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A COMMON FIXED POINT THEOREM FOR SINGLE AND MULTI-VALUED MAPPINGS IN MENGER SPACES

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Abstract. The aim of the present paper is to establish a common fixed point theorem for two single-valued and two multi-valued mappings using weak compatibility in Menger spaces..

Keywords: Menger spaces, Multi-valued maps, weakly compatible maps.

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1. Introduction

Many fixed point theorems have been developed after establishment of Banach's fixed point theorem given by Polish mathematician Stefan Banach in 1922.In [1,2,3,4,5,11]authors have developed fixed point theorems in metric spaces for two set-valued mappings and two single-valued mappings in many ways using implicit relations, contractive conditions, strict contractive conditions.In 1942 Menger [8]introduced probabilistic metric spaces (briefly PM-spaces)as a generalization of metric spaces. Sehgal[12]initiated study of contraction mappings in PM-spaces.As in metric spaces fixed point theorems developed for set-valued and single -valued mappings, in a similar vein fixed point theorems have

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been developed by authors[6,7,9]in PM-spaces. In the present paper our aim is to develop fixed point theorem for two single-valued and two set-valued maps in PM-spaces using weak compatibility.In the paper let R denotes set of real numbers and R^+ denotes set of non-negative reals.

2. Preliminaries

Definition 1. *A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $inf_{t\in R}F(t) = 0$ and $sup_{t\in R}F(t) = 1$.Let D denotes the set of all distribution functions whereas H stands for specific distribution function(also known as Heaviside function) defined as*

$$H(t) = \begin{cases} 0, & t \leq 0; \\ 1, & t > 0. \end{cases}$$

Definition 2. *A PM-space is an ordered pair (X, F)consisting of non-empty set $X$and a mapping F from $X \times X$ into D.The value of F at $(x, y) \in X$ is represented by $F_{x,y}$.The functions $F_{x,y}$are assumed to satisfy the following conditions:*

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(ii) $F_{x,y}(0) = 0$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$;

(iv) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$,then $F_{x,z}(t + s) = 1$for all $x, y \in X$ and $t, s \geq 0$.

Every metric (X, d) space can always be realized as a PM-space by considering F from $X \times X$ into D as $F_{u,v}(s) = H(s - d(u, v))$ for all $u, v \in X$.

Definition 3. *A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly t-norm) if the following conditions are satisfied:*

(i)$\Delta(a, 1) = a$ for all $a \in [0, 1]$;

(ii) $\Delta(a, b) = \Delta(b, a)$;

(iii) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$;

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norm are $\Delta(a,b) = min(a,b)$,$\Delta(a,b) = ab$ *and* $\Delta(a,b) = min(a+b-1,0)$ *etc.*

Definition 4. *A Menger space is a triplet*(X, F, Δ),*where*(X, F)*is a PM-space,*Δ*is t-norm and the following condition hold:*

$F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ *holds for all* $x, y, z \in X$ *and* $t, s \geq 0$.

Definition 5. *A sequence* $\{p_n\}$*in a Menger space* (X, F, Δ) *is said to converge to a point* p *in* X *if for every* $\epsilon > 0$ *and* $\lambda > 0$,*there is an integer* $N(\epsilon, \lambda)$ *such that* $F_{p_n,p}(\epsilon) > 1 - \lambda$,*for all* $n \geq N(\epsilon, \lambda)$.*The sequence is said to be Cauchy sequence if for every* $\epsilon > 0$ *and* $\lambda > 0$,*there is an integer* $N(\epsilon, \lambda)$ *such that*$F_{p_n,p_m}(\epsilon) > 1 - \lambda$,*for all* $n, m \geq N(\epsilon, \lambda)$.

Throughout this paper,Let $B(X)$ denotes the set of all non-empty bounded subsets of Menger space X.

Definition 6. *The mappings* J *from* X *into* X *and* G *from* X *into* $B(X)$ *are weakly compatible if they commute at there coincidence points, that is for each* $x \in X$ *such that* $Gx = \{Jx\}$, *we have* $GJx = JGx$.*(Note here* $Gx = \{Jx\}$ *implies that* Gx *is a singleton.)*

For all $A, B \in B(X)$ and for all $t > 0$, we define

$\delta F_{A,B}(t) = inf\{F_{a,b}(t) : a \in A, b \in B\}$.

If $A = \{a\}$ then $\delta F_{A,B}(t) = \delta F_{a,B}(t)$.

If we have also $B = \{b\}$ then $\delta F_{A,B}(t) = F_{a,b}(t)$.

It follows from the definition that $\delta F_{A,B}(t) = 1 \Leftrightarrow A = B = \{a\}$.

Let $\{A_n\}$ be a sequence in $B(X)$. we say that $\{A_n\}$ δ-converges to a set A in X if for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} \delta F_{A_n,A}(\epsilon) = 1.$$

Lemma 1. [7]*Let* (X, F, min) *be a Menger space.Let* $A, G, H \in B(X)$.*Then for* $t_1, t_2 > 0$ *we have*

$\delta F_{A,H}(t_1 + t_2) \geq min\{\delta F_{A,G}(t_1), \delta F_{G,H}(t_2)\}$.

Lemma 2. [10]*Let (X, F, min) be a Menger space.If sequence $\{a_n\}$ converges to a and sequence $\{b_n\}$ converges to b,then for $t > 0$ we have*

$$\liminf_{n\to\infty} F_{a_n,b_n}(t) = F_{a,b}(t).$$

Lemma 3. [7]*Let (X, F, min) be a Menger space.If sequence $\{A_n\}$ δ-converges to a and sequence $\{B_n\}$ δ-converges to b,then for $t > 0$ we have*

$$\liminf_{n\to\infty} \delta F_{A_n,B_n}(t) = F_{a,b}(t).$$

3. Main results

Theorem 1 Let (X, F, min) be a complete Menger space.Let H, G be two set-valued mappings from X into $B(X)$ and I, J be two single-valued mappings from X into X satisfying following conditions:

(1) $G(X) \subseteq I(X), H(X) \subseteq J(X)$.

(2) $\delta F_{Hx,Gy}(kt) \geq min\{F_{Ix,Jy}(t), \delta F_{Ix,Hx}(t), \delta F_{Jy,Gy}(t)\}$ for all $x, y \in X$, $t > 0, k \in (0, 1)$.

(3) pairs (H, I) and (G, J) are weakly compatible.

(4) one of $I(X)$ or $J(X)$ is closed.

Then H, G, I and J have a unique common fixed point.

Proof: Let x_0 be an arbitary point of X.Define a sequence $\{x_n\}$ as follows:

$Jx_{2n+1} \in Hx_{2n} = Y_{2n}$, $Ix_{2n+2} \in Gx_{2n+1} = Y_{2n+1}$,for $n = 0, 1, 2....$

Using (2),we have

$\delta F_{Hx_{2n},Gx_{2n+1}}(kt) \geq min\{F_{Ix_{2n},Jx_{2n+1}}(t), \delta F_{Ix_{2n},Hx_{2n}}(t), \delta F_{Jx_{2n+1},Gx_{2n+1}}(t)\}$.

We get $\delta F_{Y_{2n},Y_{2n+1}}(kt) \geq min\{\delta F_{Y_{2n-1},Y_{2n}}(t), \delta F_{Y_{2n-1},Y_{2n}}(t), \delta F_{Y_{2n},Y_{2n+1}}(t)\}$.

This implies $\delta F_{Y_{2n},Y_{2n+1}}(kt) \geq \delta F_{Y_{2n-1},Y_{2n}}(t)$. (5)

Again using (2),we have

$\delta F_{Hx_{2n+2},Gx_{2n+1}}(kt) \geq min\{F_{Ix_{2n+2},Jx_{2n+1}}(t), \delta F_{Ix_{2n+2},Hx_{2n+2}}(t), \delta F_{Jx_{2n+1},Gx_{2n+1}}(t)\}$.

We get $\delta F_{Y_{2n+2},Y_{2n+1}}(kt) \geq min\{\delta F_{Y_{2n+1},Y_{2n}}(t), \delta F_{Y_{2n+1},Y_{2n+2}}(t), \delta F_{Y_{2n},Y_{2n+1}}(t)\}$.

This gives $\delta F_{Y_{2n+2},Y_{2n+1}}(kt) \geq \delta F_{Y_{2n},Y_{2n+1}}(t)$. (6)

From (5) and (6),we have

$\delta F_{Y_n,Y_{n+1}}(t) \geq \delta F_{Y_{n-1},Y_n}(\frac{t}{k})$,for $n = 1, 2, 3..$ (7)

Using Lemma(1)for $m > n$ and $\epsilon > 0$,we have

$\delta F_{Y_n,Y_m}(\epsilon) \geq min\{\delta F_{Y_n,Y_{n+1}}(\epsilon - k\epsilon), \delta F_{Y_{n+1},Y_m}(k\epsilon)$.

Using (7),we have

$\delta F_{Y_n,Y_m}(\epsilon) \geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_{n+1},Y_m}(k\epsilon)$.

$\geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), min\{\delta F_{Y_{n+1},Y_{n+2}}(k\epsilon - k^2\epsilon), \delta F_{Y_{n+2},Y_m}(k^2\epsilon)\}\}$.

$\geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), min\{\delta F_{Y_0,Y_1}(\frac{k\epsilon-k^2\epsilon}{k^{n+1}}), \delta F_{Y_{n+2},Y_m}(k^2\epsilon)\}\}$.

$= min\{min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n})\}, \delta F_{Y_{n+2},Y_m}(k^2\epsilon)\}$.

$= min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_{n+2},Y_m}(k^2\epsilon)\}$.

Continuing this process,we get

$\geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_{m-1},Y_m}(k^{m-1-n}\epsilon)\}$.

$\geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_0,Y_1}(\frac{k^{m-1-n}\epsilon}{k^{m-1}})\}$

$\geq min\{\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n}), \delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n})\}$.

$= \delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^n})$.

If N be taken such that $\delta F_{Y_0,Y_1}(\frac{\epsilon-k\epsilon}{k^N}) > 1 - \lambda$,then we have
$\delta F_{Y_n,Y_m}(\epsilon) \geq 1 - \lambda$ for all $n \geq N$.

This implies $\{Y_n\}$ is a Cauchy sequence.Since X is complete,therefore for any sequence $\{y_n\}$ in Y_n there must exist a point,say,p in X such that sequence $\{y_n\}$ converges to point p.The point p is independent of choice of sequence $\{y_n\}$ in Y_n so we must have

$$\lim_{n\to\infty} Jx_{2n+1} = p, \lim_{n\to\infty} Hx_{2n} = p, \lim_{n\to\infty} Ix_{2n+2} = p, \lim_{n\to\infty} Gx_{2n+1} = p.$$

Suppose $J(X)$is closed.Then there exists some $v \in X$ such that $p = Jv \in J(X)$.

Using (2),we have

$\delta F_{Hx_{2n},Gv}(kt) \geq min\{F_{Ix_{2n},Jv}(t), \delta F_{Ix_{2n},Hx_{2n}}(t), \delta F_{Jv,Gv}(t)\}$.

Taking $\liminf$ as $n \to \infty$ and using Lemma (2)and Lemma (3),we have

$\delta F_{p,Gv}(kt) \geq min\{F_{p,p}(t), \delta F_{p,p}(t), \delta F_{p,Gv}(t)\}$.

$\delta F_{p,Gv}(kt) \geq \delta F_{p,Gv}(t)$.

This implies $Gv = \{p\}$ and so $Gv = \{p\} = \{Jv\}$.

Since $G(X) \subseteq I(X)$so there exist $u \in X$ such that $\{Iu\} = Gv = \{p\} = \{Jv\}$.

Using (2),we have

$\delta F_{Hu,Gv}(kt) \geq min\{F_{Iu,Jv}(t), \delta F_{Iu,Hu}(t), \delta F_{Jv,Gv}(t)\}$.

Or $\delta F_{Hu,Iu}(kt) \geq min\{F_{Iu,Iu}(t), \delta F_{Iu,Hu}(t), \delta F_{Iu,Iu}(t)\}$.

Implying $\delta F_{Hu,Iu}(kt) \geq \delta F_{Hu,Iu}(t)$.

This gives $Hu = \{Iu\}$,we have $Hu = \{Iu\} = Gv = \{p\} = \{Jv\}$.But{H,I} is weakly compatible,it gives $Hp = HIu = IHu = \{Ip\}$.Using (2),we have

$\delta F_{Hp,Gv}(kt) \geq min\{F_{Ip,Jv}(t), \delta F_{Ip,Hp}(t), \delta F_{Jv,Gv}(t)\}$.

Or $F_{Ip,p}(kt) \geq min\{F_{Ip,p}(t), \delta F_{Ip,Ip}(t), \delta F_{p,p}(t)\}$.

Which implies $F_{Ip,p}(kt) \geq F_{Ip,p}(t.)$.

This gives $p = Ip$ and therefore we get $Hp = \{p\} = \{Ip\}$.when (G,J)is weakly compatible we have $Gp = GJv = JGv = \{Jp\}$.Using (2),we have

$\delta F_{Hp,Gp}(kt) \geq min\{F_{Ip,Jp}(t), \delta F_{Ip,Hp}(t), \delta F_{Jp,Gp}(t)\}$.

$\delta F_{p,Jp}(kt) \geq min\{F_{p,Jp}(t), \delta F_{p,p}(t), \delta F_{Jp,Jp}(t)\}$.

$\delta F_{p,Jp}(kt) \geq F_{p,Jp}(t)$.

This gives $p = Jp$.Therefore we obtain $Hp = \{p\} = \{Ip\} = Gp = \{Jp\}$.Hence p is a common fixed point of H, G, I and J.Similarly,if $I(X)$ is taken closed result follows.

Uniqueness: Let w be another fixed point of H, G, I and J such that $w \neq p$.Then $Hw = Gw = \{Jw\} = \{Iw\} = \{w\}$.Using (2),we have

$\delta F_{Hp,Gw}(kt) \geq min\{F_{Ip,Jw}(t), \delta F_{Ip,Hp}(t), \delta F_{Jw,Gw}(t)\}$.

Or $F_{p,w}(kt) \geq min\{F_{p,w}(t), \delta F_{p,p}(t), \delta F_{w,w}(t)\}$.

Or $F_{p,w}(kt) \geq F_{p,w}(t)$.

This implies $p = w$. Hence point p is unique.

Corollory 1. *Let (X, F, min) be a complete Menger space.Let H, G be two set-valued mappings from X into $B(X)$ satisfying following condition:*

$\delta F_{Hx,Gy}(kt) \geq min\{F_{x,y}(t), \delta F_{x,Hx}(t), \delta F_{y,Gy}(t)\}$ *for all* $x, y \in X, t > 0$,
$k \in (0, 1)$.*Then F and G have a unique common fixed point.*

Corollory 2. *Let (X, F, min) be a complete Menger space.Let G be a set-valued mapping from X into $B(X)$ and I be single-valued mapping from X into X satisfying following conditions :*

(8) $G(X) \subseteq I(X)$
(9) $\delta F_{Gx,Gy}(kt) \geq min\{F_{Ix,Iy}(t), \delta F_{Ix,Gx}(t), \delta F_{Iy,Gy}(t)\}$ *for all* $x, y \in X$,
$t > 0, k \in (0, 1)$.

(10)*pair* (G, I) *is weakly compatible.*

(11)$I(X)$ *is closed.*

Then G *and* I *have a unique common fixed point.*

Example 1. *Let* $X = [0, 2]$ *with the metric* $d(u, v) = |u - v|$ *and define* $F_{u,v}(s) = H(s - d(u, v))$ *for all* $u, v \in X$*.Then* (X, F, min) *is a complete Menger space.Define* G, H, I *and* J *as follows:*

$$G(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}]; \\ (\frac{3}{8}, \frac{1}{2}], & x \in (\frac{1}{2}, 1]. \end{cases}$$

$$I(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}]; \\ \frac{(x+1)}{4}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

$$J(x) = \begin{cases} (1 - x), & x \in [0, \frac{1}{2}]; \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

$$H(x) = \{\tfrac{1}{2}\}, x \in X.$$

$G(X) = (\frac{3}{8}, \frac{1}{2}] = I(X)$ *and* $H(X) = \{\frac{1}{2}\} \subseteq J(X) = \{0\} \cup [\frac{1}{2}, 1]$ *and therefore condition* (1) *of Theorem*(1)*is satisfied.*

Taking $t = 1 > 0, k = .7 \in (0, 1)$*,we have*

$\delta F_{Hx,Gy}(.7) = inf\{F_{u_1,v_1}(.7) : u_1 \in Hx, v_1 \in Gy\}.$

$= inf\{H(.7 - d(u_1, v_1) : u_1 \in Hx, v_1 \in Gy\}.$

Since $.7 - d(u_1, v_1) > 0$ *for all* $u_1 \in Hx, v_1 \in Gy$*, we have*

$\delta F_{Hx,Gy}(.7) = 1.$

$F_{Ix,Jy}(1) = H(1 - d(u_2, v_2))$ *for all* $u_2 \in Ix, v_2 \in Jy.$

Since $1 - d(u_2, v_2) > 0$ *for all* $u_2 \in Ix, v_2 \in Jy$,*we have*

$F_{Ix,Jy}(1) = 1.$

$\delta F_{Ix,Hx}(1) = inf\{F_{u_2,u_1}(1) : u_2 \in Ix, u_1 \in Hx\}.$

$= inf\{H(1 - d(u_2, u_1)) : u_2 \in Ix, u_1 \in Hx\}.$

Since $1 - d(u_2, u_1) > 0$ *for all* $u_2 \in Ix, u_1 \in Hx$,*we have*

$\delta F_{Ix,Hx}(1) = 1.$

$\delta F_{Jy,Gy}(1) = inf\{F_{v_2,v_1}(1) : v_2 \in Jy, v_1 \in Gy\}.$

$= inf\{H(1 - d(v_2, v_1)) : v_2 \in Jy, v_1 \in Gy\}.$

Since $1 - d(v_2, v_1) > 0$ *for all* $v_2 \in Jy, v_1 \in Gy$,*we have*

$\delta F_{Jy,Gy}(1) = 1.$

Now for $t > 0, k = .7 \in (0, 1)$*and* $x, y \in X$,*we have*

$\delta F_{Hx,Gy}(.7) = 1, F_{Ix,Jy}(1) = 1, \delta F_{Ix,Hx}(1) = 1, \delta F_{Jy,Gy}(1) = 1.$ *Thus condition* (2) *of Theorem*(1)*is satisfied.*

$\frac{1}{2}$ *is coincidence point of* H *and* I. *Also* H *and* I *commute at* $\frac{1}{2}$. *Similarly* $\frac{1}{2}$ *is coincidence pint of* G *and* J, *and* G *and* J *commute at coincidence point* $\frac{1}{2}$. *Therefore pairs* (H, I) *and* (G, J) *are weakly compatible.* $J(X)$ *is closed subset of* X.*Thus all the conditions of Theorem*(1)*are satisfied and* $\frac{1}{2}$ *is unique fixed point of* G, H, I *and* J.

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