

Taking the limit of $G(\eta_n, \eta_m, \eta_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} G(\eta_n, \eta_m, \eta_l) = 0. \tag{3.8}$$

Hence $\{\eta_n\}$ is a G-cauchy sequence. By utilizing the completeness property of (X,G), it follows that there exist $\tau \in X$ such that

$$\lim_{n\to\infty} \eta_n = \tau \ and \lim_{n\to\infty} \eta_n = \lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_{n+1} = \tau.$$

Given that either f or g is continuous, we can assume, without loss of generality, that g is continuous.

$$\lim_{n\to\infty} gf\xi_n = g\tau. \tag{3.9}$$

Furthermore, since f and g exhibit compatiblilty, therefore

$$\lim_{n\to\infty} G(gf\xi_n, fg\xi_n, fg\xi_n) = 0.$$

$$\implies \lim_{n\to\infty} fg\xi_n = g\tau.$$

From (3.3.3), we have

$$G(fg\xi_n, f\xi_n, f\xi_n) \leq \alpha\ G(gg\xi_n, g\xi_n, g\xi_n). \tag{3.10}$$

Proceeding limit as $n \to \infty$, we have $g\tau = \tau$. Again from (3.3.3), we have

$$G(f\xi, f\tau, f\tau) \leq \alpha\ G(g\xi, g\tau, g\tau).$$

Taking limit as $n \to \infty$, we have $f\tau = \tau$. Therefore, we have $\tau = f\tau = g\tau$. Thus τ is a common fixed point of both f and g.

Uniqueness: Let us assume that $\tau_1 (\neq \tau)$ is another fixed point that is common to both the functions f and g. Then $G(\tau, \tau_1, \tau_1) > 0$ and

$$G(\tau, \tau_1, \tau_1) = G(f\tau, f\tau_1, f\tau_1) \leq \alpha G(g\tau, g\tau_1, g\tau_1) = \alpha G(\tau, \tau_1, \tau_1) < G(\tau, \tau_1, \tau_1),$$

a contradiction, therefore $\tau = \tau_1$. Hence uniqueness follows. □

Example 3.4. Let $X = [-1, 1]$ with the generalized rectangular metric $G(\xi, \eta, \tau) = |\xi - \eta| + |\eta - \tau| + |\tau - \xi|$ for all $\xi, \eta, \tau \in X$. Define mappings $f, g : X \to X$ as follows:

$$f(\xi) = \frac{\xi}{4}, and\ \ g(\xi) = \xi.$$

Then (X,G) is a generalized rectangular metric space and $f(X) \subseteq g(X)$. Moreover,

$$G(f\xi, f\eta, f\tau) = \frac{1}{4}\left[|\xi-\eta|+|\eta-\tau|+|\tau-\xi|\right] \leq \frac{1}{2}\left[G(g\xi, g\eta, g\tau)\right].$$

Consider the sequence $\{\xi_n\} = \frac{1}{n}$. Clearly,

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 0.$$

Also,

$$\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = 0.$$

However, maps are compatible at $\xi = 0$ and the only common fixed point of f and g is 0. Thus all the conditions of theorem 3.3 are satisfied.

Proposition 3.5. *Let f and g be compatible mappings from a generalized rectangular metric space (X,G) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = gft = ggt$.*

Proof. Let $\{\xi_n\}$ be a sequence in X defined by $\xi_n = t$, $n = 1,2,3,\ldots$ for some $t \in X$ and $ft = gt$. Then $f\xi_n, g\xi_n \to ft$ as $n \to \infty$. Since both f and g are compatible, we have

$$G(fgt, gft, gft) = \lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 0.$$

Hence we have $fgt = gft$. Therefore, since $ft = gt$, we have $fgt = fft = gft = ggt$. □

Proposition 3.6. *Let (X,G) be a generalized rectangular metric space. Consider f and g, which are compatible self-mappings on X. If $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Then*

(1) If f exhibit continuity at t, then $\lim_{n\to\infty} gf\xi_n = ft$.
(2) If g exhibit continuity at t, then $\lim_{n\to\infty} fg\xi_n = gt$.
(3) If f, g are continuous at t, then $f(t) = g(t)$ and $fg(t) = gf(t)$.

Proof. (1) Suppose that f is continuous at t. Since $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$, we have $\lim_{n\to\infty} fg\xi_n \to ft$. Since f and g are compatible, we have

$$\begin{aligned}\lim_{n\to\infty} G(gf\xi_n, ft, ft) &\leq \lim_{n\to\infty} G(gf\xi_n, fg\xi_n, fg\xi_n) + \lim_{n\to\infty} G(fg\xi_n, ff\xi_n, ff\xi_n) + \lim_{n\to\infty} G(ff\xi_n, ft, ft),\\ &= 0.\end{aligned}$$

Hence, as a result $\lim_{n\to\infty} gf\xi_n = ft$. This concludes the proof.

(2) The proof of $\lim_{n\to\infty} fg\xi_n = gt$ follows by similar argument as in (1).

(3) Assume that both f and g are continuous at t. Since $g\xi_n \to t$ as $n \to \infty$, and f is continuos at t, by (1), we can conclude that $gf\xi_n \to ft$ as $n \to \infty$. Furthermore, since g is also continuous at t, $gf\xi_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limits and so by proposition 3.5 $fgt = gft$. This concludes the proof.

□

3.1. Compatible mappings of type (A).

In this section, motivated by the concept of compatible mappings on metric spaces, we are introducing the notion of compatible mappings of type (A) on generalized rectangular metric spaces. We demonstrate that compatible mappings and type (A) compatible mappings are equivalent under certain conditions.

Definition 3.7. ([9]) A pair of self-mappings f,g on a metric space (X,d) is said to be compatible maps of type (A) if

$$\lim_{n\to\infty} d(gf\xi_n, ff\xi_n) = 0 \textit{ and } \lim_{n\to\infty} d(fg\xi_n, gg\xi_n) = 0,$$

whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

Definition 3.8. A pair of self-mappings f,g on a generalized rectangular metric space (X,G) is said to be compatible maps of type (A) if

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 0 \textit{ and } \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0,$$

whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

Theorem 3.9. *Let (X,G) be a complete generalized rectangular metric space . Suppose f and g are self mappings of X that satisfy the following conditions:*

(3.9.1) $f(X) \subseteq g(X)$,

(3.9.2) f *or* g *is continuous,*

(3.9.3) $G(f\xi, f\eta, f\tau) \leq \alpha G(g\xi, g\eta, g\tau)$ *for every* $\xi,\eta,\tau \in X$ *and* $0 \leq \alpha < 1$.

If f and g are compatible maps of type (A) in X, then under the given conditions, they possess a unique common fixed point in X.

Proof. We deduce that $\{\eta_n\}$ is a cauchy sequence in X from Theorem 3.3. Leveraging the completeness property of (X,G), we can conclude that there exist $\tau \in X$ such that

$$\lim_{n\to\infty} \eta_n = \tau \; and \lim_{n\to\infty} \eta_n = \lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_{n+1} = \tau.$$

Given that either f or g is continuous, for definiteness, we can assume that g is continuous, therefore $\lim_{n\to\infty} gf\xi_n = \lim_{n\to\infty} gg\xi_n = g\tau$. Further f as well as g are compatible maps of type (A), therefore

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 0.$$

$$\implies \lim_{n\to\infty} ff\xi_n = g\tau.$$

Further from (3.9.3), we have

$$G(ff\xi_n, f\xi_n, f\xi_n) \leq \alpha\, G(gf\xi_n, g\xi_n, g\xi_n). \tag{3.11}$$

Proceeding limit as $n \to \infty$, we have $g\tau = \tau$. Again from (3.9.3), we have

$$G(f\xi, f\tau, f\tau) \leq \alpha\, G(g\xi, g\tau, g\tau).$$

Taking limit as $n \to \infty$, we have $f\tau = \tau$. Therefore, we have $\tau = f\tau = g\tau$. Thus τ is a common fixed point of f and g.

Uniqueness: Let us assume that $\tau_1 (\neq \tau)$ is another fixed point that is common to both the functions f and g. Then $G(\tau, \tau_1, \tau_1) > 0$ and

$$G(\tau,\tau_1,\tau_1) = G(f\tau, f\tau_1, f\tau_1) \leq \alpha G(g\tau, g\tau_1, g\tau_1) = \alpha G(\tau,\tau_1,\tau_1) < G(\tau,\tau_1,\tau_1),$$

a contradiction, therefore $\tau = \tau_1$. Hence uniqueness follows. □

Example 3.10. Let $X = [-1,1]$ with the generalized rectangular metric $G(\xi,\eta,\tau) = |\xi-\eta| + |\eta-\tau| + |\tau-\xi|$ for all $\xi,\eta,\tau \in X$. Define mappings $f,g : X \to X$ as follows:

$$f(\xi) = \frac{\xi}{8}, and \;\; g(\xi) = \frac{\xi}{2}.$$

Then (X,G) is a generalized rectangular metric space and $f(X) \subseteq g(X)$. Moreover,

$$G(f\xi, f\eta, f\tau) = \frac{1}{8}\left[|\xi-\eta| + |\eta-\tau| + |\tau-\xi|\right] \leq \frac{1}{2}\left[G(g\xi, g\eta, g\tau)\right]$$

Consider the sequence $\{\xi_n\} = \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0 \textit{ and } \lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 0..$$

Also,

$$\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = 0.$$

However, maps are compatible of type (A) at $\xi = 0$ and the only common fixed point of f and g is 0. Thus, all the requirements specified in theorem 3.9 are satisfied.

Proposition 3.11. *Let f and g be compatible mappings of type (A) from a generalized rectangular metric space (X,G) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = gft = ggt$.*

Proof. Let $\{\xi_n\}$ be a sequence in X defined by $\xi_n = t$, $n = 1,2,3,...$ for some $t \in X$ and $ft = gt$. Then we have $f\xi_n, g\xi_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (A), we have

$$G(fgt, ggt, ggt) = \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0.$$

Similarly, we have $gft = fft$. But$ft = gt$ implies $ggt = gft$. Therefore, $fgt = fft = gft = ggt$. This concludes the proof. □

Proposition 3.12. *Let f and g be compatible mappings of type (A) from a generalized rectangular metric space (X,G) into itself. If $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Then*

(1) If f exhibit continuity at t, then $\lim_{n\to\infty} gg\xi_n = ft$.

(2) If g exhibit continuity at t, then $\lim_{n\to\infty} ff\xi_n = gt$.

(3) If f, g are continuous at t, then $f(t) = g(t)$ and $fg(t) = gf(t)$.

Proof. (1) Assume that f exhibits continuity at t. Since $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$, we have $\lim_{n\to\infty} ff\xi_n, \lim_{n\to\infty} fg\xi_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (A), we have

$$G(ft, gg\xi_n, gg\xi_n) = \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0.$$

Hence, as a result $\lim_{n\to\infty} gg\xi_n = ft$. This concludes the proof.

(2) The proof of $\lim_{n\to\infty} ff\xi_n = gt$ follows by similar argument as in (1).

(3) Let us assume that f and g are both continuous at t. Since $g\xi_n \to t$ as $n \to \infty$, and f is continuos at t, by (1), we can conclude that $gg\xi_n \to ft$ as $n \to \infty$. Moreover, since g is also continuous at t, $gg\xi_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limits and so by proposition 3.11 $fgt = gft$. This concludes the proof.

□

The subsequent propositions demonstrate that definition 3.2 and 3.8 are equivalent under certain conditions.

Proposition 3.13. *Let (X,G) be a generalized rectangular metric space and let f, g: $(X,G) \to (X,G)$ be continuous mappings. If both f and g are compatible, then they are also compatible mappings of type (A).*

Proof. Suppose that f and g both are compatible mappings. Let $\{\xi_n\}$ denote a sequence in X such that, for some $t \in X$, $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$. By (4) of definition 2.3,

$$\begin{aligned}
G(gf\xi_n, ff\xi_n, ff\xi_n) \leq\ & G(gf\xi_n, fg\xi_n, fg\xi_n) + G(fg\xi_n, gg\xi_n, gg\xi_n) \\
& + G(gg\xi_n, ff\xi_n, ff\xi_n) + G(ff\xi_n, ff\xi_n, ff\xi_n), \\
& \leq G(gf\xi_n, fg\xi_n, fg\xi_n) + G(fg\xi_n, gg\xi_n, gg\xi_n) + G(gg\xi_n, ff\xi_n, ff\xi_n).
\end{aligned}$$

Given that f is continuous and f, g are compatible, we obtain

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 0.$$

Similarly, assuming g is continuous, we obtain

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0.$$

□

Proposition 3.14. *Let (X,G) be a generalized rectangular metric space and let f, g: $(X,G) \to (X,G)$ be continuous mappings. If both f and g are compatible mappings of type (A) then they are compatible also.*

Proof. Suppose f and g both are compatible maps of type (A). Let $\{\xi_n\}$ denote a sequence in X such that, for some $t \in X$, $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$. As g is continuous,

$$\lim_{n\to\infty} gf\xi_n = \lim_{n\to\infty} gg\xi_n = gt.$$

By (4) of definition 2.3,

$$G(fg\xi_n, gf\xi_n, gf\xi_n) \le G(fg\xi_n, gg\xi_n, gg\xi_n) + G(gg\xi_n, ff\xi_n, ff\xi_n) + G(ff\xi_n, gf\xi_n, gf\xi_n) + G(gf\xi_n, gf\xi_n, gf\xi_n).$$

Taking $n \to \infty$, given that g is continuous and f, g are compatible mappings of type (A), we obtain

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 0.$$

Similarly, assuming f is continuous, we obtain

$$\lim_{n\to\infty} G(gf\xi_n, fg\xi_n, fg\xi_n) = 0.$$

Therefore f and g are compatible. □

From proposition 3.13 and 3.14, we have:

Proposition 3.15. *Let f and g be continuous mappings from a generalized rectangular metric space (X, G) into itself. Then, f and g are considered compatible if and only if they are compatible of type(A).*

Example 3.16. Let $X = \mathbb{R}$, the set of all real numbers, with the metric $G(\xi, \eta, \tau) = |\xi - \eta| + |\eta - \tau| + |\tau - \xi|$ for all $\xi, \eta, \tau \in X$. Define f, g as follows:

$$f(\xi) = \begin{cases} \frac{1}{\xi} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \quad \textit{and} \quad g(\xi) = \begin{cases} \frac{1}{\xi^2} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Thus, f and g fail to be continuous at $t = 0$. Consider a sequence $\{\xi_n\}$ in X defined by $\xi_n = n^2$, $n = 1, 2, 3, \ldots$. Then for $n \to \infty$, we have

$$f(\xi_n) = \frac{1}{n^2} \to 0,\ g(\xi_n) = \frac{1}{n^4} \to 0$$

and

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = \lim_{n\to\infty} G(n^4, n^4, n^4) = 0.$$

However, the following limit do not exist

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = \lim_{n\to\infty} G(n^4, n^8, n^8) = \infty,$$

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = \lim_{n\to\infty} G(n^4, n^2, n^2) = \infty.$$

Therefore, f and g are compatible, but they are not type (A) compatible.

Example 3.17. Let $X = [0,1]$, endowed with a generalized rectangular metric $G(\xi, \eta, \tau) = |\xi - \eta| + |\eta - \tau| + |\tau - \xi|$ for all $\xi, \eta, \tau \in X$. Define f, $g : [0,1] \to [0,1]$ by:

$$f(\xi) = \begin{cases} \xi & \text{if} \quad \xi \in [0, \frac{1}{2}), \\ 1 & \text{if} \quad \xi \in [\frac{1}{2}, 1], \end{cases} \quad and \quad g(\xi) = \begin{cases} 1 - \xi & \text{if} \quad \xi \in [0, \frac{1}{2}), \\ 1 & \text{if} \quad \xi \in [\frac{1}{2}, 1]. \end{cases}$$

Thus, f and g fail to be continuous at $t = \frac{1}{2}$. Although f and g are compatible mappings of type (A), we assert that they are not compatible. To see this, suppose that $\{\xi_n\} \subseteq [0,1]$ and that $f(\xi_n)$, $g(\xi_n) \to t$. By definition of f and g consider $t = \frac{1}{2}$. Since f and g agree on $[\frac{1}{2}, 1]$, our analysis is limited to the case where $t = \frac{1}{2}$. So we can assume that $\{\xi_n\}$ converges to $\frac{1}{2}$ and $\xi_n < \frac{1}{2}$ for all values of n. Then $g(\xi_n) = 1 - \xi_n$ converges to $\frac{1}{2}$ from the right and $f(\xi_n) = \xi_n$ converges to $\frac{1}{2}$ from the left. Thus, since $1 - \xi_n > \frac{1}{2}$ for all n,

$$fg(\xi_n) = f(1 - \xi_n) = 1$$

and since $\xi_n < \frac{1}{2}$,

$$gf(\xi_n) = g(\xi_n) = 1 - \xi_n \to \frac{1}{2}.$$

Consequently,

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = \lim_{n\to\infty} G(1, \frac{1}{2}, \frac{1}{2}) \to 1.$$

Further, we have

$$G(fg\xi_n, gg\xi_n, gg\xi_n) = G(1,1,1) \to 0.$$

and

$$G(gf\xi_n, ff\xi_n, ff\xi_n) = 2|1-2\xi_n| \to 0.$$

as $\xi_n \to \frac{1}{2}$. Thus, both f and g are compatible mappings of type (A). It's crucial to emphasize that they are not compatible.

3.2. Compatible mappings of type (B). This section introduces the notion of compatible mappings of type (B) and demonstrate how, under certain circumstances, these mappings are equivalent to compatible mappings and compatible mappings of type (A) in generalized rectangular metric space.

Definition 3.18. ([10]) A pair of self-mappings f, g on a metric space (X, d) is said to be compatible maps of type (B) if

$$\lim_{n\to\infty} d(fg\xi_n, gg\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} d(fg\xi_n, ft) + \lim_{n\to\infty} d(ft, ff\xi_n)\right),$$

$$\lim_{n\to\infty} d(gf\xi_n, ff\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} d(gf\xi_n, gt) + \lim_{n\to\infty} d(gt, gg\xi_n)\right),$$

whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

Definition 3.19. A pair of self-mappings f, g on a generalized rectangular metric space (X, G) is said to be compatible maps of type (B) if

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right),$$

and

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(gf\xi_n, gt, gt) + \lim_{n\to\infty} G(gt, gg\xi_n, gg\xi_n)\right),$$

whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$.

Theorem 3.20. *Let (X, G) be a complete generalized rectangular metric space . Suppose f and g are self-mappings of X that satisfy the following conditions:*

(3.20.1) $f(X) \subseteq g(X)$,

(3.20.2) f *or* g *is continuous,*

(3.20.3) $G(f\xi, f\eta, f\tau) \leq \alpha G(g\xi, g\eta, g\tau)$ *for every* $\xi, \eta, \tau \in X$ *and* $0 \leq \alpha < 1$.

If f and g are compatible maps of type (B) in X, then under the given conditions, they possess a unique common fixed point in X.

Proof. From Theorem 3.3, we conclude that $\{\eta_n\}$ is a cauchy sequence in X. Since (X,G) is complete generalized rectangular metric space, therefore there exist $\tau \in X$ such that

$$\lim_{n\to\infty} \eta_n = \tau \; and \lim_{n\to\infty} \eta_n = \lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_{n+1} = \tau.$$

Given that either f or g is continuous, for definiteness, we can assume that g is continuous, therefore $\lim_{n\to\infty} gf\xi_n = \lim_{n\to\infty} gg\xi_n = g\tau$. Further f and g are compatible maps of type (B), therefore

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(gf\xi_n, gt, gt) + \lim_{n\to\infty} G(gt, gg\xi_n, gg\xi_n)\right).$$

$$\implies \lim_{n\to\infty} ff\xi_n = g\tau.$$

Further from (3.20.3), we have

$$G(ff\xi_n, f\xi_n, f\xi_n) \leq \alpha\, G(gf\xi_n, g\xi_n, g\xi_n). \tag{3.12}$$

Proceeding limit as $n \to \infty$, we have $g\tau = \tau$. Again from (2.20.3), we have

$$G(f\xi_n, f\tau, f\tau) \leq \alpha\, G(g\xi_n, g\tau, g\tau).$$

Taking limit as $n \to \infty$, we have $f\tau = \tau$. Therefore, we have $\tau = f\tau = g\tau$. Thus τ is a common fixed point of f and g.

Uniqueness: Let us assume that $\tau_1 (\neq \tau)$ is another fixed point that is common to both the functions f and g. Then $G(\tau, \tau_1, \tau_1) > 0$ and

$$G(\tau, \tau_1, \tau_1) = G(f\tau, f\tau_1, f\tau_1) \leq \alpha G(g\tau, g\tau_1, g\tau_1) = \alpha G(\tau, \tau_1, \tau_1) < G(\tau, \tau_1, \tau_1),$$

a contradiction, therefore $\tau = \tau_1$. Hence uniqueness follows. □

Example 3.21. Let $X = [-1,1]$ with the generalized rectangular metric $G(\xi, \eta, \tau) = |\xi - \eta| + |\eta - \tau| + |\tau - \xi|$ for all $\xi, \eta, \tau \in X$. Define mappings $f, g : X \to X$ as follows:

$$f(\xi) = \frac{\xi}{6}, and \;\; g(\xi) = \xi.$$

Then $f(X) \subseteq g(X)$. Moreover,

$$G(f\xi, f\eta, f\tau) = \frac{1}{6}\left[|\xi-\eta|+|\eta-\tau|+|\tau-\xi|\right] \leq \frac{1}{3}\left[G(g\xi, g\eta, g\tau)\right].$$

Consider the sequence $\{\xi_n\} = \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly,

$$\begin{aligned}
&\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 0,\\
&\frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n, f(0), f(0)) + \lim_{n\to\infty} G(f(0), ff\xi_n, ff\xi_n)\right] \to 0.
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 0,\\
&\frac{1}{2}\left[\lim_{n\to\infty} G(gf\xi_n, g(0), g(0)) + \lim_{n\to\infty} G(g(0), gg\xi_n, gg\xi_n)\right] \to 0.
\end{aligned}$$

Also,

$$\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = 0.$$

However, maps are compatible of type (B) at ξ = 0 and the only common fixed point of f and g is 0. Thus all the conditions of the theorem 3.20 are satisfied.

Proposition 3.22. *Let f, g be compatible maps of type (B) from a generalized rectangular metric space (X,G) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = gft = ggt$.*

Proof. Let $\{\xi_n\}$ be a sequence in X defined by $\xi_n = t$, $n = 1,2,3,...$ for some $t \in X$ and $ft = gt$. Then we have $f\xi_n, g\xi_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned}
G(fgt, ggt, ggt) &= \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n),\\
&\leq \frac{1}{2}\left(\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right),\\
&= 0.
\end{aligned}$$

Hence we have $fgt = gft$. Therefore, since $ft = gt$, we have $fgt = fft = gft = ggt$. This concludes the proof. □

Proposition 3.23. *Let f, g be compatible maps of type (B) from a generalized rectangular metric space (X,G) into itself. If $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Then*

(1) If f exhibit continuity at t, then $\lim_{n\to\infty} gg\xi_n = ft$.

(2) If g exhibit continuity at t, then $\lim_{n\to\infty} ff\xi_n = gt$.

(3) If f, g are continuous at t , then $f(t) = g(t)$ and $fg(t) = gf(t)$.

Proof. (1) Suppose that f exhibit continuity at t. Since $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$, we have $\lim_{n\to\infty} ff\xi_n, \lim_{n\to\infty} fg\xi_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned} G(ft, gg\xi_n, gf\xi_n) &= \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n), \\ &\leq \frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right], \\ &= 0. \end{aligned}$$

Hence, as a result $\lim_{n\to\infty} gg\xi_n = ft$. This concludes the proof.

(2) The proof of $\lim_{n\to\infty} ff\xi_n = gt$ follows by similar argument as in (1).

(3) Assume that both f and g are continuous at t. Since $g\xi_n \to t$ as $n \to \infty$, and f is continuos at t, by (1), we can conclude that $gg\xi_n \to ft$ as $n \to \infty$. Furthermore, since g is also continuous at t, $gg\xi_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limits and so by proposition 3.22 $fgt = gft$. This concludes the proof.

□

Proposition 3.24. *Every pair compatible mappings of type (A) is also compatible mappings of type (B).*

Proof. Suppose f and g are compatible mappings of type (A), then we have

$$0 = \lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right),$$

and

$$0 = \lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(gf\xi_n, gt, gt) + \lim_{n\to\infty} G(gt, gg\xi_n, gg\xi_n)\right),$$

as derived. □

The subsequent propositions demonstrate that definition 3.2 and 3.19 are equivalent under certain conditions.

Proposition 3.25. *Let f and g be continuous mappings on a generalized rectangular metric space (X,G) into itself. If f and g are compatible mappings of type (B) then they are compatible of type(A).*

Proof. Let $\{\xi_n\}$ be a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Given the continuity of f and g, we have

$$\lim_{n\to\infty} ff\xi_n = \lim_{n\to\infty} fg\xi_n = ft,$$

and

$$\lim_{n\to\infty} gg\xi_n = \lim_{n\to\infty} gf\xi_n = gt.$$

By definition, we have

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) \leq \frac{1}{2}\left[(\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right] = 0.$$

and

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) \leq \frac{1}{2}\left[(\lim_{n\to\infty} G(gf\xi_n, gt, gt) + \lim_{n\to\infty} G(gt, gg\xi_n, gg\xi_n)\right] = 0.$$

Therefore, f and g are compatible mappings of type (A). □

Proposition 3.26. *Let f and g be continuous mappings on a generalized rectangular metric space (X,G) into itself. If f and g are compatible mappings of type (B) then they are compatible.*

Proof. Let $\{\xi_n\}$ be a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Given the continuity of f and g, it follows that

$$\lim_{n\to\infty} ff\xi_n = \lim_{n\to\infty} fg\xi_n = ft,$$

and

$$\lim_{n\to\infty} gg\xi_n = \lim_{n\to\infty} gf\xi_n = gt.$$

By definition, we have

$$\begin{aligned} G(fg\xi_n, gf\xi_n, gf\xi_n) \leq{} & G(fg\xi_n, ff\xi_n, ff\xi_n) + G(ff\xi_n, gg\xi_n, gg\xi_n) \\ & + G(gg\xi_n, gf\xi_n, gf\xi_n) + G(gf\xi_n, gf\xi_n, gf\xi_n). \end{aligned}$$

Letting $n \to \infty$ and taking into account f and g are compatible mappings of type (B), we have

$$\begin{aligned}\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) &\leq \lim_{n\to\infty} G(fg\xi_n, ff\xi_n, ff\xi_n) + \lim_{n\to\infty} G(ff\xi_n, gg\xi_n, gg\xi_n) + \lim_{n\to\infty} G(gg\xi_n, gf\xi_n, gf\xi_n),\\ &\leq \frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)) + \lim_{n\to\infty} G(ff\xi_n, gg\xi_n, gg\xi_n)\right],\\ &\leq 0.\end{aligned}$$

Therefore, f and g are compatible. This concludes the proof. □

Proposition 3.27. *Let f and g be continuous mappings on a generalized rectangular metric space (X, G) into itself. If f and g are compatible, then they are compatible of type (B).*

Proof. Let $\{\xi_n\}$ be a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$. Given the continuity of f and g, we have

$$\lim_{n\to\infty} ff\xi_n = \lim_{n\to\infty} fg\xi_n = ft,$$

and

$$\lim_{n\to\infty} gg\xi_n = \lim_{n\to\infty} gf\xi_n = gt,$$

so

$$\lim_{n\to\infty} ff\xi_n = \lim_{n\to\infty} fg\xi_n = \lim_{n\to\infty} gg\xi_n = \lim_{n\to\infty} gf\xi_n.$$

Since f and g are compatible, we have

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 0.$$

Now

$$\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(fg\xi_n, ft, ft) + \lim_{n\to\infty} G(ft, ff\xi_n, ff\xi_n)\right) = 0,$$

and

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) \leq \frac{1}{2}\left(\lim_{n\to\infty} G(gf\xi_n, gt, gt) + \lim_{n\to\infty} G(gt, gg\xi_n, gg\xi_n)\right) = 0.$$

which implies f and g are compatible maps of type (B). □

By unifying proposition 3.24 - 3.27, we have

Proposition 3.28. *Let f and g be continuous mappings on a generalized rectangular metric space (X,G) into itself. Then*

(1) f, g are compatible if and only if they are compatible of type (B).

(2) f, g are compatible of type (A) if and only if they are compatible of type (B).

Proof. (1) One can easily prove it using proposition 3.26 and 3.27.

(2) One can easily prove it using proposition 3.24 and 3.25.

□

Example 3.29. Let $X=\mathbb{R}$, the set of all real numbers, with the metric $G(\xi,\eta,\tau)=|\xi-\eta|+|\eta-\tau|+|\tau-\xi|$ for all $\xi,\eta,\tau\in X$. Define f and g as follows:

$$f(\xi)=\begin{cases}\dfrac{1}{\xi^4} & \text{if} \quad \xi\neq 0,\\ 1 & \text{if} \quad \xi=0,\end{cases} \quad \textit{and} \quad g(\xi)=\begin{cases}\dfrac{1}{\xi^2} & \text{if} \quad \xi\neq 0,\\ 1 & \text{if} \quad \xi=0.\end{cases}$$

Then f and g are not continuous at $t=0$. Consider a sequence $\{\xi_n\}$ in X defined by $\xi_n=n$, $n=1,2,3,\ldots$. Then for $n\to\infty$ we have

$$f(\xi_n)=\frac{1}{n^4}\to 0,\ g(\xi_n)=\frac{1}{n^2}\to 0$$

and

$$\lim_{n\to\infty} G(fg\xi_n,gf\xi_n,gf\xi_n)=\lim_{n\to\infty} G(n^8,n^8,n^8)=0.$$

However, the following limit do not exist

$$\lim_{n\to\infty} G(fg\xi_n,gg\xi_n,gg\xi_n)=\lim_{n\to\infty} G(n^8,n^4,n^4)=\infty,$$
$$\frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n,ft,ft)+\lim_{n\to\infty} G(ft,ff\xi_n,ff\xi_n)\right]=\frac{1}{2}\left[\lim_{n\to\infty} G(n^8,1,1)+\lim_{n\to\infty} G(1,n^{16},n^{16})\right]=\infty.$$

and

$$\lim_{n\to\infty} G(gf\xi_n,ff\xi_n,ff\xi_n)=\lim_{n\to\infty} G(n^8,n^{16},n^{16})=\infty,$$
$$\frac{1}{2}\left[\lim_{n\to\infty} G(gf\xi_n,gt,gt)+\lim_{n\to\infty} G(gt,gg\xi_n,gg\xi_n)\right]=\frac{1}{2}\left[\lim_{n\to\infty} G(n^8,1,1)+\lim_{n\to\infty} G(1,n^4,n^4)\right]=\infty.$$

Therefore f and g are compatible but they are neither compatible type (A) nor compatible type (B).

Example 3.30. Let $X = [0,8]$, endowed with a generalized rectangular metric $G(\xi,\eta,\tau) = |\xi-\eta|+|\eta-\tau|+|\tau-\xi|$ for all $\xi,\eta,\tau \in X$. Define f, $g : [0,8] \to [0,8]$ by:

$$f(\xi) = \begin{cases} \xi & \text{if} \quad \xi \in [0,4), \\ 8 & \text{if} \quad \xi \in [4,8], \end{cases} \quad and \quad g(\xi) = \begin{cases} 8-\xi & \text{if} \quad \xi \in [0,4), \\ 1 & \text{if} \quad \xi \in [4,8]. \end{cases}$$

Then f and g are not continuous at $t = 4$. Now we assert that f and g are not compatible but they are compatible of type (A) and hence compatible of type (B). To see this, suppose that $\{\xi_n\} \subseteq [0,8]$ and that $f(\xi_n)$, $g(\xi_n) \to t$. By definition of f and g consider $t \in [4,8]$. Since f and g agree on $[4,8]$, our analysis is limited to the case where $t = 4$. So we can assume that $\{\xi_n\}$ converges to 4 and $\xi_n < 4$ for all values of n. Then $g(\xi_n) = 8-\xi_n$ converges to 4 from the right and $f(\xi_n) = \xi_n$ converges to 4 from the left. Thus, since $8-\xi_n > 4$ for all n,

$$fg(\xi_n) = f(8-\xi_n) = 8,$$

and since $\xi_n < 4$,

$$gf(\xi_n) = g(\xi_n) = 8-\xi_n \to 4.$$

Consequently,

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 2\lim_{n\to\infty} |8-(8-\xi_n)| \to 8.$$

Further, we have

$$\begin{aligned}
&\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 2\lim_{n\to\infty} |fg\xi_n - gg\xi_n)| \to 0, \\
&\frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n, f(4), f(4)) + \lim_{n\to\infty} G(f(4), ff\xi_n, ff\xi_n)\right], \\
&\frac{1}{2}\left[2\lim_{n\to\infty} |fg\xi_n - 8| + 2\lim_{n\to\infty} |8 - ff\xi_n|\right] \to 4.
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 2\lim_{n\to\infty} |gf\xi_n - ff\xi_n| = 2|8-2\xi_n| \to 0, \\
&\frac{1}{2}\left[\lim_{n\to\infty} G(gf\xi_n, g(4), g(4)) + \lim_{n\to\infty} G(g(4), gg\xi_n, gg\xi_n)\right], \\
&\frac{1}{2}\left[2\lim_{n\to\infty} |gf\xi_n - 8| + 2\lim_{n\to\infty} |8 - gg\xi_n|\right] \to 8.
\end{aligned}$$

as $\xi_n \to 4$. Hence, both f and g are compatible mappings of type (A) and compatible mappings of type (B). However, it is important to note that they are not compatible.

Example 3.31. Let $X = [0,2]$, endowed with a generalized rectangular metric $G(\xi,\eta,\tau) = |\xi-\eta|+|\eta-\tau|+|\tau-\xi|$ for all $\xi,\eta,\tau \in X$. Define f, $g : [0,2] \to [0,2]$ by:

$$f(\xi) = \begin{cases} \frac{1}{2}+\xi & \text{if } \xi \in [0,\frac{1}{2}), \\ 2 & \text{if } \xi = \frac{1}{2}, \\ 1 & \text{if } \xi \in (\frac{1}{2},2], \end{cases} \quad \textit{and} \quad g(\xi) = \begin{cases} \frac{1}{2}-\xi & \text{if } \xi \in [0,\frac{1}{2}), \\ 1 & \text{if } \xi = \frac{1}{2}, \\ 0 & \text{if } \xi \in (\frac{1}{2},2]. \end{cases}$$

Then f and g are not continuous at $t = \frac{1}{2}$. Now we assert that f and g are compatible of type (B) but they are neither compatible nor compatible of type (A). To see this, suppose that $\{\xi_n\} \subseteq$ [0, 2] and that $f(\xi_n)$, $g(\xi_n) \to t = \frac{1}{2}$. By definition of f and g $t \in \frac{1}{2}$. So we can assume that $\{\xi_n\}$ converges to 0. Then $g(\xi_n) = \frac{1}{2} - \xi_n$ converges to $\frac{1}{2}$ from the left and $f(\xi_n) = \frac{1}{2}+\xi_n$ converges to $\frac{1}{2}$ from the right. Also,

$$fg(\xi_n) = f(\frac{1}{2}-\xi_n) = 1-\xi_n,$$

and

$$gf(\xi_n) = g(\frac{1}{2}+\xi_n) = 0.$$

Consequently,

$$\lim_{n\to\infty} G(fg\xi_n, gf\xi_n, gf\xi_n) = 2\lim_{n\to\infty} |(1-\xi_n)-0| \to 2.$$

Further, we have

$$\begin{aligned} &\lim_{n\to\infty} G(fg\xi_n, gg\xi_n, gg\xi_n) = 2\lim_{n\to\infty} |(1-\xi_n)-\xi_n)| \to 2, \\ &\frac{1}{2}\left[\lim_{n\to\infty} G(fg\xi_n, f(\frac{1}{2}), f(\frac{1}{2})) + \lim_{n\to\infty} G(f(\frac{1}{2}), ff\xi_n, ff\xi_n)\right], \\ &\frac{1}{2}\left[2\lim_{n\to\infty}|1+\xi_n|+2\right] \to 2. \end{aligned}$$

and

$$\lim_{n\to\infty} G(gf\xi_n, ff\xi_n, ff\xi_n) = 2\lim_{n\to\infty} |gf\xi_n - ff\xi_n| = 2|0-1| \to 2,$$

$$\frac{1}{2}\left[\lim_{n\to\infty} G(gf\xi_n, g(\frac{1}{2}), g(\frac{1}{2})) + \lim_{n\to\infty} G(g(\frac{1}{2}), gg\xi_n, gg\xi_n)\right],$$
$$\frac{1}{2}\left[2\lim_{n\to\infty}|0-1| + 2\lim_{n\to\infty}|1-\xi_n|\right] \to 2.$$

Hence, both f and g are compatible mappings of type (B) but they are neither compatible nor compatible mappings of type (A).

Conflict of Interests

The authors declare that there is no conflict of interests.

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