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PRACTICAL STABILITY OF CAPUTO FRACTIONAL DYNAMIC EQUATIONS ON TIME SCALE

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Abstract. This paper presents a novel approach to analyzing the practical stability of Caputo fractional dynamic equations on time scales, utilizing a new generalized derivative known as the Caputo fractional delta derivative and the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$. This generalized derivative provides a unified framework for analyzing dynamic systems across both continuous and discrete time domains, making it suitable for hybrid systems exhibiting both gradual and abrupt changes. By incorporating memory effects inherent in fractional-order systems, this derivative is particularly suited to practical stability analysis, where deviations from equilibrium are permitted within acceptable limits. The established practical stability results are demonstrated through an illustrative example.

Keywords: fractional calculus; practical stability; Caputo fractional derivative; time scales; Lyapunov stability; dynamic systems..

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1. INTRODUCTION

The theory of stability in the sense of Lyapunov is well-established and extensively applied to real-world problems. While Lyapunov stability is important, in many practical applications, asymptotic stability holds greater significance. Understanding the size of the region of asymptotic stability is crucial for determining whether a given system is stable enough to function properly and for identifying ways to enhance its stability. However, there are situations where a system may be inherently unstable, yet its performance remains acceptable due to oscillations occurring near the desired state. In such cases, the classical notion of Lyapunov stability may not be adequate. Instead, a more suitable concept of stability is required, one that accounts for the system's behavior within an acceptable range. This concept is known as practical stability, and it plays a vital role in evaluating the performance of systems under less stringent stability requirements. ([22]).

Practical stability is particularly valuable when dealing with real-world systems where exact stability is either too restrictive or unnecessary. In many practical applications, it is sufficient for the system state to remain within a certain bounded region over time, rather than converging exactly to an equilibrium point. This approach allows for flexibility in the stability requirements, acknowledging that small deviations from equilibrium may be acceptable, especially in the presence of uncertainties or disturbances. Practical stability thus provides a more realistic and less restrictive framework compared to asymptotic or uniform stability, making it highly applicable to engineering, control systems, and applied sciences.

Time scale calculus, introduced in [6], offers a unified framework for analyzing systems that evolve in both discrete and continuous domains. By bridging the gap between these domains, time scale calculus extends the tools of continuous analysis to discrete cases and vice versa, providing a broader view of dynamic behaviors. When combined with fractional calculus, the result is a versatile and powerful framework for the study of dynamic systems that can exhibit both gradual and abrupt changes, as often seen in hybrid systems.

This paper presents a novel approach to analyzing the Lyapunov practical stability of Caputo fractional dynamic equations on time scales, utilizing a new generalized derivative. The generalized derivative, known as the Caputo fractional delta derivative and the Caputo fractional delta Dini derivative of order $\alpha \in (0,1)$, extends the traditional Caputo fractional derivative to arbitrary time scales. This allows for a unified analysis of practical stability that is applicable across different time domains, accommodating both continuous and discrete systems.

Previous research, such as [2, 3, 4, 5, 8, 9, 10], has primarily focused on stability, uniform, asymptotic stability, and variational stability for delay, and impulsive, all on continuous time systems. On the other hand, studies like [17, 18] have considered stability in discrete domains. This work aims to address some gaps by developing a practical stability framework that is suitable for both continuous and discrete time scales, thereby providing a more comprehensive and realistic assessment of the behavior of fractional dynamic systems.

An interesting recent study [11, 12] analyzed Lyapunov stability for Caputo fractional dynamical systems but did not consider practical stability.

This paper aims to expand these analyses by establishing practical stability criteria, thus offering a new perspective that allows systems to operate within acceptable bounds despite uncertainties or transient disturbances.

By establishing comparison results and practical stability criteria for Caputo fractional dynamic equations, this paper extends the stability analysis in [12] and introduces new methods for addressing practical constraints in dynamic systems. The resulting framework bridges continuous and discrete time scales, providing a versatile tool for researchers and practitioners to ensure that system behavior remains within acceptable limits under real-world conditions. This work contributes significantly to the understanding of dynamic systems by providing a realistic approach to stability analysis, crucial for reliable applications in various fields such as engineering and control theory.

Consider the Caputo fractional dynamic system of order α with $0 < \alpha < 1$

$$\begin{aligned} {}^{C}\Delta^{\alpha} x &= f(t,x),\ t \in \mathbb{T}, \\ x(t_0) &= x_0,\ t_0 \geq 0, \end{aligned} \tag{1}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$ and ${}^{C}\Delta^{\alpha} x$ is the Caputo fractional delta derivative of $x \in \mathbb{R}^n$ of order α with respect to $t \in \mathbb{T}$. Let $x(t) = x(t,t_0,x_0) \in C_{rd}^{\alpha}[\mathbb{T},\mathbb{R}^n]$ be a solution of (1) and Suppose that the function f is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions. (results on existence and uniqueness of (1) are contained in [1, 21]), this work aims to investigate the practical stability of the system (1) using the comparison system of the form:

$$ {}^{C}\Delta^{\alpha}u = g(t,u),\ u(t_0) = u_0 \geq 0, \tag{2} $$

where $u \in \mathbb{R}_+$, $g \in C_{rd}[\mathbb{R}^2_+, \mathbb{R}]$ and $g(t,0) \equiv 0$. For this work, we will assume that the function $g \in [\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$, is such that for any initial data $(t_0, u_0) \in \mathbb{T} \times \mathbb{R}_+$, the system (2) with $u(t_0) = u_0$ has a unique solution $u(t) = u(t;t_0,u_0) \in C^{\alpha}_{rd}[\mathbb{T}, \mathbb{R}_+]$, see [1, 9, 19, 24].

The content of this research is organized as follows: Section 2 introduces essential terminologies, remarks, and fundamental lemmas that form the foundation for the subsequent developments. It also presents key definitions and important remarks relevant to the study. Section 3 presents the main results of our research, including the theoretical advancements and findings. In Section 4, a practical example is provided to demonstrate the relevance and application of our proposed approach. Finally, Section 5 summarizes the key findings of this study and discusses their implications in the conclusion.

2. Preliminaries

Definition 2.1 ([16]). *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as*

$$ \sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, $$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$ \rho(t) = sup\{s \in \mathbb{T} : s < t\}. $$

(i) if $\sigma(t) > t$, t is right scattered,

(ii) if $\rho(t) < t$, t is left scattered,

(iii) if $t < max\mathbb{T}$ and $\sigma(t) = t$, then t is called right dense,

(iv) if $t > min\mathbb{T}$ and $\rho(t) = t$, then t is called left dense.

Definition 2.2 ([16]). *The graininess function $\mu : \mathbb{T} \to [0,\infty)$ for $t \in \mathbb{T}$ is defined as*

$$ \mu(t) = \sigma(t) - t. $$

The derivative uses the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows.

If $\mathbb{T}$ has a left scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.3 ([16]). *Let $h : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. We define the delta derivative h^Δ also known as the Hilger derivative as*

$$h^\Delta(t) = \lim_{s \to t} \frac{h(\sigma(t)) - h(s)}{\sigma(t) - s}, \quad s \neq \sigma(t).$$

provided the limit exists.

The function $h^\Delta : \mathbb{T} \to \mathbb{R}$ is called the (Delta) derivative of h on $\mathbb{T}^k$.

If t is right dense, the delta derivative of $h : \mathbb{T} \to \mathbb{R}$, becomes

$$h^\Delta(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s},$$

and if t is right scattered, the Delta derivative becomes

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)}.$$

For a function $h : \mathbb{T} \to \mathbb{R}$, h^σ denotes $h(\sigma(t))$.

Definition 2.4 ([7]). *A function $h : \mathbb{T} \to \mathbb{R}$ is right dense continuous if it is continuous at all right dense points of $\mathbb{T}$ and its left sided limits exist and is finite at left dense points of $\mathbb{T}$. The set of all right dense continuous functions are denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 2.5 ([7]). *Assume $[a,b]$ is a closed and bounded interval in $\mathbb{T}$. Then a function $H : [a,b] \to \mathbb{R}$ is called a delta antiderivative of $h : [a,b] \to \mathbb{R}$ provided H is continuous on $[a,b]$, delta differentiable on $[a,b)$, and $H^\Delta(t) = h(t)$ for all $t \in [a,b)$. Then, we define the Delta integral by*

$$\int_a^b h(t) = H(b) - H(a) \quad \forall a, b \in \mathbb{T}.$$

Remark 2.1 ([7]). *All right dense continuous functions are delta integrable.*

Definition 2.6 ([7]). *A function $\phi : [0,r] \to [0,\infty)$ is of class $\mathcal{K}$ if it is continuous, and strictly increasing on $[0,r]$ with $\phi(0) = 0$.*

Definition 2.7 ([7]). *A continuous function $\mathscr{V}:\mathbb{R}^n \to \mathbb{R}$ with $\mathscr{V}(0)=0$ is called positive definite(negative definite) on the domain D if there exists a function $\phi \in \mathscr{K}$ such that $\phi(|x|) \leq \mathscr{V}(x)$ $(\phi(|x|) \leq -\mathscr{V}(x))$ for $x \in D$.*

Definition 2.8 ([7]). *A continuous function $\mathscr{V}:\mathbb{R}^n \to \mathbb{R}$ with $\mathscr{V}(0)=0$ is called positive semi-definite (negative semi-definite) on D if $\mathscr{V}(x) \geq 0$ $(\mathscr{V}(x) \leq 0)$ for all $x \in D$ and it can also vanish for some $x \neq 0$.*

Definition 2.9 ([16]). *Let $a,b \in \mathbb{T}$ and $h \in C_{rd}$, then we define the integration on a time scale $\mathbb{T}$ as follows:*

(i) If $\mathbb{T}=\mathbb{R}$, then

$$\int_a^b h(t)\Delta t = \int_a^b h(t)dt.$$

where $\int_a^b h(t)dt$ is the usual Riemann integral from calculus.

(ii) If $[a,b]$ consists of only isolated points, then

$$\int_a^b h(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)h(t) & \text{if} \quad a<b \\ 0 & \text{if} \quad a=b \\ -\sum_{t\in[b,a)} \mu(t)h(t) & \text{if} \quad a>b \end{cases}$$

(iii) If there exists a point $\sigma(t)>t$, then

$$\int_t^{\sigma(t)} h(s)\Delta s = \mu(t)f(t).$$

Definition 2.10 ([15]). *Assume $V \in C[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}_+]$, $h \in C_{rd}[\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n]$ and $\mu(t)$ is the graininess function then we define the dini derivative of $V(t,x)$ as:*

$$D_{-}V^{\Delta}(t,x) = \liminf_{\mu(t)\to 0} \frac{V(t,x)-V(t-\mu(t),x-\mu(t)h(t,x))}{\mu(t)} \tag{3}$$

$$D^{+}V^{\Delta}(t,x) = \limsup_{\mu(t)\to 0} \frac{V(t+\mu(t),x+\mu(t)h(t,x))-V(t,x)}{\mu(t)}. \tag{4}$$

If V is differentiable, then $D_{-}V^{\Delta}(t,x) = D^{+}V^{\Delta}(t,x) = V^{\Delta}(t,x)$.

Definition 2.11 ([1]). **(Fractional Integral on Time Scales)**. *Let $\alpha \in (0,1)$, $[a,b]$ be an interval on $\mathbb{T}$ and h an integrable function on $[a,b]$. Then the fractional integral of order α of h is defined by*

$$^{\mathbb{T}}_{a}I^{\alpha}_{t}h^{\Delta}(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\Delta s.$$

Definition 2.12 ([1]). **(Caputo Derivative on Time Scale)** *Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The Caputo fractional derivative of order α of h is defined by*

$${}_a\Delta^{\alpha}_{t}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}h^{\Delta^n}(s)\Delta s.$$

Definition 2.13. *Let $\mathbb{T}$ be a time scale. A point $t_0 \in \mathbb{T}$ is said to be a minimal element of $\mathbb{T}$ if, for any $t \in \mathbb{T}$, $t > t_0$ whenever $t \neq t_0$.*

Remark 2.2. *The concept of minimal element is essential in studying dynamic equations because it establishes a starting point, a reference time from which the dynamics of the system evolve. In the study of difference equations (a discrete-time setting), t_0 represents the initial time step. Similarly, in differential equations (a continuous-time setting), t_0 represents the initial time instant.*

Lemma 2.1 ([23]). *Let $\mathbb{T}$ be a time scale with the minimal element $t_0 \geq 0$. Suppose that for each $t \in \mathbb{T}$, there exists a statement $\mathbf{S}(t)$ such that the following conditions hold:*

(i) $\mathbf{S}(t_0)$ *is true;*

(ii) if t is right-scattered and $\mathbf{S}(t)$ is true, then $\mathbf{S}(\sigma(t))$ is also true;

(iii) for every right-dense t, there exists a neighborhood $\mathscr{U}$ such that if $\mathbf{S}(t)$ is true, then $\mathbf{S}(t^*)$ is also true for all $t^* \in \mathscr{U}$ with $t^* \geq t$;

(iv) for left-dense t, if $\mathbf{S}(t^*)$ is true for all t^* in $[t_0, t)$, then $\mathbf{S}(t)$ is true.

Then the statement $\mathbf{S}(t)$ is true for all $t \in \mathbb{T}$.

Remark 2.3. *When $\mathbb{T} = \mathbb{N}$, then Lemma 2.1 reduces to the well-known principle of mathematical induction. That is,*

(1) $\mathbf{S}(t_0)$ *is true is equivalent to the statement is true for $n = 1$;*

(2) $\mathbf{S}(t)$ *is true then $\mathbf{S}(\sigma(t))$ is true is equivalent to if the statement is true for $n = k$, then the statement is true for $n = k+1$.*

Definition 2.14. *Let $h \in C_{rd}^{\alpha}[\mathbb{T},\mathbb{R}^n]$, the Grunwald-Letnikov fractional delta derivative is given by*

$$^{GL}\Delta_0^{\alpha}h(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)]. \quad t\geq t_0, \tag{5}$$

and the Grunwald-Letnikov fractional delta dini derivative is given by

$$^{GL}\Delta_{0^+}^{\alpha}h(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)]. \quad t\geq t_0, \tag{6}$$

where $0<\alpha<1$, $^{\alpha}C_r = \frac{q(q-1)\dots(q-r+1)}{r!}$, *and* $[\frac{(t-t_0)}{\mu}]$ *denotes the integer part of the fraction* $\frac{(t-t_0)}{\mu}$.

Observe that if the domain is $\mathbb{R}$, *then* (6) *becomes*

$$^{GL}\Delta_{0^+}^{\alpha}h(t) = \limsup_{d\to 0+}\frac{1}{d^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{d}]}(-1)^{r\alpha}C_r[h(t-rd)], \quad t\geq t_0.$$

Remark 2.4. *It is necessary to note that the relationship between the Caputo fractional delta derivative and the Grunwald-Letnikov fractional delta derivative is given by*

$$^{C}\Delta_0^{\alpha}h(t) = ^{GL}\Delta_0^{\alpha}[h(t)-h(t_0)], \tag{7}$$

substituting (5) into (7) we have that the Caputo fractional delta derivative becomes

$$^{C}\Delta_0^{\alpha}h(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)], \quad t\geq t_0,$$

$$^{C}\Delta_0^{\alpha}h(t) = \lim_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\left\{h(\sigma(t))-h(t_0)+\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\right\}, \tag{8}$$

and the Caputo fractional delta Dini derivative becomes

$$^{C}\Delta_{0^+}^{\alpha}h(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)], \quad t\geq t_0. \tag{9}$$

Which is equivalent to

$$^{C}\Delta_{0^+}^{\alpha}h(t) = \limsup_{\mu\to 0+}\frac{1}{\mu^{\alpha}}\left\{h(\sigma(t))-h(t_0)+\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\right\}, \quad t\geq t_0. \tag{10}$$

For notation simplicity, we shall represent the Caputo fractional delta derivative of order α as ${}^{C}\Delta^{\alpha}$ and the Caputo fractional delta dini derivative of order α as ${}^{C}\Delta^{\alpha}_{+}$.

Given that $\lim\limits_{N\to\infty}\sum_{r=0}^{N}(-1)^{r\alpha}C_r = 0$ where $\alpha \in (0,1)$, and $\lim\limits_{\mu\to 0^+}[\frac{(t-t_0)}{\mu}] = \infty$ then it is easy to see that

$$\lim_{\mu\to 0^+}\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = -1. \tag{11}$$

Also from (9) and since the Caputo and Riemann-Liouville formulations coincide when $h(t_0) = 0$, then we have that

$$\limsup_{\mu\to 0^+}\frac{1}{\mu^{\alpha}}\sum_{r=0}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r = {}^{RL}\Delta^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{12}$$

Now we define the notation of practical stability below:

Definition 2.15. *System* (1) *is said to be:*

- *(P_1) practically stable if, given (λ, A) with $0 < \lambda < A$, we have $|x_0| < \lambda$ implies $|x(t)| < A$, $t \geq t_0$ for some $t_0 \in \mathbb{T}$;*
- *(P_2) uniformly practically stable if (P_1) holds for every $t_0 \in \mathbb{T}$;*
- *(P_3) practically quasi stable if given $(\lambda, B, T) > 0$ and some $t_0 \in \mathbb{T}$, we have $|x_0| < \lambda$ implies $|x(t)| < B$, $t \geq t_0 + T$;*
- *(P_4) uniformly practically quasi stable if (P_3) holds for all $t_0 \in \mathbb{T}$;*
- *(P_5) strongly practically stable if (P_1) and (P_3) hold simultaneously;*
- *(P_6) strongly uniformly practically stable if (P_2) and (P_4) hold together;*
- *(P_7) practically asymptotically stable if (P_1) and (S_1) hold with $a = A$;*
- *(P_8) uniformly practically asymptotically stable if (P_2) and (S_2) hold at the same time with $a = \lambda$;*
- *(P_9) practically unstable if (P_1) does not hold.*

Definition 2.16. *Corresponding to definition 2.15, the scalar fractional dynamic system* (2)*, is said to be practically stable if given $0 < \lambda < A$, we have*

$$u_0 < \lambda \implies u(t) < A,\ t \geq t_0, \tag{13}$$

for some $t_0 \in \mathbb{R}_+$.

Lemma 2.2. *Assume h and $m \in C_{rd}(\mathbb{T},\mathbb{R})$. Suppose there exists $t_1 > t_0$, where $t_1 \in \mathbb{T}$, such that $h(t_1) = m(t_1)$ and $h(t) < m(t)$ for $t_0 \leq t < t_1$. Then, if the Caputo fractional delta Dini derivatives of h and m exist at t_1, the inequality ${}^{C}\Delta_{+}^{\alpha}h(t_1) >^{C} \Delta_{+}^{\alpha}m(t_1)$ holds.*

Proof. Applying (9), we have

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}(h(t)-m(t)) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg\{ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-m(\sigma(t)-r\mu)] \\
&-[h(t_0)-m(t_0)]\Bigg\} \\
{}^{C}\Delta_{+}^{\alpha}h(t) -^{C}\Delta_{+}^{\alpha}m(t) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg\{ \sum_{r=0}^{[\frac{t-t_0}{\mu}]} (-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-m(\sigma(t)-r\mu)] \\
&-[h(t_0)-m(t_0)]\Bigg\},
\end{aligned}
$$

at t_1, we have

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) = -\limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg\{ \sum_{r=0}^{[\frac{t_1-t_0}{\mu}]} (-1)^{r\alpha}C_r[h(t_0)-m(t_0)]\Bigg\} +^{C}\Delta_{+}^{\alpha}m(t_1). \tag{14}
$$

Applying (12) to (14), we have

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) = -\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0)-m(t_0)] +^{C}\Delta_{+}^{\alpha}m(t_1),
$$

however, based on the Lemma's statement, we know that

$$
h(t) < m(t), \text{ for } t_0 \leq t < t_1,
$$

$$
\implies h(t)-m(t) < 0, \text{ for } t_0 \leq t < t_1,
$$

then, we obtain

$$
-\frac{(t_1-t_0)^{-\alpha}}{\Gamma(1-\alpha)}[h(t_0)-m(t_0)] > 0,
$$

implying

$$
{}^{C}\Delta_{+}^{\alpha}h(t_1) >^{C}\Delta_{+}^{\alpha}m(t_1).
$$

□

Theorem 2.1. *Assume that*

(i) $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t,u)\mu$ *is non-decreasing in* u.

(ii) $h \in C_{rd}[\mathbb{T}, \mathbb{R}_+]$ *is such that*

$$^{C}\Delta_{+}^{\alpha} h(t) \leq g(t, m(t)), \tag{15}$$

(iii) $z(t) = z(t; t_0, u_0)$ *is the maximal solution of (2) existing on* $\mathbb{T}$.

Then

$$h(t) \leq z(t), \quad t \geq t_0 \tag{16}$$

provided that

$$h(t_0) \leq u_0 \tag{17}$$

where $t \in \mathbb{T}, \quad t \geq t_0$

Proof. Utilizing the principle of induction as outlined in Lemma 2.1 for the assertion

$$\mathbf{S}(\mathbf{t}) : h(t) \leq z(t), \quad t \in \mathbb{T},\, t \geq t_0,$$

(i) $\mathbf{S}(\mathbf{t_0})$ is true since $h(t_0) \leq \upsilon_0$

(ii) Let t be right-scattered and $\mathbf{S}(\mathbf{t})$ be true. We need to show that $\mathbf{S}(\sigma(t))$ is true; that is

$$h(\sigma(t)) \leq z(\sigma(t)), \tag{18}$$

but from (9), we have

$$^{C}\Delta_{+}^{\alpha} h(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)], \quad t \geq t_0.$$

Also,

$$^{C}\Delta_{+}^{\alpha} z(t) = \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)], \quad t \geq t_0,$$

so that

$$\begin{aligned} ^{C}\Delta_{+}^{\alpha} z(t) - {}^{C}\Delta_{+}^{\alpha} h(t) \quad &= \quad \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [z(\sigma(t) - r\mu) - z(t_0)] \\ &\quad - \limsup_{\mu \to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r [h(\sigma(t) - r\mu) - h(t_0)] \end{aligned}$$

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}z(t) - {}^{C}\Delta_{+}^{\alpha}h(t) &= \limsup_{\mu\to 0+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r\Big[[z(\sigma(t)-r\mu)-z(t_0)] \\
&\quad -[h(\sigma(t)-r\mu)-h(t_0)]\Big] \\
\Big({}^{C}\Delta_{+}^{\alpha}z(t) - {}^{C}\Delta_{+}^{\alpha}h(t)\Big)\mu^{\alpha} &= \limsup_{\mu\to 0+} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\,\alpha}C_r\Big[[z(\sigma(t)-r\mu)-z(t_0)] \\
&\quad -[h(\sigma(t)-r\mu)-h(t_0)]\Big] \\
\Big({}^{C}\Delta_{+}^{\alpha}z(t) - {}^{C}\Delta_{+}^{\alpha}h(t)\Big)\mu^{\alpha} &\leq [z(\sigma(t))-h(\sigma(t))]-[z(t_0)-h(t_0)] \\
&\quad [h(\sigma(t))-z(\sigma(t))] \leq \Big({}^{C}\Delta_{+}^{\alpha}h(t) - {}^{C}\Delta_{+}^{\alpha}z(t)\Big)\mu^{\alpha} \\
&\quad +[h(t_0)-z(t_0)] \\
&\leq \Big(g(t,h(t))-g(t,z(t))\Big)\mu^{\alpha}+[h(t_0)-z(t_0)].
\end{aligned}
$$

Given that $g(t,\upsilon)\mu^{\alpha}$ is non-decreasing in υ and $\mathbf{S}(\mathbf{t})$ holds, it follows that $h(\sigma(t)) - z(\sigma(t)) \leq 0$, ensuring that (18) is satisfied.

(iii) Let t be right-dense and $\mathscr{N}$ denote the right neighborhood of $t \in \mathbb{T}$. We need to demonstrate that $\mathbf{S}(\mathbf{t}^*)$ holds for $t^* \in \mathscr{N}$. This can be established by applying the comparison theorem for Caputo fractional differential equations, where the Lyapunov function $V(t,x) = h(t)$; since at every right-dense point $t^* \in \mathscr{N}$, $\sigma(t^*) = t^*$. see [12].

Therefore by induction principle, the statement $\mathbf{S}(\mathbf{t})$ is true. Completing the proof.

□

3. Main Result

Theorem 3.1 (Practical Stability)**.** *Assume that*

(1) $g \in C_{rd}[\mathbb{R}_+^2, \mathbb{R}]$ *and* $g(t,u)$ *is non-decreasing in* u *with* $g(t,u) \equiv 0$.

(2) $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *and*

$$[x, f(t,x)]_+ \leq g(t,|x|). \tag{19}$$

(3) The FrDE (2) *is practically stable.*

Then the system (1) *is practically stable.*

Proof. By the assumption of the practical stability of (2), we have that for some $t_0 \in \mathbb{R}_+$ and $0 < \lambda < A$, (13) holds. The it is easy to show that for these λ and A, the system (1) is also practically stable.

If this where false, there would exists a solution $x(t) = x(t;t_0,x_0)$ of (1) with $|x_0| < \lambda$ and a time $t_1 > t_0$ such that

$$|x(t_1)| = A \text{ and } |x(t)| \leq A \text{ for } t \in [t_0,t_1). \tag{20}$$

Setting

$$h(t) = |x(t)|, \tag{21}$$

then from (19), we obtain

$$^{C}\Delta^{\alpha}_{+}h(t) \leq g(t,h(t)),\ t \in [t_0,t_1). \tag{22}$$

Also from (16) of the comparison theorem 2.1, we have

$$h(t) \leq z(t),\ t \in [t_0,t_1), \tag{23}$$

where $z(t)$ is the maximal solution of (2). Combining this estimates; (20), (21), (23) and assumption (3) of the Theorem, that is (13), we obtain,

$$A = |x(t_1)| \leq z(t_1,t_0,|x_0|) < A. \tag{24}$$

(24) is a contradiction, so the assumption of the practical stability of system (1) is true.

The prove of other concepts of practical stability in definition 2.15 follows a similar pattern and hence the proof is complete. □

4. Application

Consider the system of dynamic equations

$$\begin{aligned} ^{C}\Delta^{\alpha}\chi_1(t) &= -\chi_1\sin(\chi_2) - 2\chi_2 - 2\chi_1 \\ ^{C}\Delta^{\alpha}\chi_2(t) &= 2\chi_1 + \chi_2 - \chi_2\sin(\chi_1) \end{aligned} \tag{25}$$

for $t \geq t_0$, with initial conditions

$$\chi_1(t_0) = \chi_{10} \quad \text{and} \quad \chi_2(t_0) = \chi_{20},$$

where $\chi = (\chi_1, \chi_2)$ and $f = (f_1, f_2)$.

Consider $V(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$, for $t \in \mathbb{T}$, $(\chi_1, \chi_2) \in \mathbb{R}^2$ and choose $\alpha = 1$, so that (25) becomes an integer (first) order system. Then we compute the delta Dini derivative of $V(t, \chi_1, \chi_2) = \chi_1^2 + \chi_2^2$ along the solution path of (25) as follows:

From (4) we have that

$$\begin{aligned}
D^+V^{\Delta}(t,\chi) &= \limsup_{\mu(t)\to 0} \frac{V(t+\mu(t), \chi+\mu(t)f(t,\chi)) - V(t,\chi)}{\mu(t)} \\
&= \limsup_{\mu(t)\to 0} \frac{(\chi_1+\mu(t)f_1(t,\chi_1,\chi_2))^2 + (\chi_2+\mu(t)f_2(t,\chi_1,\chi_2))^2 - [\chi_1^2+\chi_2^2]}{\mu(t)} \\
&= \limsup_{\mu(t)\to 0} \frac{\chi_1^2 + 2\chi_1\mu(t)f_1 + \mu^2(t)f_1^2 + \chi_2^2 + 2\chi_2\mu(t)f_2 + \mu^2(t)f_2^2 - [\chi_1^2+\chi_2^2]}{\mu(t)} \\
&= \limsup_{\mu(t)\to 0} \frac{2\chi_1\mu(t)f_1 + \mu^2(t)f_1^2 + 2\chi_2\mu(t)f_2 + \mu^2(t)f_2^2}{\mu(t)} \\
&\leq 2\chi_1 f_1 + 2\chi_2 f_2 \\
&= 2[\chi_1(-\chi_1\sin(\chi_2) - 2\chi_2 - 2\chi_1) + \chi_2(2\chi_1 + \chi_2 - \chi_2\sin(\chi_1))] \\
&\leq 2\left[2\chi_1^2 + \chi_2^2\right].
\end{aligned}$$

Now, consider the consider the comparison equation

$$D^+u^{\Delta} = 4u > 0,\ u(0) = u_0. \tag{26}$$

Even though conditions (i)-(iii) of [15] are satisfied that is $V \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}_+]$, $D^+V^{\Delta}(t,\chi) \leq g(t, V(t,\chi))$ and $\sqrt{\chi_1^2+\chi_2^2} \leq \chi_1^2+\chi_2^2 \leq 2(\chi_1^2+\chi_2^2)$, for $b(\|\chi\|) = r$ and $a(\|\chi\|) = 2r^2$, it is obvious to see that the solution of the comparison system (26) is not practically stable, so we can not deduce the practical stability properties of the system (25) by applying the basic definition of the integer order Dini-derivative time scale.

Let us consider (25) with $\alpha \in (0,1)$ and apply the new definition (10).

Let $h(\chi_1, \chi_2) = \chi_1^2 + \chi_2^2$, for $(\chi_1, \chi_2) \in \mathbb{R}^2$.

$${}^{C}\Delta^{\alpha}_{0^+} h(t)$$

$$
\begin{aligned}
&= \limsup_{\mu\to 0+}\frac{1}{\mu^\alpha}\left\{h(\sigma(t))-h(t_0)+\sum_{r=1}^{[\frac{(t-t_0)}{\mu}]}(-1)^{r\alpha}C_r[h(\sigma(t)-r\mu)-h(t_0)]\right\} t\geq t_0\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\chi_1(\sigma(t)))^2+(\chi_2^2(\sigma(t)))^2\right]-\left[(\chi_{10})^2+(\chi_{20})^2\right]\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[(\chi_1(\sigma(t))-\mu^\alpha f_1(t,\chi_1,\chi_2))^2\\
&\quad+(\chi_2(\sigma(t))-\mu^\alpha f_2(t,\chi_1,\chi_2))^2((\chi_{10})^2+(\chi_{20})^2)]\Bigg\}\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\chi_1(\sigma(t)))^2+(\chi_2^2(\sigma(t)))^2\right]-\left[(\chi_{10})^2+(\chi_{20})^2\right]\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[(\chi_1(\sigma(t)))^2-2\chi_1(\sigma(t))\mu^\alpha f_1(t,\chi_1,\chi_2)\\
&\quad+\mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2+(\chi_2(\sigma(t)))^2\\
&\quad-2\chi_2(\sigma(t))\mu^\alpha f_2(t,\chi_1,\chi_2)+\mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2-((\chi_{10})^2+(\chi_{20})^2)]\Bigg\}\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{-\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[(\chi_{10})^2+(\chi_{20})^2\right]\\
&\quad+\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[(\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2\right]\\
&\quad-\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[2\chi_1(\sigma(t))\mu^\alpha f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))\mu^\alpha f_2(t,\chi_1,\chi_2)\right]\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[\mu^{2\alpha}(f_1(t,\chi_1,\chi_2))^2+\mu^{2\alpha}(f_2(t,\chi_1,\chi_2))^2\right]\Bigg\}\\
&\quad=-\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[(\chi_{10})^2+(\chi_{20})^2\right]\right\}\\
&\quad+\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)\left[(\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2\right]\right\}\\
&\quad-\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=1}^{[\frac{t-t_0}{\mu}]}(-1)^r({}^\alpha C_r)[2\chi_1(\sigma(t))\mu^\alpha f_1(t,\chi_1,\chi_2+2\chi_2(\sigma(t))\mu^\alpha f_2(t,\chi_1,\chi_2)\right\}.
\end{aligned}
$$

Applying (11) and (12), we obtain

$$\begin{aligned} & -\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left((\chi_{10})^2+(\chi_{20})^2\right)+\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2) \\ & -[2x_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)] \\ \leq\ & \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2) \\ & -[2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)]. \end{aligned}$$

As $t \to \infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}((\chi_1(\sigma(t)))^2+(\chi_2(\sigma(t)))^2) \to 0$, then

$$\begin{aligned} {}^{C}\Delta_{0^+}^{\alpha} h(t) & \leq -[2\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+2\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)] \\ & = -2\left[\chi_1(\sigma(t))f_1(t,\chi_1,\chi_2)+\chi_2(\sigma(t))f_2(t,\chi_1,\chi_2)\right], \end{aligned}$$

applying $\chi(\sigma(t)) \leq \mu^{C}\Delta^{\alpha}x(t)+x(t)$,

$$\begin{aligned} & -2\Big[\mu(t)f_1^2(t,\chi_1,\chi_2)+\chi_1(t)f_1(t,\chi_1,\chi_2)+\mu(t)f_2^2(t,\chi_1,\chi_2)+\chi_2(t)f_2(t,\chi_1,\chi_2)\Big] \\ = & -2\Big[\mu(t)(-\chi_1\sin(\chi_2)-2\chi_2-2\chi_1)^2+\chi_1(-\chi_1\sin(\chi_2)-2\chi_2-2\chi_1) \\ & +\mu(t)(2\chi_1+\chi_2-\chi_2\sin(\chi_1))^2+\chi_2(2\chi_1+\chi_2-\chi_2\sin(\chi_1))\Big]. \end{aligned}$$

Therefore,

$$ {}^{C}\Delta_{+}^{\alpha} h(\chi_1,\chi_2) \leq -4h(\chi_1,\chi_2). $$

Consider the comparison system

$$ {}^{C}\Delta^{\alpha}u^{\Delta} = g(t,u) \leq -4u. \tag{27} $$

It is obvious to see that all the conditions of Theorem 3.1 are satisfied, so we conclude that the system (25) is practically stable.

5. Conclusion

In conclusion, practical stability offers a more flexible and realistic approach for analyzing the performance of dynamic systems, particularly in real-world scenarios where classical Lyapunov stability may not be sufficient. While traditional stability concepts require equilibrium states, practical stability accounts for systems that may not be perfectly stable but can still function effectively within acceptable bounds. This concept is particularly relevant in applications such as control systems, where disturbances or environmental changes can cause deviations from stability without critically affecting the system's overall performance. By providing novel a framework that allows for system behavior near a desired state without requiring absolute stability, practical stability broadens the scope of stability analysis. This approach proves valuable for systems that need to operate reliably under varying conditions, such as aircraft, missiles, space vehicles, and industrial processes. The development of practical stability theory therefore fills a crucial gap in stability analysis, offering insights into systems that oscillate or deviate from equilibrium but still perform their intended functions within acceptable limits. We have also shown the theoretical applicability of this definition in Theorem 3.1 and the practical applicability as well as effectiveness in system (25).

AUTHORS' CONTRIBUTIONS

All authors contributed equally to the manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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