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NEW ITERATIVE ALGORITHM FOR GENERAL TRIFUNCTION EQUILIBRIUM VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose and analyse several iterative solutions to a new class of equilibrium problems and variational inequalities known as the trifunction equilibrium variational inequality using the auxiliary principle technique. Under extremely reasonable and appropriate assumptions, convergence of these iterative approaches is demonstrated. For these well-known and novel kinds of equilibrium and variational inequalities, the conclusions of this study remain valid.

Keywords: mixed variational inequality; equilibrium problems; convergence; auxiliary principle technique; variational inequalities.

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1. Introduction

Variational inequalities have emerged as an important and active field of both pure and applied mathematics in recent years. Different methods are being employed to conduct the research. A broad category of issues with industrial applicability, economics, quantitative finance and structural engineering transportation, optimization and optimization issues. This has inspired us to introduce and research a number of variations inequality classes. Blum and

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Oettili[1] explained the equilibrium issues and proved that the variational inequalities are part of the equilibrium. concerns with optimization and fixed issues as distinct cases. Inspired by Blum and Oettili's [1] research, the equilibrium difficulties were examined by Noor et al.[8]. The trifunction equilibrium is a situation involving the trifunction challenges, such as equilibrium issues and unique situations.

The problems of trifunction equilibrium and variational inequalities can be solved numerically using an extensive range of techniques. The trifunction equilibrium variational inequality cannot be solved using Wiener-Hopf equations or other projection methods due to the nature of the trifunction variational inequality problem. This feature led us to apply the Glowinski et al.[4] auxiliary principle technique, as developed by Noor [12, 13] and Noor et al. [7, 12, 13, 15]. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results. These may be extended to other classes of variational inequalities and equilibrium problems. Comparison of these methods with other methods is an interesting problem for future research.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a closed and convex set in H.

For a given trifunction $F(.,.,.) : H \times H \times H \to R$ and an operator $T : H \to H$, we consider the problem of finding $u \in K$ such that:

$$F(g(u), Tu, g(v)) + \langle Tu, g(v) - g(u) \rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \forall v \in K, \tag{2.1}$$

which is called the general trifunction equilibrium variational inequality. We note that if F(.,.,.) ≅ F(.,.), then the problem of Equation (2.1) is studied (Takhashi and Takhashi [28]; Yao et al. [30, 29]; Noor et al.[22, 26]) and Gupta et al. [5]. We now discuss some important special cases of the problem of equation (2.1).

Special Cases

(I) We note that, If $F(g(u), Tu, g(v)) = 0$, then the problem of (2.1) is equivalent to finding $u \in K$ such that:

$$\langle Tu, g(v)-g(u)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \forall v \in K, \tag{2.2}$$

which is known as mixed general variational inequality. A wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of above equation (see [2, 3, 4, 12, 18, 20, 7, 11]).

(II) If $\langle Tu, g(v)-g(u)\rangle = 0$, then the problem of equation (2.1) turns into the problem of finding $u \in K$ such that:

$$F(g(u), Tu, g(v)) + \phi(g(v)) - \phi(g(u)) \geq 0, \forall v \in K,$$

which is known as general trifunction equilibrium Problem.

For $\phi = 0$, the above equation reduces to the following form.

$$F(g(u), Tu, g(v)) \geq 0, \forall v \in K, \tag{2.3}$$

which is known as trifunction variational inequality.

(III) If $\phi(.)$ is an indicator function on the closed convex set K in H, then equation (2.1) is equivalent to finding $u \in H : g(u) \in K$ satisfying:

$$F(g(u), Tu, g(v)) + \langle Tu, g(v)-g(u)\rangle \geq 0, \forall v \in H : g(v) \in K, \tag{2.4}$$

which is known as general trifunction equilibrium problems.

(IV) From equation (2.1) reduces to the following form for $g = 1$ and $\phi = 0$.

$$\langle Tu, v-u\rangle \geq 0, \forall v \in K, \tag{2.5}$$

which is known as a variational inequality, introduced and studied by Stampacchia [27]. Within the overall framework of the problems of equation (2.1), a wide class of problems occurring in elasticity, fluid flow through porous media and optimization can be studied. Variational inequalities and their generalizations are discussed in terms of applications, numerical result formulation, and other features (Giannessi et al. [2, 3]; Noor [6, 10, 12, 13, 14]; Noor et al. [7, 8, 11, 13, 15]; Yao et al. [30, 29]).

Definition 2.1. An operator $T : H \to H$ is said to be:

(i) monotone, if and only if,

$$\langle Tu - Tv, g(v) - g(u)\rangle \geq 0, \forall u, v \in H,$$

(ii) Partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v\rangle \geq -\alpha\|z - u\|^2, \forall u, v, z \in H,$$

For z = u , partially relaxed strong monotonicity reduces to monotonicity of the operator T.

Definition 2.2. A trifunction $F(.,.,.) : H \times H \times H \to R$ with respect to an operator T is said to be:

1. Jointly monotone, if and only if,

$$F(u, Tu, v) + F(v, Tv, u) \leq 0 \qquad \forall u, v \in H.$$

2. Partially relaxed strongly jointly monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v) + F(v, Tv, z) \leq \mu\|z - u\|^2, \qquad \forall u, v, z \in H.$$

It is clear that for z = u, partially relaxed strongly jointly monotone trifunction is simply jointly monotone.

We would like to draw attention to the fact that the general trifunction equilibrium problem and variational inequalities are quite different. It makes sense to consider these problems to be connected. Motivated and inspired by the work being done in all these fields, we describe and study a novel class of variational inequalities and trifunction equilibrium issues called general trifunction equilibrium variational inequality. This new class includes trifunction equilibrium issues and variational inequalities as specific examples.

3. Main Results

Here, utilizing the auxiliary principle technique, we propose and examine an iterative solution for resolving the mixed equilibrium variational inequality problem of Equation (2.1). Glowinski et al.[4] are primarily responsible for this technique, which Noor [12, 13] and Noor et al.[18, 19, 20, 21] developed in their work.

For a given $u \in K$ consider the problem of finding $w \in K$ such that:

$$\rho F(g(u), Tu, g(v)) + \langle \rho Tu, g(v) - g(w)\rangle + \langle g(w) - g(u), g(v) - g(w)\rangle \geq \rho\phi(g(w)) - \rho\phi(g(v)) \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

If $w = u$, then $w \in H$ is a solution of equation (2.1). This observation enables us to suggest and analyze the following iterative method for solving the general trifunction equilibrium variational inequality of equation (2.1).

Algorithm 1. For a given $u_0 \in H$, compute the approximate solution $u_{n+1} \in H$ by the iterative scheme

$$\rho F(g(u_n), Tu_n, g(v)) + \langle \rho Tu_n, g(v) - g(u_{n+1})\rangle + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \geq \rho\phi(g(u_{n+1})) - \rho\phi(g(v)), \quad \forall v \in H. \tag{3.2}$$

If $F(g(u_n), Tu, g(v)) = 0$, then Algorithm 1 reduces to the following scheme for variational inequalities of equation (2.2).

Algorithm 2. For a given $u_0 \in H$, compute $u_{n+1} \in H$ the iterative scheme

$$\langle \rho Tu_n, g(v) - g(u_{n+1})\rangle + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \geq \rho\phi(g(u_{n+1})) - \rho\phi(g(v)), \quad \forall v \in H.$$

If $\langle Tu, g(v) - g(u)\rangle = 0$, then Algorithm 1 reduces to Algorithm 3.

Algorithm 3. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from iterative scheme

$$F(g(u_n), Tu_n, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \geq \rho\phi(g(u_{n+1})) - \rho\phi(g(v)), \quad \forall v \in H.$$

If $\phi(.)$ is an indicator function on the closed convex set K in H, then Algorithm 1 can be used to find the approximate solution of equation (2.4), which appears to be a new one.

Algorithm 4. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$F(g(u_n), Tu_n, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \geq 0, \quad \forall v \in H,$$

For a suitable and appropriate choice of F(.,.), T, ϕ (.) and spaces, one can define iterative algorithms to find the solutions to different classes of equilibrium problems and variational inequalities.

We now study the convergence analysis of Algorithm 1 using the technique of Noor et al. [18].

Theorem 3.1. *Let $u \in H$ be a solution of equation (2.1) and $u_{n+1} \in H$ be an approximate solution obtained from Algorithm 1. If the trifunction F(.,.,.) and the operator T(.) are partially relaxed, strongly monotone operators with constants $\mu > 0$ and $\sigma > 0$, respectively, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2\rho(\mu + \sigma))\|u_{n+1} - u_n\|^2. \tag{3.3}$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (1). Then replacing v by u_{n+1} in equation (2.1), we have

$$\begin{aligned} \rho F(g(u), Tu, g(u_{n+1})) + \rho\langle Tu, g(u_{n+1}) - g(u)\rangle \geq \rho\phi(g(u_n)) - \rho\phi(g(u_{n+1})), \\ \rho > 0. \end{aligned} \tag{3.4}$$

Let $u_{n+1} \in K$ be the approximate solution obtained from Algorithm 1. Taking $v = u$ in equation (3.1), we have

$$\begin{aligned} \rho F(g(u_n), Tu_n, g(u)) + \rho\langle Tu_n, g(u) - g(u_{n+1})\rangle + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ \geq \rho\phi(g(u_{n+1})) - \rho\phi(g(u_n)) \quad \forall v \in H. \end{aligned} \tag{3.5}$$

Adding equations (3.3) and (3.4), we have

$$\rho[F(g(u), Tu, g(u_{n+1})) + F(g(u_n), Tu_n, g(u))] + \rho\langle Tu, g(u_{n+1}) - g(u)\rangle + \rho\langle Tu_n, g(u) - g(u_{n+1})\rangle$$
$$+\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \geq 0,$$

which implies that

$$\rho[F(g(u), Tu, g(u_{n+1})) + F(g(u_n), Tu_n, g(u))] + \rho\langle Tu_n - Tu, g(u_{n+1}) - g(u)\rangle$$
$$+ \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \geq 0$$

which implies that

$$\begin{aligned}\langle g(u_{n+1})-g(u_n),g(u)-g(u_{n+1})\rangle &\geq -\rho[F(g(u),Tu,g(u_{n+1}))+F(g(u_n),Tu_n,g(u))]\\ &\quad+\rho\langle Tu_n-Tu,g(u_{n+1})-g(u)\rangle\\ &\geq -\rho\mu\|g(u_{n+1})-g(u_n)\|^2+\sigma\|g(u_{n+1})-g(u_n)\|^2\\ &\geq -\rho(\mu+\sigma)\|g(u_{n+1})-g(u_n)\|^2,\end{aligned} \tag{3.6}$$

where, we have used partially relaxed strong monotonicity of the trifunction $F(.,.,.)$ and operator $T(.)$.

Using the relation

$$2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2, \qquad \forall u,v\in H,$$

and from equation (3.5), taking $g=I$, we get

$$\|u-u_{n+1}\|^2\leq\|u-u_n\|^2-(1-2\rho(\mu+\sigma))\|u_{n+1}-u_n\|^2,$$

which is the required result of equation (3.2).

Theorem 3.2. *Let H be a finite-dimensional space and let u_{n+1} be the approximate solution obtained from Algorithm 1 and $u\in K$ be a solution to the problem of equation (2.1). Then* $\lim_{n\to\infty}u_n=u$.

Proof. Let $u\in H$ be a solution of (2.1). For $0<\rho<\frac{1}{2(\mu+\sigma)}$, we see that the sequence $\{|u-u_n\|\}$ is non increasing and consequently $\{u_n\}$ is bounded. Also from Equation (2.1), taking $g=I$, we have:

$$\sum_{n=0}^{\infty}(1-2(\mu+\sigma))\|u_{n+1}-u_n\|^2\leq\|u-u_0\|^2,$$

which implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{3.7}$$

Let $\hat{u}$ be a cluster point of the sequence u_n and let the subsequence $\{u_n\}$ of the sequence u_n converge to $\hat{u}\in H$. Replacing u_n by u_{n_j} in (3.1) and taking the limit as $n_j\to\infty$ and using (3.6), taking $g=I$, we have

$$F(\hat{u},T\hat{u},v)+\langle T\hat{u},v-\hat{u}\rangle\geq 0, \qquad \forall v\in H, g(v)\in K,$$

which implies that $\hat{u}$ solves the mixed equilibrium variational inequality (1) and

$$\|u_{n+1} - \hat{u}\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus, it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$, the required result.

We again use the auxiliary principle technique to suggest and analyze several proximal point algorithms for solving the mixed equilibrium-variational inequalities of equation (2.1). For a given $u \in K$, consider the problem of finding $w \in K$, such that

$$\begin{aligned} &\rho F(g(w), Tw, g(v)) + \langle \rho Tw, g(v) - g(w)\rangle \\ &+ \langle g(w) - g(u) - \gamma(g(u) - g(u)), g(v) - g(w)\rangle \geq 0, \\ &\forall v \in H : g(v) \in K, \end{aligned} \tag{3.8}$$

where $\rho \geq 0$ and $\gamma \geq 0$ are constants.
If $w = u$, then $w \in K$ is a solution of equation (2.1). This observation enables us to suggest and analyze the following iterative method for solving trifunction equilibrium variational inequalities of equation (2.1).

Algorithm 5. For a given $u_0 \in H$, compute the approximate solution $u_{n+1} \in H$ by the iterative scheme

$$\begin{aligned} &\rho F(g(u_{n+1}), Tu_{n+1}, g(v)) + \langle \rho Tu_{n+1}, g(v) - g(u_{n+1})\rangle \\ &+ \langle g(u_{n+1}) - g(u_n) - \gamma(g(u_n) - g(u_{n-1}), g(v) - g(u_{n+1}))\rangle \geq 0, \\ &\forall v \in H : g(v) \in K. \end{aligned} \tag{3.9}$$

Algorithm 6. For a given $u_0 \in H$, compute the approximate solution $u_{n+1} \in H$ by the iterative scheme

$$\begin{aligned} &\rho F(g(u_{n+1}), Tu_{n+1}, g(v)) + \langle \rho Tu_{n+1}, g(v) - g(u_{n+1})\rangle \\ &+ \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \geq 0. \end{aligned}$$

For a given $u \in K$, consider the problem of finding a $w \in K$ such that:

$$\begin{aligned} &\rho F(g(u), Tu, g(v)) + \langle \rho Tw, g(v) - g(w)\rangle \\ &+ \langle g(w) - g(u) - \gamma(g(u) - g(u)), g(v) - g(w)\rangle \geq 0, \end{aligned} \tag{3.10}$$

$$\forall v \in H : g(v) \in K.$$

Note that if $w = u$, then $w \in K$ is a solution of (2.1). This observation enables us to suggest and analyze the following proximal iterative method for solving trifunction equilibrium variational inequalities of (2.1).

Algorithm 7. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}&\rho F(g(u), Tu_n, g(v)) + \langle \rho Tu_{n+1}, g(v) - g(u_{n+1})\rangle\\&+ \langle g(u_{n+1}) - g(u_n) - \gamma(g(u_n) - g(u_{n-1}), g(v) - g(u_{n+1}))\rangle \geq 0,\\&\forall v \in H; g(v) \in K.\end{aligned}$$

For a given $u \in K$, consider the problem of finding a $w \in K$ such that

$$\begin{aligned}&\rho F(g(w), Tw, g(v)) + \langle \rho Tu, g(v) - g(w)\rangle\\&+ \langle g(w) - g(u) - \gamma(g(u) - g(u)), g(v) - g(w)\rangle \geq 0,\\&\forall v \in H : g(v) \in K,\end{aligned} \tag{3.11}$$

If $w = u$, then $w \in K$ is a solution of (2.1). This observation enables us to suggest and analyze the following iterative method for solving mixed equilibrium variational inequalities of (2.1).

Algorithm 8. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\rho F(g(u_{n+1}), Tu_{n+1}, g(v)) + \langle \rho Tu_n, g(v) - g(u_{n+1})\rangle + \langle g(u_{n+1}) - g(u_n)&\\- \gamma(g(u_n) - g(u_{n-1})), g(v) - g(u_{n+1})\rangle \geq 0.&\end{aligned}$$

Some special cases of these algorithms are as follows:
If $F(g(u), Tu, g(v)) = 0$, then Algorithm 5 reduces to the following scheme for mixed variational inequalities given as (2.2).

Algorithm 9. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}&\langle \rho Tu_{n+1}, g(v) - g(u_{n+1})\rangle\\&+ \langle g(u_{n+1}) - g(u_n) - \gamma(g(u_n) - g(u_{n-1}), g(v) - g(u_{n+1}))\rangle \geq 0,\end{aligned}$$

$$\forall v \in H : g(v) \in K.$$

If $\langle Tu, g(v) - g(u) \rangle = 0$, then Algorithm 1 reduces to Algorithm 10.

Algorithm 10. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}
&\rho F(g(u_{n+1}), Tu_{n+1}, g(v)) \\
&+ \langle g(u_{n+1}) - g(u_n) - \gamma(g(u_n) - g(u_{n-1}), g(v) - g(u_{n+1})) \rangle \geq 0, \\
&\forall v \in H : g(v) \in K
\end{aligned}$$

which is used for finding the solution of general trifunction equilibrium problem of (2.3).

Iterative algorithms can be defined as special cases of Algorithms 6 and Algorithms 7 for appropriate and suitable choices of F(.,.,.), T, and spaces to discover solutions to various classes of equilibrium problems and variational inequalities.

Theorem 3.3. *Let $u \in H$ be a solution of (2.1) and $u_{n+1} \in H$ be an approximate solution obtained from Algorithm 6. If trifunction F(.,.,.) and operator T are monotone, then*

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2. \tag{3.12}$$

Proof. Let $u \in H : g(v) \in K_r$ be a solution of (2.1). Then replacing v by u_{n+1} in (1), we have

$$\begin{aligned}
&\rho F(g(u), Tu, g(u_{n+1})) \\
&+ \rho \langle Tu, g(u_{n+1}) - g(u) \rangle + \rho \phi(g(u_{n+1})) - \rho \phi(g(u)) \geq 0, \forall v \in K.
\end{aligned} \tag{3.13}$$

Let $u_{n+1} \in H$ be the approximate solution obtained from Algorithm 5. Taking $v = u$ in equation (13), we have

$$\begin{aligned}
&\rho F(g(u_{n+1}), Tu_{n+1}, g(u)) + \langle \rho Tu_{n+1}, g(u) - g(u_{n+1}) \rangle \\
&+ \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}), \rangle \geq 0, \forall v \in H : g(v) \in K.
\end{aligned} \tag{3.14}$$

Adding equations (3.13) and (3.14), we have

$$\begin{aligned}
&\rho[F(g(u), Tu, g(u_{n+1})) + F(g(u_{n+1}), Tu_{n+1}, g(u))] + \rho \langle Tu, g(u_{n+1}) - g(u) \rangle \\
&+\rho \langle Tu_{n+1}, g(u) - g(u_{n+1}) \rangle + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}), \rangle \geq 0.
\end{aligned}$$

Which implies that

$$
\begin{aligned}
\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1}),\rangle \geq & -\rho[F(g(u),Tu,g(u_{n+1})) \\
& + F(g(u_{n+1}),Tu_{n+1},g(u))] \\
& +\rho\langle Tu, g(u_{n+1})-g(u)\rangle \\
& + \rho\langle Tu_{n+1}, g(u)-g(u_{n+1})\rangle.
\end{aligned}
\tag{3.15}
$$

We have used the monotonicity of the operator T and the trifunction F(.,.,.).

Using the relation

$2\langle u,v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \qquad u,v \in H$

and from (3.14), we have

$$\|g(u)-g(u_{n+1})\|^2 \leq \|g(u)-g(u_n)\|^2 - \|g(u_{n+1})-g(u_n)\|^2.$$

Which is the required result.

4. Conclusion

In this paper, we have introduced and considered a new class of general trifunction equilibrium variational inequalities. It is shown that the optimum of the sum of differentiable and directional differentiable nonconvex functions can be characterized by means of this class. We have used the auxiliary principle technique for suggesting and analyzing some explicit and inertial proximal point algorithms for solving the trifunction equilibrium variational inequality problem. Some special cases are also discussed. Our findings can be seen as an enhancement and improvement over the outcomes that were previously known. Note that the projection and resolvent techniques are not used in this technique. Readers are encouraged to find novel applications of the general trifunction equilibrium variational inequalities in pure and applied sciences. The ideas and techniques of this paper stimulate further research in this area of pure and applied sciences.

Conflict of Interests

The authors declare that there is no conflict of interests.

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