Available online at http://scik.org
Adv. Fixed Point Theory, 2024, 14:59
https://doi.org/10.28919/afpt/8864
ISSN: 1927-6303

APPLICATION AND THEOREMS OF COMMON FIXED POINTS IN INTUITIONISTIC MENGER SPACE UNDER CLR PROPERTY

IQBAL M. BATIHA[1,2,*], IQBAL H. JEBRIL[1], LEILA BENAOUA[3], KHALED BERRAH[4], TAKI-EDDINE OUSSAEIF[3], ABDELKRIM ALIOUCHE[3], SHAWKAT ALKHAZALEH[5]

[1]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan
[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE
[3]Department of Mathematics and Computer Sciences, Larbi Ben M'hidi University, Oum El Bouaghi, Algeria
[4]Laboratory of Mathematics, Informatics and Systems (LAMIS), Larbi Tebessi University, Tebessa, Algeria
[5]Department of Mathematics, Jadara University, Irbid, Jordan

Abstract. In this work, we prove some common fixed point theorems for weakly compatible mappings in intuitionistic Menger space using the notion of the common limit range property (CLR property). Our results extend and generalize fixed point theorems on metric spaces, Menger metric spaces. we establish our result by proving the existence and uniqueness of a common solution of Fredhom integral equations. Illustrative examples are also furnished to support our aim results.

Keywords: common fixed point theorems; intuitionistic Menger space; weakly compatible; CLR property.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Among numerous generalizations of metric spaces. We are interested in Menger space which is introduced in 1942 [1] by Menger who replaced for non-negative real numbers as values of

*Corresponding author
E-mail address: i.batiha@zuj.edu.jo
Received August 24, 2024

the metric by distribution functions. This space was expanded the original works of Schweizer and Sklar [2, 3]. Developing the idea of Kramosil and Michalek [4], George and Veeramani [5] introduced fuzzy metric spaces which are very similar that of Menger space. After that, Park [6] innovated the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces. Kutukcu et al [7] generated the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [1]. Further they presented the notion of Cauchy sequences and establish a necessary and sufficient condition for an intuitionistic Menger space to be complete. Due to Jungck [8] which introduced the notion of compatible mappings in metric spaces which allowed him to prove a common fixed point theorem. Mishra [9] extended the notion of compatibility to probabilistic metric spaces. Jungck and Rhoades [10] generalized the weakened notion named weakly compatible mappings.
Fixed point theorems are the useful instruments in many applied areas such as mathematical economics, non-cooperative game theory, dynamic optimization and stochastic games, functional analysis, and variation calculus [11, 12, 13, 14, 15]. Therefore, extending the fixed point theorems to intuitionistic Menger inner product spaces can be applicable in the other fields as integral equations.

Sintunavarat and Kuman [16] introduced a new concept of (CLR property). The importance of CLR property ensures that one does not require the closedness of range subspaces. The intent of this paper is to establish the notion of (CLR) property in intuitionistic Menger space and prove a common fixed point theorem for weakly compatible mappings using this property. In this work, we use the notion of the common limit range property (CLR property) in intuitionistic Menger space for prove a common fixed point theorem for four mappings. we have asserted the existence and uniqueness of a common solution of Fredhom integral equations by applying our result. Demonstrative examples are also furnished to support our aim results.

2. Preliminaries

Definition 1. [17] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *is satisfying conditions:*

a) $*$ *is an commutative and associative,*

b) $*$ *is continuous,*

c) $a * 1 = a$*, for all* $a \in [0,1]$,

d) $a * b \leqslant c * d$ *wherever* $a \leqslant c$, $b \leqslant d$ *and* $a,b,c,d \in [0,1]$.

Examples of t-norm are $a * b = \min\{a,b\}$ and $a * b = ab$.

Definition 2. *A binary operation* $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if* $\diamondsuit$ *is satisfying conditions:*

a) $\diamondsuit$ *is an commutative and associative;*

b) $\diamondsuit$ *is continuous;*

c) $a \diamondsuit 0 = a$*, for all* $a \in [0,1]$,

d) $a \diamondsuit b \geq c \diamondsuit d$ *wherever* $a \geq c$, $b \geq d$ *and* $a,b,c,d \in [0,1]$.

Remark 1. *The concept of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the ascionatic sketlons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger* [1] *in his study of statistical metric spaces.*

Definition 3. *A distance distribution function is a function* $F : \mathbb{R} \to \mathbb{R}^+$ *which is left continuous on* $\mathbb{R}$, *non-decreasing and* $\inf_{t \in \mathbb{R}} F(t) = 0$, $\sup_{t \in \mathbb{R}} F(t) = 1$. *We will denote by D the family of all distance distribution functions and by H a spatial of D defined by* $H(t) = \begin{cases} 0, & \text{if } t \leqslant 0 \\ 1, & \text{if } t > 0 \end{cases}$.

If X is a non-empty set, $F : X \times X \to D$ *is called a probabilistic distance on X and* $F(x,y)$ *is usually denote by* $F_{x,y}$.

Definition 4. *A non-distance distribution function is a function* $L : \mathbb{R} \to \mathbb{R}^+$ *which is right continuous on R, non-increasing and* $\inf_{t \in \mathbb{R}} L(t) = 1$, $\sup_{t \in \mathbb{R}} L(t) = 0$. *We will denote by E the family of all distance distribution functions and by G a spatial of E defined by* $G(t) = \begin{cases} 1, & \text{if } t \leqslant 0 \\ 0, & \text{if } t > 0 \end{cases}$.

If X is a non-empty set, $L : X \times X \to E$ *is called a probabilistic distance on X and* $L(x,y)$ *is usually denote by* $L_{x,y}$.

Definition 5. [17] *A 5-tuple* $(X, F, L, *, \diamondsuit)$ *is said to be an intuitionistic Menger metric space if X is an arbitrary set,* $*$ *is a continuous t-norm,* $\diamondsuit$ *is a continuous t-conorm, F is a probabilistic* *distance and L is a probabilistic non-distance on X satisfying the following conditions. For all $x,y,z \in X$ and $t,s \geqslant 0$,*

1) $F_{x,y}(t) + L_{x,y}(t) \leqslant 1$,

2) $F_{x,y}(0) = 0$,

3) $F_{x,y}(t) = H(t)$ if and only if $x = y$,

4) $F_{x,y}(t) = F_{y,x}(t)$

5) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then, $F_{x,z}(t+s) = 1$,

6) $F_{x,y}(t) * F_{y,z}(s) \leqslant F_{x,z}(t+s)$,

7) $L_{x,y}(0) = 1$,

8) $L_{x,y}(t) = G(t)$, if and only if, $x = y$,

9) $L_{x,y}(t) = L_{y,x}(t)$,

10) If $L_{x,y}(t) = 0$ and $L_{y,z}(s) = 0$, then, $L_{x,z}(t+s) = 0$,

11) $L_{x,y}(t) \Diamond L_{y,z}(s) \leqslant L_{x,z}(t+s)$.

The functions $F_{x,y}(t)$ and $L_{x,y}(t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2. *Every Menger space $(X,F,*)$ is intuitionistic Menger space of the form $(X,F,1-F,*,\Diamond)$ such that $*$ t-norm and $\Diamond$ t-conorm are associated $[11]$, that is $x\Diamond y = 1-(1-x)*(1-y)$ for any $x,y \in X$.*

Definition 6. *Let $(X,F,L,*,\Diamond)$ be an intuitionistic Menger space on X.*

(a) $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to a point to $x \in X$, if

$$\lim_{n\to+\infty} F_{x_n,x}(t) = 1,\ \lim_{n\to+\infty} L_{x_n,x}(t) = 0, \text{ for } t > 0.$$

(b) $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence if for all $t > 0$ and $p > 0$:

$$\lim_{n\to+\infty} F_{x_{n+p},x_n}(t) = 1,\ \lim_{n\to+\infty} L_{x_{n+p},x_n}(t) = 0.$$

(c) X is complete if every Cauchy sequence converges in X.

In this section, X is considered to be the intuitionistic Menger space with the following condition

$$\lim_{t\to+\infty} F_{x,y}(t) = 1,\ \lim_{t\to+\infty} L_{x,y}(t) = 0, \text{ for all } x,y \in X \text{ and } t > 0.$$

Lemma 1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X with condition (2.1). If there exist a number $k \in (0,1)$ such that*

$$F_{x_{n+2},x_{n+1}}(kt) \geq F_{x_{n+1},x_n}(t),\ L_{x_{n+2},x_{n+1}}(kt) \leq L_{x_{n+1},x_n}(t),\ \text{for } x,y \in X,\ t>0 \text{ and } n=1,2,...$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Definition 7. *Let A,B be a self-mappings in X, A and B are said to be compatible if*

$$\lim_{n\to+\infty} F_{ABx_n,BAx_n}(t)=1,\ \lim_{n\to+\infty} L_{ABx_n,BAx_n}(t)=0, \text{ for all } t>0,$$

whenever $\{x_n\} \subset X$ such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Bx_n = x, \text{ for some } x\in X.$$

Definition 8. *Two self-mappings A and B of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Ax = Bx$ some $x \in X$, then, $ABx = BAx$.*

Remark 3. *Two compatible self-mapping are weakly compatible, however the inverse is not true in general, therefore the concept of weak compatibility is more general than that of compatibility.*

Definition 9. *A pair of self mappings A and B of a intuitionistic Menger space $(X,F,L,*,\diamondsuit)$ is said to satisfy the (E.A) property, if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Bx_n = z.$$

for some $z \in X$.

Definition 10. [16] *A pair of self-mappings A and S of a intuitionistic Menger space $(X,F,L,*,\diamondsuit)$ is said to satisfy the common limit in the range of S property (briefly: CLR_S property) , if there exists a sequence $\{x_n\}$ in X, such that*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z,$$

where $z \in S(X)$

For clarification, we give some examples upon CLR_S property.

Example 1. *Let $(X,F,L,*,\Diamond)$ be a intuitionistic Menger space, where $X=[0,\infty)$, and*

$$F_{x,y}(t)=\frac{t}{t+|x-y|};\quad L_{x,y}(t)=\frac{|x-y|}{t+|x-y|},$$

for all $x,y\in X,\ t>0$. Define A and S on X by $Ax=x+4,\ Sx=5x$. Let a sequence $\left\{x_n=1+\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X; we have

$$\lim_{n\to+\infty}Ax_n=\lim_{n\to+\infty}Sx_n=5.$$

where $5\in S(X)$ and $t>0$. Therefore the mappings A,S satisfy the CLR_S property

From the example 1, it is evident that a pair (A,S) satisfying the property (E,A) along with closedness the subspace $S(X)$ always enjoys the CLR_S property. On the work of Imdad et al. [18] which define the CLR_{ST} property as follows

Definition 11. *Two pairs (A,S) and (B,T) of self-mappings of a intuitionistic Menger space $(X,F,L,*,\Diamond)$ are said to satisfy the common limit range property with respect to mappings S and T (briefly, CLR_{ST} property), if there exists a sequences $\{x_n\}$ and $\{y_n\}$ in X, such that*

$$\lim_{n\to+\infty}Ax_n=\lim_{n\to+\infty}Sx_n=\lim_{n\to+\infty}By_n=\lim_{n\to+\infty}Ty_n=z,$$

where $z\in S(X)\cap T(X)$, $t>0$.

3. Main Result

Let φ be a set of all increasing continuous functions, $\varphi:(0,1]\to(0,1]$, where, $\varphi(t)>t$ for every $t\in(0,1)$ with ψ be a set of all decreasing continuous functions, $\psi:(0,1]\to(0,1]$, where $\psi(t)<t$ for every $t\in(0,1)$.

we start this section by proving the following Lemma.

Lemma 2. *Let $(X,F,L,*,\Diamond)$ be a intuitionistic Menger space and A,B,S and T are self mappings of X satisfying the following conditions:*

C_1) The pair (A,S) satisfies the (CLR_S) property (or The pair (B,T) satisfies the (CLR_T) property).

C_2) $A(X)\subseteq T(X)$; (Or $B(X)\subseteq S(X)$).

C_3) $T(X)$, (Or $S(X)$) are a closed subset of X.

C_4) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges (or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges).

C_5)

$$F_{Ax,By}(t) \geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t);\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{F_{Ax,Sx}(t_1),F_{By,Ty}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{F_{Sx,By}(t_3),F_{Ax,Ty}(t_4)\right\}\end{array}\right\}\right)$$
$$L_{Ax,By}(t) \leqslant \psi\left(\max\left\{\begin{array}{c} L_{Sx,Ty}(t);\ \inf_{t_1+t_2=\frac{2}{k}t}\max\left\{L_{Ax,Sx}(t_1),L_{By,Ty}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t}\min\left\{L_{Sx,By}(t_3),L_{Ax,Ty}(t_4)\right\}\end{array}\right\}\right)$$

for all $x,y\in X$, $t>0$ and for some $1<k<2$. Therefore the pairs (A,S) and (B,T) share the CLR_{ST} property.

Proof. Assume that (A,S) satisfies (CLR$_S$) property. So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z, \text{ where } z\in S(X).$$

Using C_1 where $A(X)\subseteq T(X)$ (wherein $T(X)$ is a closed subset of X), for each $\{x_n\}$ in X, there corresponds a sequence $\{y_n\}$ in X, we have $Ax_n = Ty_n$, therefore

$$\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} Ax_n = z, \text{ where } z\in S(X)\cap T(X).$$

Thus in all, we have $Ax_n\to z$, $Sx_n\to z$ and $Ty_n\to z$. Now we show that $By_n\to z$.

Let $\lim_{n\to+\infty} F_{By_n,l}(t_0)=1$ and $\lim_{n\to+\infty} L_{By_n,l}(t_0)=0$, we assert that $(l=z)$. On the contrary, suppose that $l\neq z$, so, there exists $t_0>0$, where

$$(1)\qquad \begin{aligned} F_{z,l}\left(\frac{2}{k}t_0\right) &> F_{z,l}(t_0), \\ L_{z,l}\left(\frac{2}{k}t_0\right) &< L_{z,l}(t_0). \end{aligned}$$

To support the claim, let it be untrue, then we have

$$\begin{aligned} F_{z,l}\left(\frac{2}{k}t\right) &> F_{z,l}(t), \\ L_{z,l}\left(\frac{2}{k}t\right) &< L_{z,l}(t). \end{aligned}$$

Repeatedly using this equality, we obtain

$$\begin{aligned} F_{z,l}(t) &= F_{z,l}\left(\frac{2}{k}t\right) = ... = F_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 1, \\ L_{z,l}(t) &= L_{z,l}\left(\frac{2}{k}t\right) = ... = L_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 0, \end{aligned}$$

as $n \to +\infty$, we get $F_{z,l}(t) = 1$, $L_{z,l}(t) = 0$, for all $t > 0$, which contradicts $l \neq z$, hence (3.2) is hold.

Using (C_5) with $x = x_n$, $y = y_n$, for some $t_0 > 0$, we have

$$\begin{aligned} F_{Ax_n,By_n}(t_0) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Sx_n,Ty_n}(t_0);\ \sup_{t_1+t_2=\frac{2}{k}t_0}\min\left\{F_{Ax_n,Sx_n}(t_1), F_{By_n,Ty_n}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t_0}\max\left\{F_{Sx_n,By_n}(t_3), F_{Ax_n,Ty_n}(t_4)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Sx_n,Ty_n}(t_0);\ \min\left\{F_{Ax_n,Sx_n}(\varepsilon), F_{By_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}; \\ \max\left\{F_{Sx_n,By_n}(\varepsilon), F_{Ax_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\} \end{array}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} L_{Ax_n,By_n}(t_0) &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Sx_n,Ty_n}(t_0);\ \inf_{t_1+t_2=\frac{2}{k}t_0}\max\left\{L_{Ax_n,Sx_n}(t_1), L_{By_n,Ty_n}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t_0}\min\left\{L_{Sx_n,By_n}(t_3), L_{Ax_n,Ty_n}(t_4)\right\} \end{array}\right\}\right) \\ &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Sx_n,Ty_n}(t_0);\ \max\left\{L_{Ax_n,Sx_n}(\varepsilon), L_{By_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}; \\ \min\left\{L_{Sx_n,By_n}(\varepsilon), L_{Ax_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\} \end{array}\right\}\right). \end{aligned}$$

Suppose that $\lim_{n\to+\infty} F_{By_n,l}(t_0) = 1$ and $\lim_{n\to+\infty} L_{By_n,l}(t_0) = 0$, where $(l \neq z)$, for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$, $\forall t_0 > 0$.

passing to the limit as $n \to +\infty$, we get

$$\begin{aligned} F_{z,l}(t_0) &\geqslant \varphi\left(\min\left\{F_{z,z}(t_0), F_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}\right) \\ &\geqslant \varphi\left(F_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) > F_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), \\ L_{z,l}(t_0) &\leqslant \psi\left(\max\left\{L_{z,z}(t_0), L_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}\right) \\ &\leq \psi\left(L_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) < L_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), \end{aligned}$$

Putting $\varepsilon = 0$, we get

$$\begin{aligned} F_{z,l}(t_0) &> F_{z,l}\left(\frac{2}{k}t_0\right), \\ L_{z,l}(t_0) &< L_{z,l}\left(\frac{2}{k}t_0\right), \end{aligned}$$

which contradicts to (3.1), hence $z = l$. Therefore, the pairs (A,S) and (B,T) share the CLR$_{ST}$ property. □

Theorem 1. *Let A,B,S and T are self mappings of a intuitionistic Menger space $(X,F,L,*,\diamondsuit)$ satisfying the condition (C_5) of Lemma 2. If the pairs (A,S) and (B,T) share the CLR$_{ST}$ property, then (A,S) and (B,T) have a coincidence point each. Moreover, if the pairs (A,S) and (B,T) are weakly compatible. Then the mappings A,B,S and T have a unique common fixed point.*

Proof. Since both the pairs (A,S) and (B,T) share the CLR$_{ST}$ property, there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X, such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. Hence there exist a points $u, v \in X$, where $Su = z$ and $Tv = z$. We show that $Au = Su$.

Putting $x = u$ and $y = y_n$ in (C_5), we have, for some $t_0 > 0$,

$$\begin{aligned} F_{Au,By_n}(t_0) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Su,Ty_n}(t_0);\ \sup_{t_1+t_2=\frac{2}{k}t_0}\min\left\{F_{Au,Su}(t_1), F_{By_n,Ty_n}(t_2)\right\};\\ \sup_{t_3+t_4=\frac{2}{k}t_0}\max\left\{F_{Su,By_n}(t_3), F_{Au,Ty_n}(t_4)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Su,Ty_n}(t_0);\ \min\left\{F_{Au,Su}\left(\frac{2}{k}t_0-\varepsilon\right), F_{By_n,Ty_n}(\varepsilon)\right\};\\ \max\left\{F_{Su,By_n}(\varepsilon), F_{Au,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\} \end{array}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} L_{Au,By_n}(t_0) &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Su,Ty_n}(t_0);\ \inf_{t_1+t_2=\frac{2}{k}t_0}\max\left\{L_{Au,Su}(t_1), L_{By_n,Ty_n}(t_2)\right\};\\ \inf_{t_3+t_4=\frac{2}{k}t_0}\min\left\{L_{Su,By_n}(t_3), L_{Au,Ty_n}(t_4)\right\} \end{array}\right\}\right) \\ &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Su,Ty_n}(t_0);\ \max\left\{L_{Au,Su}\left(\frac{2}{k}t_0-\varepsilon\right), L_{By_n,Ty_n}(\varepsilon)\right\};\\ \min\left\{L_{Su,By_n}(\varepsilon), L_{Au,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\} \end{array}\right\}\right), \end{aligned}$$

Passing to the limit as $n \to +\infty$, we get $\forall \varepsilon \in \left(0, \frac{2}{k}t_0\right)$,

$$\begin{aligned} F_{Au,z}(t_0) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{z,z}(t_0), \min\left\{F_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right), F_{z,z}(\varepsilon)\right\} \\ \max\left\{F_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right), F_{z,z}(\varepsilon)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(F_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) > F_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right) \end{aligned}$$

and

$$\begin{aligned} L_{Au,z}(t_0) &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{z,z}(t_0), \max\left\{L_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right), L_{z,z}(\varepsilon)\right\} \\ \min\left\{L_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right), L_{z,z}(\varepsilon)\right\} \end{array}\right\}\right) \\ &\leqslant \psi\left(L_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) > L_{Au,z}\left(\frac{2}{k}t_0-\varepsilon\right), \end{aligned}$$

putting $\varepsilon = 0$, we have

$$\begin{aligned} F_{Au,z}(t_0) &> F_{Au,z}\left(\frac{2}{k}t_0\right), \\ L_{Au,z}(t_0) &< L_{Au,z}\left(\frac{2}{k}t_0\right), \end{aligned}$$

which contradicts to (3.1), so $Au = z$. Hence $Au = Su = z$, which implies that u is a coincidence point of the pair (A,S).

Now, we show that $Bv = Tv$. Putting $x = u$ and $y = v$ in (C_5), $\forall \varepsilon \in \left(0, \frac{2}{k}t_0\right)$, we have, fore some $t_0 > 0$,

$$\begin{aligned} F_{Au,Bv}(t_0) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Su,Tv}(t_0);\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Au,Su}(t_1), F_{Bv,Tv}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Su,Bv}(t_3), F_{Au,Tv}(t_4)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{z,z}(t_0), \min\left\{F_{z,z}(\varepsilon), F_{Bv,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right\} \\ \max\left\{F_{z,Bv}\left(\frac{2}{k}t_0-\varepsilon\right), F_{z,z}(\varepsilon)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(F_{Bv,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) \\ &> F_{Bv,z}\left(\frac{2}{k}t_0-\varepsilon\right) \end{aligned}$$

and

$$L_{Au,Bv}(t_0) \leqslant \psi\left(\max\left\{\begin{array}{c} L_{Su,Tv}(t_0);\ \inf_{t_1+t_2=\frac{2}{k}t_0} \max\left\{L_{Au,Su}(t_1), L_{Bv,Tv}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t_0} \min\left\{L_{Su,Bv}(t_3), L_{Au,Tv}(t_4)\right\} \end{array}\right\}\right)$$

$$\begin{aligned} &\leqslant \psi\left(L_{Bv,z}\left(\frac{2}{k}t_0-\varepsilon\right)\right) \\ &< L_{Bv,z}\left(\frac{2}{k}t_0-\varepsilon\right), \end{aligned}$$

Putting $\varepsilon=0$, we get

$$\begin{aligned} F_{z,Bv}(t_0) &> F_{z,Bv}\left(\frac{2}{k}t_0\right), \\ L_{z,Bv}(t_0) &< L_{z,Bv}\left(\frac{2}{k}t_0\right). \end{aligned}$$

Which is a contradiction. Therefore, $Bv=z$, hence $Bv=Tv=z$, which implies that v is a coincidence point of the pair (B,T).

Since (A,S) is weakly compatible and $Au=Su$, hence $Az=ASu=SAu=Sz$. we prove that z is a common fixed point of A and S.

Using (C_5), $\forall\varepsilon\in\left(0,\frac{2}{k}t_0\right)$, with $x=z$ and $y=v$, we obtain

$$\begin{aligned} F_{Az,Bv}(t) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Sz,Tv}(t);\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{F_{Az,Sz}(t_1),F_{Bv,Tv}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{F_{Sz,Bv}(t_3),F_{Az,Tv}(t_4)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Az,z}(t),\min\left\{F_{Az,Az}(\varepsilon),F_{z,z}\left(\frac{2}{k}t-\varepsilon\right)\right\} \\ \max\left\{F_{Az,z}(\varepsilon),F_{Az,z}\left(\frac{2}{k}t-\varepsilon\right)\right\} \end{array}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} L_{Az,Bv}(t) &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Sz,Tv}(t);\ \inf_{t_1+t_2=\frac{2}{k}t}\max\left\{L_{Az,Sz}(t_1),L_{Bv,Tv}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t}\min\left\{L_{Sz,Bv}(t_3),L_{Az,Tv}(t_4)\right\} \end{array}\right\}\right) \\ &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Az,z}(t),\max\left\{L_{Az,Az}(\varepsilon),L_{z,z}\left(\frac{2}{k}t-\varepsilon\right)\right\} \\ \min\left\{L_{Az,z}(\varepsilon),L_{Az,z}\left(\frac{2}{k}t-\varepsilon\right)\right\} \end{array}\right\}\right). \end{aligned}$$

Putting $\varepsilon=0$, we get

$$\begin{aligned} F_{Az,z}(t) &\geqslant \varphi\left(\min\left\{F_{Az,z}(t),F_{Az,z}\left(\frac{2}{k}t\right)\right\}\right)\geqslant\varphi\left(F_{Az,z}(t)\right)\geqslant F_{Az,z}(t), \\ L_{Az,z}(t) &\leqslant \psi\left(\max\left\{L_{Az,z}(\varepsilon),L_{Az,z}\left(\frac{2}{k}t\right)\right\}\right)\leqslant\psi\left(L_{Az,z}(t)\right)\leqslant L_{Az,z}(t). \end{aligned}$$

Which is a contradiction. Hence $Az=z=Sz$, it follows that z is a common fixed point of A and S.

Also the pair (B,T) is weakly compatible, therefore $Bz = BTv = TBv = Tz$.

Now we prove that z is a common fixed point of B and T. Putting $x = u$ and $y = z$ in (C_5), $\forall\varepsilon \in \left(0, \frac{2}{k}t_0\right)$, we have

$$\begin{aligned} F_{Au,Bz}(t) &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Su,Tz}(t);\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{F_{Au,Su}(t_1), F_{Bz,Tz}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{F_{Su,Bz}(t_3), F_{Au,Tz}(t_4)\right\} \end{array}\right\}\right) \\ &\geqslant \varphi\left(\min\left\{\begin{array}{c} F_{z,Bz}(t), \min\left\{F_{z,z}(\varepsilon), F_{Bz,Bz}\left(\frac{2}{k}t-\varepsilon\right)\right\} \\ \max\left\{F_{z,Bz}(\varepsilon), F_{z,Bz}\left(\frac{2}{k}t-\varepsilon\right)\right\} \end{array}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} L_{Au,Bz}(t) &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{Su,Tz}(t);\ \inf_{t_1+t_2=\frac{2}{k}t}\max\left\{L_{Au,Su}(t_1), L_{Bz,Tz}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t}\min\left\{L_{Su,Bz}(t_3), L_{Au,Tz}(t_4)\right\} \end{array}\right\}\right) \\ &\leqslant \psi\left(\max\left\{\begin{array}{c} L_{z,Bz}(t), \max\left\{L_{z,z}(\varepsilon), L_{Bz,Bz}\left(\frac{2}{k}t-\varepsilon\right)\right\} \\ \min\left\{L_{z,Bz}(\varepsilon), L_{z,Bz}\left(\frac{2}{k}t-\varepsilon\right)\right\} \end{array}\right\}\right). \end{aligned}$$

Putting $\varepsilon = 0$, we obtain

$$\begin{aligned} F_{z,Bz}(t) &\geqslant \varphi\left(\min\left\{F_{z,Bz}(t), F_{z,Bz}\left(\frac{2}{k}t\right)\right\}\right) \geqslant \varphi\left(F_{z,Bz}(t)\right) > F_{z,Bz}(t), \\ L_{z,Bz}(t) &\leqslant \psi\left(\max\left\{L_{z,Bz}(\varepsilon), L_{z,Bz}\left(\frac{2}{k}t\right)\right\}\right) \leqslant \psi\left(L_{z,Bz}(t)\right) < L_{Az,z}(t). \end{aligned}$$

Which is a contradiction. Hence $Bz = z = Tz$, it follows that z is a common fixed point of B and T. Then, z is a common fixed point of A, B, S and T.

Now, the uniqueness of the common fixed point is an easy consequence of the condition (C_5). □

We show some illustrative examples which demonstrate the validity of the hypothesis and the utility of our results.

Example 2. *Let $(X, F, L, *, \diamondsuit)$ be a intuitionistic Menger space, where $X = [3, 11[$, $a * b = ab$ and $a \diamondsuit b = \min\{a+b, 1\}$ with*

$$F_{x,y}(t) = \frac{t}{t+|x-y|}, \quad L_{x,y}(t) = \frac{|x-y|}{t+|x-y|}$$

for all $x, y \in X$, $t > 0$. *Define the self-mapping* A, B, S *and* T *by*

$$Ax = \begin{cases} 3, & x \in \{3\} \cup]5, 11[\\ 10, & x \in]3, 5] \end{cases}$$

$$Bx = \begin{cases} 3, & x \in \{3\} \cup]5, 11[\\ 9, & x \in]3, 5] \end{cases}$$

$$Sx = \begin{cases} 3, & \text{if} \quad x = 3 \\ 7, & \text{if} \quad x \in]3, 5] \\ \frac{x+1}{2}; & \text{if} \quad x \in]5, 11[\end{cases}$$

$$Tx = \begin{cases} 3, & \text{if} \quad x = 3 \\ x+4, & \text{if} \quad x \in]3, 5] \\ x-2, & \text{if} \quad x \in]5, 11[\end{cases} .$$

Choosing $\{x_n = 3\}$, $\left\{y_n = 5 + \frac{1}{n}\right\}$ *or* $\left\{x_n = 5 + \frac{1}{n}\right\}$, $\{y_n = 3\}$, *it is easy to show that both the pair* (A, S) *and* (B, T) *satisfy the property* CLR_{ST},

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} Bx_n = \lim_{n \to +\infty} Tx_n = 3 \in S(X) \cap T(X).$$

We note that $A(X) = \{3, 10\} \subsetneq]3, 9[= T(X)$ *and* $B(X) = \{3, 9\} \subsetneq (\{7\} \cup]3, 6[) = S(X)$. *Thus, all the conditions of Theorem 1 are satisfied and the limit* 3 *is a unique common fixed point of the pairs* (A, S) *and* (B, T), *which also remains a point of coincidence as well. Also, all the mappings are discontinuous at their unique common fixed* 3. *We note that* $S(X)$ *and* $T(X)$ *are not closed subsets of* X *also.*

Theorem 2. *Let* A, B, S *and* T *are self-mappings of a intuitionistic Menger space* $(X, F, L, *, \diamondsuit)$ *satisfying the conditions* $(C_1) - (C_5)$ *of Lemma 2. Then* A, B, S *and* T *have a unique common fixed point if both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. In view of Lemma 2, both the pairs (A, S) and (B, T) enjoy the CLR_{ST} property, therefore there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof in the same manner as the proof of Theorem 1. □

Example 3. *We replace the self-mappings S and T in example 2 by*

$$Sx = \begin{cases} 3, & if \quad x = 3 \\ 6, & if \quad x \in]3,5] \\ \frac{x+1}{2}, & if \quad x \in [5,11) \end{cases}$$

$$Tx = \begin{cases} 3, & if \quad x = 3 \\ 9, & if \quad x \in]3,5[\\ x-2, & if \quad x \in [5,11) \end{cases}.$$

Then $A(X) = \{3,4\} \subset [3,9] = T(X)$ and $B(X) = \{3,5\} \subset [3,6] = S(X)$. Thus, all the conditions of Theorem 2 are satisfied and the limit 3 is a unique common fixed point of the pairs (A,S) and (B,T). We note that Theorem 1 can not be used in the context of this example while $S(X)$ and $T(X)$ are closed subsets of X.

Taking $A = B$ and $S = T$ in Theorem 1 we obtain a fixed point theorem for a pair of self-mappings.

Corollary 1. *Let A,S be self-mappings of a intuitionistic Menger space $(X,F,L,*,\diamondsuit)$, If the following conditions*

$Q_1)$ *The pair (A,S) satisfies the (CLR_S) property.*

$Q_2)$

$$F_{Ax,Ay}(t) \geqslant \varphi\left(\min\left\{\begin{array}{c} F_{Sx,Sy}(t);\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{F_{Sx,Ax}(t_1), F_{Sy,Ay}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{F_{Sx,Ay}(t_3), F_{Sy,Ax}(t_4)\right\} \end{array}\right\}\right)$$

$$L_{Ax,Ay}(t) \leqslant \psi\left(\max\left\{\begin{array}{c} L_{Sx,Sy}(t);\ \inf_{t_1+t_2=\frac{2}{k}t}\max\left\{L_{Sx,Ax}(t_1), L_{Sy,Ay}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t}\min\left\{L_{Sx,Ay}(t_3), L_{Sy,Ax}(t_4)\right\} \end{array}\right\}\right)$$

are satisfying for all $x,y \in X$, $t > 0$ and for some $1 \leqslant k < 2$. Then (A,S) has a coincidence point. Moreover if the pair (A,S) is weakly compatible. Then A and S have a unique common fixed point in X.

In view of Theorem 1, we can prove a common fixed point theorem for four finite families of self-mappings.

Corollary 2. *Let* $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ *and* $\{T_h\}_{h=1}^q$ *be four finite families of self-mappings of a intuitionistic Menger space* $(X,F,L,*,\diamondsuit)$, *where* $*$ *is a continuous t-norm and* $\diamondsuit$ *is a continuous t-conorm with* $A = A_1A_2....A_m$, $B = B_1B_2....B_n$, $S = S_1S_2....S_p$ *and* $T = T_1T_2....T_q$ *satisfying the condition* (C_5) *of Lemma 2 such that the pairs* (A,S) *and* (B,T) *share the* (CLR_{ST}) *property. Then* $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ *and* $\{T_h\}_{h=1}^q$ *have a unique common fixed point provided the pairs of families* $\left(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p\right)$, $\left(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q\right)$ *commute pairwise.*

Putting $A_1 = A_2 = ... = A_m = A$, $B_1 = B_2 = ... = B_n = B$, $S_1 = S_2 = ... = S_p = S$ and $T_1 = T_2 = ... = T_q = T$ in Corollary 2, we deduce the following

Corollary 3. *Let* A,B,S *and* T *are self-mappings of a intuitionistic Menger space* $(X,F,L,*,\diamondsuit)$ *where* $*$ *is a continuous t-norm and* $\diamondsuit$ *is a continuous t-conorm. If the following conditions*

Q_1) $\left(CRL_{S^p,T^q}\right)$, *where* m,n,p,q *are fixed positive integers.*

Q_2)

$$F_{A^mx,B^ny}(t) \geqslant \varphi\left(\min\left\{\begin{array}{c} F_{S^px,T^qy}(t);\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{F_{A^mx,S^px}(t_1), F_{B^ny,T^qy}(t_2)\right\}; \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{F_{S^px,B^ny}(t_3), F_{A^mx,T^qy}(t_4)\right\} \end{array}\right\}\right)$$

$$L_{A^mx,B^ny}(t) \leqslant \psi\left(\max\left\{\begin{array}{c} L_{S^px,T^qy}(t);\ \inf_{t_1+t_2=\frac{2}{k}t}\max\left\{L_{A^mx,Sx}(t_1), L_{B^ny,T^qy}(t_2)\right\}; \\ \inf_{t_3+t_4=\frac{2}{k}t}\min\left\{L_{S^px,B^ny}(t_3), L_{A^mx,T^qy}(t_4)\right\} \end{array}\right\}\right)$$

are satisfying for all $x,y \in X$, $t > 0$ *and for some* $1 \leqslant k < 2$. *Then* (A,B,S) *and* T *have a unique common fixed point if* $AS = SA$ *and* $BT = TB$.

4. Application to Integral Equations

In this section, we establish the solution of the system of Fredholm integral equations satisfying the main result

Theorem 3. *Let* $X = C([a,b],\mathbb{R}^n)$ *and* $F,L : X\times X\times\mathbb{R}^+ \to \mathbb{R}^+$ *and* $\theta > 0$ *defined as follows*

$$F_{x,y}(\theta) = \frac{\theta}{\theta + \max_{t\in[a,b]}\|x(t)-y(t)\|_\infty};\quad L_{x,y}(\theta) = \frac{\max_{t\in[a,b]}\|x(t)-y(t)\|_\infty}{\theta + \max_{t\in[a,b]}\|x(t)-y(t)\|_\infty}.$$

Consider the system of Fredholm integral equations

$$x(t) = \int_a^b K_1(t,s)x(s)ds + g(t), \tag{2}$$

$$x(t) = \int_a^b K_2(t,s)x(s)ds + h(t), \tag{3}$$

$$x(t) = \int_a^b K_3(t,s)x(s)ds + \rho(t), \tag{4}$$

$$x(t) = \int_a^b K_4(t,s)x(s)ds + \sigma(t), \tag{5}$$

where $\theta > 0, t \in [a,b] \subset \mathbb{R}$ *and* $x, g, h, \rho, \sigma \in X$. *Assume that* $K_1, K_2, K_3, K_4 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* $H_x, G_x, P_x, Q_x \in X$ *for each* $x \in X$, *where*

$$\begin{aligned} H_x(t) &= \int_a^b K_1(t,s)x(s)ds,\ G_x(t) = \int_a^b K_2(t,s)x(s)ds, \\ P_x(t) &= \int_a^b K_3(t,s)x(s)ds,\ Q_x(t) = \int_a^b K_4(t,s)x(s)ds,\ \text{for all}\, t \in [a,b]. \end{aligned}$$

Assume that φ *is decreasing continuous functions and* ψ *is increasing continuous functions. If* $1 < k \leqslant 2$ *such that the inequalities*

$$\begin{cases} \frac{\theta}{\theta + \|H_x + g - G_y - h\|} \geqslant \varphi\left(\min\left\{\begin{array}{c} R(x,y,\theta);\ \sup_{\theta_1+\theta_2=\frac{2}{k}\theta} \min\{R_1(x,y,\theta_1), R_2(x,y,\theta_2)\}; \\ \sup_{\theta_3+\theta_4=\frac{2}{k}\theta} \max\{R_3(x,y,\theta_3), R_4(x,y,\theta_4)\} \end{array}\right\}\right) \\ \frac{\|H_x + g - G_y - h\|}{\theta + \|H_x + g - G_y - h\|} \leq \psi\left(\max\left\{\begin{array}{c} M(x,y,\theta);\ \inf_{\theta_1+\theta_2=\frac{2}{k}\theta} \max\{M_1(x,y,\theta_1), M_2(x,y,\theta_2)\}; \\ \inf_{\theta_3+\theta_4=\frac{2}{k}\theta} \min\{M_3(x,y,\theta_3), M_4(x,y,\theta_4)\} \end{array}\right\}\right) \end{cases}$$

where,

$$\begin{aligned} R(x,y,\theta) &= \frac{\theta}{\theta + \|P_x + \rho - Q_y - \sigma\|}, \\ R_1(x,y,\theta_1) &= \frac{\theta_1}{\theta_1 + \|H_x + g - P_x - \rho\|}, \\ R_2(x,y,\theta_2) &= \frac{\theta_2}{\theta_2 + \|G_y + h - Q_y - \sigma\|}, \\ R_3(x,y,\theta_3) &= \frac{\theta_3}{\theta_3 + \|P_x + \rho - G_y - h\|}, \\ R_4(x,y,\theta_4) &= \frac{\theta_4}{\theta_4 + \|H_x + g(t) - Q_y - \sigma\|}, \end{aligned}$$

$$\begin{aligned} M(x,y,\theta) &= \frac{\|P_x+\rho-Q_y-\sigma\|}{\theta+\|P_x+\rho-Q_y-\sigma\|}, \\ M_1(x,y,\theta_1) &= \frac{\|H_x+g(t)-P_x-\rho\|}{\theta_1+\|H_x+g(t)-P_x-\rho\|}, \\ M_2(x,y,\theta_2) &= \frac{\|G_y+h(t)-Q_y-\sigma\|}{\theta_2+\|G_y+h(t)-Q_y-\sigma\|}, \\ M_3(x,y,\theta_3) &= \frac{\|P_x+\rho-G_y-h\|}{\theta_3+\|P_x+\rho-G_y-h\|}, \\ M_4(x,y,\theta_4) &= \frac{\|H_x+g(t)-Q_y-\sigma\|}{\theta_4+\|H_x+g(t)-Q_y-\sigma\|}, \end{aligned}$$

holds for each $\theta,\theta_1,\theta_2,\theta_3,\theta_4>0$ *and for all* $x,y\in X$. *Then the system of Fredholm integral equations has a unique common solution in* X.

Proof. Define $S,T,A,B:X\to X$ by:

$$Ax=H_x+g\ ,\ Bx=G_x+h\ ,\ Sx=P_x+\rho\ ,\ Tx=Q_x+\sigma.$$

Then,

$$\begin{aligned} F_{Ax,By}(\theta) &= \frac{\theta}{\theta+\max_{t\in[a,b]}\|H_x+g-G_y-h\|_\infty}, \\ F_{Sx,Ty}(\theta) &= \frac{\theta}{\theta+\max_{t\in[a,b]}\|P_x+\rho-Q_y-\sigma\|_\infty}, \\ F_{Ax,Sx}(\theta_1) &= \frac{\theta_1}{\theta_1+\max_{t\in[a,b]}\|H_x+g-P_x-\rho\|_\infty}, \\ F_{By,Ty}(\theta_2) &= \frac{\theta_2}{\theta_2+\max_{t\in[a,b]}\|G_y+h-Q_y-\sigma\|_\infty}, \\ F_{Sx,By}(\theta_3) &= \frac{\theta_3}{\theta_3+\max_{t\in[a,b]}\|P_x+\rho-G_y-h\|_\infty}, \\ F_{Ax,Ty}(\theta_4) &= \frac{\theta_4}{\theta_4+\max_{t\in[a,b]}\|H_x+g(t)-Q_y-\sigma\|_\infty}, \\ L_{Ax,By}(\theta) &= \frac{\max_{t\in[a,b]}\|H_x+g-G_y-h\|_\infty}{\theta+\max_{t\in[a,b]}\|H_x+g-G_y-h\|_\infty} \\ L_{Sx,Ty}(\theta) &= \frac{\max_{t\in[a,b]}\|P_x+\rho-Q_y-\sigma\|_\infty}{\theta+\max_{t\in[a,b]}\|P_x+\rho-Q_y-\sigma\|_\infty}, \end{aligned}$$

$$\begin{aligned} L_{Ax,Sx}(\theta_1) &= \frac{\max_{t\in[a,b]}\|H_x+g(t)-P_x-\rho\|_\infty}{\theta_1+\max_{t\in[a,b]}\|H_x+g(t)-P_x-\rho\|_\infty}, \\ L_{By,Ty}(\theta_2) &= \frac{\max_{t\in[a,b]}\|G_y+h(t)-Q_y-\sigma\|_\infty}{\theta_2+\max_{t\in[a,b]}\|G_y+h(t)-Q_y-\sigma\|_\infty}, \\ L_{Sx,By}(\theta_3) &= \frac{\max_{t\in[a,b]}\|P_x+\rho-G_y-h\|_\infty}{\theta_3+\max_{t\in[a,b]}\|P_x+\rho-G_y-h\|_\infty}, \\ L_{Ax,Ty}(\theta_4) &= \frac{\max_{t\in[a,b]}\|H_x+g(t)-Q_y-\sigma\|_\infty}{\theta_4+\max_{t\in[a,b]}\|H_x+g(t)-Q_y-\sigma\|_\infty}. \end{aligned}$$

We can show easily that for all $x,y\in X$

$$\begin{cases} F_{Ax,By}(\theta)\geqslant\varphi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t);\ \sup_{\theta_1+\theta_2=\frac{2}{k}\theta}\min\left\{F_{Ax,Sx}(\theta_1),F_{By,Ty}(\theta_2)\right\}; \\ \sup_{\theta_3+\theta_4=\frac{2}{k}t}\max\left\{F_{Sx,By}(\theta_3),F_{Ax,Ty}(\theta_4)\right\}\end{array}\right\}\right) \\ L_{Ax,By}(\theta)\leq\psi\left(\max\left\{\begin{array}{c} L_{Sx,Ty}(\theta);\ \inf_{\theta_1+\theta_2=\frac{2}{k}\theta}\max\left\{L_{Ax,Sx}(\theta_1),L_{By,Ty}(\theta_2)\right\}; \\ \inf_{\theta_3+\theta_4=\frac{2}{k}\theta}\min\left\{L_{Sx,By}(\theta_3),L_{Ax,Ty}(\theta_4)\right\}\end{array}\right\}\right)\end{cases}$$

By applying Theorem 1, the system of Fredholm integral equations has a unique common solution. □

Example 4. *Let $X=C([1,3],\mathbb{R})$ and $F,L:X\times X\times\mathbb{R}^+\to\mathbb{R}^+$ and $\theta>0$ defined as follows*

$$F_{x,y}(\theta)=\frac{\theta}{\theta+\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty}\quad L_{x,y}(\theta)=\frac{\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty}{\theta+\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty}.$$

Define $A,S:X\to X$ by

$$Ax(t)=4+\int_1^t x(s)s^2e^{s-1}ds,t\in[1,3],$$

$$Sx(t)=2+\int_1^t x(s)(s^2+s)e^{2s-1}ds,t\in[1,3].$$

For every $x,y\in X$, putting $\lambda=\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty$.

$$\begin{aligned} F_{Ax,Ay}(\theta) &= \frac{\theta}{\theta+\max_{t\in[1,3]}\|Ax(t)-Ay(t)\|_\infty} \\ &= \frac{\theta}{\theta+\int_1^3 e^2ds\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty} \\ &= \frac{\theta}{\theta+2e^2\lambda}, \\ F_{Sx,Sy} &= (\theta)=\frac{\theta}{\theta+\max_{t\in[1,3]}\|Sx(t)-Sy(t)\|_\infty} \\ &= \frac{\theta}{\theta+\int_1^3 e^5ds\max_{t\in[1,3]}\|x(t)-y(t)\|_\infty} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta}{\theta + 2e^5\lambda} \\
L_{Ax,Ay} &= \frac{\max_{t\in[1,3]} \|Ax(t) - Ay(t)\|_\infty}{\theta + \max_{t\in[1,3]} \|Ax(t) - Ay(t)\|_\infty} \\
&= \frac{\theta + \int_1^3 e^2 ds \max_{t\in[1,3]} \|x(t) - y(t)\|_\infty}{\theta + \int_1^3 e^2 ds \max_{t\in[1,3]} \|x(t) - y(t)\|_\infty} \\
&= \frac{2e^2\lambda}{\theta + 2e^2\lambda} \\
L_{Sx,Sy} &= \frac{\max_{t\in[1,3]} \|Sx(t) - Sy(t)\|_\infty}{\theta + \max_{t\in[1,3]} \|Sx(t) - Sy(t)\|_\infty}, \\
&= \frac{\int_1^3 e^5 ds \max_{t\in[1,3]} \|x(t) - y(t)\|_\infty}{\theta + \int_1^3 e^5 ds \max_{t\in[1,3]} \|x(t) - y(t)\|_\infty}, \\
&= \frac{2e^5\lambda}{\theta + 2e^5\lambda}.
\end{aligned}
$$

For all $\theta, \lambda > 0$ *we have*

$$F_{Ax,Ay}(\theta) \geqslant \varphi(F_{Sx,Sy}(\theta)), \tag{6}$$

$$L_{Ax,Ay}(\theta) \leqslant \psi(L_{Sx,Sy}(\theta)) \tag{7}$$

Thus, for $\theta, \lambda > 0$ *in* (4.5) *and* (4.6)*, all conditions of Corollary 1 are satisfied and so A and S have a unique fixed point, which is the unique solution of the integral equations*

5. CONCLUSION

In this work, we have used the concept of the common limit range property (CLR property) to prove various common fixed point theorems for weakly compatible mappings in intuitionistic Menger space. Our findings have builded upon and broaden fixed point theorems on Menger metric spaces. By demonstrating the presence and singularity of a shared solution for Fredhom integral equations, we have validated our conclusion. We have also provided illustrative instances to bolster the goals we have set.

Future work could explore the application of fixed point theorems in intuitionistic Menger spaces to solve fractional partial differential equations, particularly under the CLR property, see [19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. This approach could lead to new insights in modeling complex, memory-dependent systems with uncertainty. Additionally, extending the theoretical framework to incorporate fractional calculus may provide more general and robust solutions, potentially impacting various scientific and engineering fields. This research could pave the way for novel analytical techniques in the study of nonlinear systems.

ACKNOWLEDGEMENT

The authors are thankful to the learned referees for their valuable comments which helped in bringing this paper to its present form.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] K. Menger, Statistical metrics, in: B. Schweizer, A. Sklar, K. Sigmund, P. Gruber, E. Hlawka, L. Reich, L. Schmetterer (Eds.), Selecta Mathematica, Springer Vienna, Vienna, 2003: pp. 433–435. https://doi.org/10.1007/978-3-7091-6045-9_35.

[2] B. Schweizer, A. Sklar, Statistical metric spaces, Pac. J. Math. 10 (1960), 313–334.

[3] B. Schweizer, A. Sklar, Probabilistic metric spaces, North Holland, New York, 1983.

[4] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975), (336)–344. http://dml.cz/dmlcz/125556.

[5] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7.

[6] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22 (2004), 1039–1046. https://doi.org/10.1016/j.chaos.2004.02.051.

[7] S. Kutukcu, A. Tuna, A.T. Yakut, Generalized contraction mapping principle in intuitionistic Menger spaces and application to differential equations, Appl. Math. Mech. 28 (2007), 799–809. https://doi.org/10.1007/s10483-007-0610-z.

[8] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), 771–779. https://doi.org/10.1155/s0161171286000935.

[9] S.N. Mishra, Common fixed points of compatible mappings in probabilistic metric spaces, Math. Japon. 36 (1991), 283–289.

[10] G. Jungck, B.E. Rhoades, Fixed points for set valued functions without continuity, Indian. J. Pure. Appl. Math. 29 (1998), 227–238.

[11] J.H. He, Nonlinear dynamics and the Nobel Prize in physics, Int. J. Nonlinear Sci. Numer. Simul. 8 (2007), 1–4. https://doi.org/10.1515/ijnsns.2007.8.1.1.

[12] M. A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, Oxford, 1964.

[13] M.R.S. Rahmat, M.S.M. Noorani, Approximate fixed point theorems in probabilistic normed (metric) spaces, in: Proceedings of the 2nd IMT-GT Regional Conference on Mathematics, Statistics and Applications, Universiti Sains Malaysia, 2006.

[14] R. Saadati, S.M. Vaezpour, Some results on fuzzy Banach spaces, J. Appl. Math. Comput. 17 (2005), 475–484. https://doi.org/10.1007/bf02936069.

[15] R. Saadati, S. Sedghi, N. Shobe, Modified intuitionistic fuzzy metric spaces and some fixed point theorems, Chaos Solitons Fractals 38 (2008), 36–47. https://doi.org/10.1016/j.chaos.2006.11.008.

[16] W. Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. 2011 (2011), 637958. https://doi.org/10.1155/2011/637958.

[17] O. Hadzic, E. Pap, V. Radu, Generalized contraction mapping principles in probabilistic metric spaces, J. Acta Math. Hungar. 101 (2003), 131–148. https://doi.org/10.1023/B:AMHU.0000003897.39440.d8.

[18] M. Imdad, J. Ali, M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Chaos Solitons Fractals 42 (2009), 3121–3129. https://doi.org/10.1016/j.chaos.2009.04.017.

[19] H. Qawaqneh, Fractional analytic solutions and fixed point results with some applications, Adv. Fixed Point Theory 14 (2024), 1. https://doi.org/10.28919/afpt/8279.

[20] A. Fatma, I. Batiha, R. Imad, et al. Solvability and weak controllability of fractional degenerate singular problem, J. Robot. Control 5 (2024), 542–550.

[21] I.M. Batiha, A. Benguesmia, T.E. Oussaeif, et al. Study of a superlinear problem for a time fractional parabolic equation under integral over-determination condition, IAENG Int. J. Appl. Math. 54 (2024), 187–193.

[22] I.M. Batiha, S.A. Njadat, R.M. Batyha, et al. Design Fractional-order PID controllers for single-joint robot arm model, Int. J. Adv. Soft Comput. Appl. 14 (2022), 97–114. https://doi.org/10.15849/ijasca.220720.07.

[23] I.M. Batiha, J. Oudetallah, A. Ouannas, et al. Tuning the fractional-order PID-controller for blood glucose level of diabetic patients, Int. J. Adv. Soft Comput. Appl. 13 (2021), 1–10.

[24] Z. Chebana, T.E. Oussaeif, S. Dehilis, et al. On nonlinear Neumann integral condition for a semilinear heat problem with blow up simulation, Palestine J. Math. 12 (2023), 378–394.

[25] A. Benguesmia, I.M. Batiha, T.E. Oussaeif, et al. Inverse problem of a semilinear parabolic equation with an integral overdetermination condition, Nonlinear Dyn. Syst. Theory. 23 (2023), 249–260.

[26] I.M. Batiha, T.-E. Oussaeif, A. Benguesmia, A.A. Abubaker, A. Ouannas, S. Momani, A study of a superlinear parabolic dirichlet problem with unknown coefficient, Int. J. Innov. Comput. Inform. Control. 20 (2024), 541–556. https://doi.org/10.24507/ijicic.20.02.541.

[27] I.M. Batiha, L.B. Aoua, T.E. Oussaeif, et al. Common fixed point theorem in non-Archimedean Menger PM-spaces using CLR property with application to functional equations, IAENG Int. J. Appl. Math. 53 (2023), 360–368.

[28] G.H. Laid, I.M. Batiha, L. Benaoua, et al. On a common fixed point theorem in intuitionistic Menger space via C class and inverse C class functions with CLR property, Nonlinear Funct. Anal. Appl. 29 (2024), 899–912.