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HYERS-ULAM STABILITY AND CONTINUOUS DEPENDENCE OF A FIXED POINT PROBLEM OF A DELAY INTEGRO-DIFFERENTIAL EQUATION

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Abstract. Here, we study an initial value problem of a delay integro-differential equation. The existence of the unique solution will be proved in the spaces $C^1[0,T]$ and $AC[0,T]$. Hyers-Ulam stability of the problem will be introduced. The continuous dependence of the unique solution will be studied.

Keywords: delay integro-differential equation; existence of solutions; continuous dependence; Banach fixed point theorem; Hyers-Ulam stability.

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1. INTRODUCTION

Differential equations and integral equations are fundamental tools in mathematical modeling, used to describe systems and processes that change over time. Differential equations involve functions and their derivatives, capturing how a system evolves based on its current rate of change [2, 10]. Integral equations, on the other hand, focus on the accumulated effects over

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time, making them essential in scenarios where the current state depends on the history of the system [15, 16].

A delay integro-differential equation is an equation where the present state of a system depends on its previous states with a time delay. This framework is valuable for modeling situations where the impact of past events or conditions extends over time, such as in mechanical systems with response delays or biological systems with developmental lags [6, 7, 17, 19].

In studying these equations, stability is a key concern [1, 3], and two important concepts are Hyers-Ulam stability and continuous dependence. Hyers-Ulam stability examines how small deviations in the problem affect the solution [4, 5, 8, 9, 13, 14]. Continuous dependence, meanwhile, ensures that solutions change smoothly with respect to initial data and parameters [18, 20]. These concepts together offer a comprehensive view of the stability of systems described by delay differential and integral equations.

Now consider the initial value problem of a delay integro-differential equation

$$\frac{du(t)}{dt} = g(t)+\int_0^{\phi(t)} k(t,s)f\big(s,\frac{du(s)}{ds}\big)ds,\ t\in(0,T] \tag{1}$$

$$u(0) = u_0, \tag{2}$$

Let $\frac{du(t)}{dt}=x\in C[0,T]$, then the problem (1)-(2) is equivalent to the integral equation

$$u(t)=u_0+\int_0^t x(s)ds,\ t\in[0,T], \tag{3}$$

where $x(t)$ is given by the delay integro-functional equation

$$x(t)=g(t)+\int_0^{\phi(t)} k(t,s)f(s,x(s))ds. \tag{4}$$

Our aim in this paper is to study the existence of a unique solution of (1)-(2) in the space $C^1[0,T]$. To understand the stability of the problem we apply the Hyers-Ulam stability to the problem, then we study the continuous dependence of the unique solution on the initial data u_0 and the functions g, k, and f, and on the delay function $\phi(t)$. Subsequently, we extend our investigation to the space $AC[0,T]$ with the relaxation of equation (1) to be exist almost every where in $t\in(0,T]$ and letting $x\in L^1[0,T]$, giving us a better understanding of how it behaves and stays stable under less assumptions.

2. Study of a Continuously Differentiable Solution

In this section, we prove the existence of a unique continuously differentiable solution $u \in C^1[0,T]$ of the problem (1)-(2), for this aim, we assume that:

(i) $g : [0,T] \to \mathbb{R}$ is continuous.

(ii) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition [12] with $b > 0$ such that

$$|f(t,x) - f(t,u)| \leq b|x-u|.$$

(iii) $k : [0,T] \times [0,T] \to \mathbb{R}$ is continuous such that

$$|k(t,s)| \leq k_1, \forall (t,s) \in [0,T] \times [0,T].$$

(iv) $\phi : [0,T] \to [0,T]$ is continuous and increasing such that $\phi(t) \leq t$.

(v) $\sup_{t\in[0,T]} |f(t,0)| = M$.

(vi) $k_1 bT < 1$.

Theorem 1. *Let the assumptions $(i)-(vi)$ be satisfied, then the problem (1)-(2) has a unique solution $u \in C^1[0,T]$.*

Proof. From assumption (ii), we have

$$|f(t,x)| - |f(t,0)| \leq |f(t,x) - f(t,0)| \leq b|x|,$$

then

$$\begin{aligned} |f(t,x)| &\leq b|x| + |f(t,0)| \\ &\leq b|x| + \sup_{t\in[0,T]} |f(t,0)| \\ &= b|x| + M. \end{aligned}$$

Define the operator F_1 associated with (4) by

$$F_1 x(t) = g(t) + \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds$$

Firstly, we prove that F_1 maps $C[0,T]$ into itself.

For this, let $x \in C[0,T]$ and $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$ and $|t_2 - t_1| \leq \delta$, then

$$
\begin{aligned}
|F_1x(t_2)-F_1x(t_1)| &= \left|g(t_2)+\int_0^{\phi(t_2)}k(t_2,s)f(s,x(s))ds-g(t_1)-\int_0^{\phi(t_1)}k(t_1,s)f(s,x(s))ds\right| \\
&\leq |g(t_2)-g(t_1)|+\left|\int_0^{\phi(t_2)}k(t_2,s)f(s,x(s))ds-\int_0^{\phi(t_1)}k(t_1,s)f(s,x(s))ds\right| \\
&= |g(t_2)-g(t_1)|+\left|\int_0^{\phi(t_2)}k(t_2,s)f(s,x(s))ds-\int_0^{\phi(t_2)}k(t_1,s)f(s,x(s))ds\right. \\
&+ \left.\int_0^{\phi(t_2)}k(t_1,s)f(s,x(s))ds-\int_0^{\phi(t_1)}k(t_1,s)f(s,x(s))ds\right| \\
&\leq |g(t_2)-g(t_1)|+\int_0^{\phi(t_2)}|k(t_2,s)-k(t_1,s)|f(s,x(s))ds \\
&+ \int_{\phi(t_1)}^{\phi(t_2)}|k(t_1,s)||f(s,x(s))|ds \\
&\leq |g(t_2)-g(t_1)|+\int_0^{\phi(t_2)}|k(t_2,s)-k(t_1,s)|(b|x(s)|+M)ds \\
&+ \int_{\phi(t_1)}^{\phi(t_2)}k_1(b|x(s)|+M)ds=\varepsilon.
\end{aligned}
$$

This proves that $F_1 : C[0,T] \to C[0,T]$.

Secondly, to prove that F_1 is contraction, we have the following.

Let $x,z \in C[0,T]$, then

$$
\begin{aligned}
|F_1x(t)-F_1z(t)| &= \left|g(t)+\int_0^{\phi(t)}k(t,s)f(s,x(s))ds-g(t)-\int_0^{\phi(t)}k(t,s)f(s,z(s))ds\right| \\
&\leq \int_0^{\phi(t)}|k(t,s)||f(s,x(s))-f(s,z(s))|ds \\
&\leq k_1b\int_0^{\phi(t)}|x(s)-z(s)|ds \\
&\leq k_1b\int_0^{\phi(t)}\sup_{s\in[0,T]}|x(s)-z(s)|ds \\
&= k_1b\phi(t)||x-z||_C \\
&\leq k_1bt||x-z||_C \\
&\leq k_1bT||x-z||_C.
\end{aligned}
$$

Then

$$||F_1x-F_1z|| \leq k_1bT||x-z||_C,$$

since $k_1bT < 1$, then F_1 is contraction. Then by using the Banach fixed point Theorem [11], there exists a unique solution $x \in C[0,T]$ of (4) and therefore (3) also has a unique solution $u \in C[0,T]$. Consequently, by Equivalence, the problem (1)-(2) has a unique solution $u \in C^1[0,T]$.

□

2.1. Hyers-Ulam stability.

Definition 1. [4, 5, 8] *Let the solution $x \in C[0,T]$ of (4) be exists. The delay integro-functional equation (4) is Hyers-Ulam stable, if $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that for any solution $x_s \in C[0,T]$ of (4) satisfying*

$$\left|x_s(t) - g(t) - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta\right| \leq \delta,$$

then

$$||x - x_s||_C \leq \varepsilon_1.$$

Theorem 2. *Let the assumptions of Theorem (1) be satisfied, then (4) is Hyers-Ulam stable.*

Proof. Let $-\delta \leq x_s(t) - g(t) - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta \leq \delta$, consider

$$\begin{aligned}
|x(t) - x_s(t)| &= \left|g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x(\theta)) d\theta - x_s(t)\right| \\
&= \left|g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x(\theta)) ds - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta\right. \\
&+ \left.\int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta - x_s(t)\right| \\
&\leq \left|\int_0^{\phi(t)} k(t,\theta)[f(\theta, x(\theta)) - f(\theta, x_s(\theta))] d\theta\right| \\
&+ \left|-x_s(t) + g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta\right| \\
&\leq \int_0^{\phi(t)} |k(t,\theta)||f(\theta, x(\theta)) - f(\theta, x_s(\theta))| d\theta \\
&+ \left|x_s(t) - g(t) - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta\right| \\
&\leq \int_0^{\phi(t)} k_1 b |x(\theta) - x_s(\theta)| d\theta + \delta \\
&\leq k_1 bT ||x - x_s||_C + \delta.
\end{aligned}$$

Hence

$$||x-x_s||_C \leq \frac{\delta}{1-k_1bT} = \varepsilon_1.$$

Since $k_1bT < 1$, then (4) is Hyers-Ulam stable. □

Corollary 1. *Let the assumptions of Theorem (2) be satisfied, then the problem (1)-(2) is Hyers-Ulam stable.*

Proof. Consider

$$\begin{aligned}
|u(t)-u_s(t)| &= \left|u_0 + \int_0^t x(\theta)d\theta - u_0 - \int_0^t x_s(\theta)d\theta\right| \\
&\leq \int_0^T |x(\theta)-x_s(\theta)|d\theta \\
&\leq T||x-x_s||_C \\
&\leq T\varepsilon_1 = \varepsilon.
\end{aligned}$$

□

2.2. Continuous Dependence.

In this section, we study the continuous dependence of the unique solution on the initial data u_0 and the functions g, k, and f, and on the delay function $\phi(t)$.

Definition 2. *The solution $u \in C^1[0,T]$ of (1)-(2) depends continuously on the initial data u_0 and the function $x \in C[0,T]$, if $\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0$ such that*

$$\max\left\{|u_0-u_0^*|, ||x-x^*||_C\right\} \leq \delta \ \rightarrow\ ||u-u^*||_C \leq \varepsilon,$$

where u^* is the unique solution of the integral equation

$$u^*(t) = u_0^* + \int_0^t x^*(s)ds,\ t \in [0,T]. \tag{5}$$

Theorem 3. *Let the assumptions of Theorem 1 be satisfied, then the solution $u \in C^1[0,T]$ of (1)-(2) depends continuously on the initial data u_0 and the function $x \in C[0,T]$.*

Proof. Let u and u^* be the two solutions of (3) and (5), then

$$|u(t)-u^*(t)| = |u_0 + \int_0^t x(s)ds - u_0^* - \int_0^t x^*(s)ds|$$

$$\begin{aligned} &\leq |u_0 - u_0^*| + \int_0^t |x(s) - x^*(s)| ds \\ &\leq \delta + ||x - x^*||_C T. \end{aligned}$$

Hence

$$||u - u^*||_C \leq (1 + T)\delta = \varepsilon.$$

□

Definition 3. *The solution $x \in C[0,T]$ of (4) depends continuously on the function g, if $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that*

$$|g(t) - g^*(t)| \leq \delta \ \rightarrow \ ||x - x^*||_C \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation

$$x^*(t) = g^*(t) + \int_0^{\phi(t)} k(t,s) f(s, x^*(s)) ds,\ t \in [0,T]. \tag{6}$$

Theorem 4. *Let the assumptions of Theorem 1 be satisfied, then the solution $x \in C[0,T]$ of (4) depends continuously on the function g.*

Proof. Let x and x^* be the two solutions of (4) and (6), then

$$\begin{aligned} |x(t) - x^*(t)| &= |g(t) + \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds - g^*(t) - \int_0^{\phi(t)} k(t,s) f(s,x^*(s)) ds| \\ &\leq |g(t) - g^*(t)| + \int_0^{\phi(t)} |k(t,s)||f(s,x(s)) - f(s,x^*(s))| ds \\ &\leq \delta + k_1 b \int_0^{\phi(t)} |x(s) - x^*(s)| ds \\ &\leq \delta + k_1 b T ||x - x^*||_C. \end{aligned}$$

Thus

$$||x - x^*||_C \leq \delta + k_1 b T ||x - x^*||.$$

Hence

$$||x - x^*||_C \leq \frac{\delta}{1 - k_1 b T} = \varepsilon.$$

□

Definition 4. *The solution $x \in C[0,T]$ of (4) depends continuously on the delay function $\phi(t)$, if $\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|\phi(t) - \phi^*(t)| \leq \delta\ \rightarrow\ ||x - x^*||_C \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation

$$x^*(t) = g(t) + \int_0^{\phi^*(t)} k(t,s) f(s,x^*(s)) ds,\ t \in [0,T]. \tag{7}$$

Theorem 5. *Let the assumptions of Theorem 1 be satisfied, then the solution $x \in C[0,T]$ of (4) depends continuously on the delay function $\phi(t)$.*

Proof. Let x and x^* be the two solutions of (4) and (7), then

$$\begin{aligned}
|x(t) - x^*(t)| &= \left| g(t) + \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds - g(t) - \int_0^{\phi^*(t)} k(t,s) f(s,x^*(s)) ds \right| \\
&= \left| \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds - \int_0^{\phi^*(t)} k(t,s) f(s,x^*(s)) ds \right| \\
&= \left| \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds - \int_0^{\phi(t)} k(t,s) f(s,x^*(s)) ds \right. \\
&+ \left. \int_0^{\phi(t)} k(t,s) f(s,x^*(s)) ds - \int_0^{\phi^*(t)} k(t,s) f(s,x^*(s)) ds \right| \\
&\leq \int_0^{\phi(t)} |k(t,s)||f(s,x(s)) - f(s,x^*(s))| ds + \int_{\phi^*(t)}^{\phi(t)} |k(t,s)||f(s,x^*(s))| ds \\
&\leq k_1 b \int_0^{\phi(t)} |x(s) - x^*(s)| ds + k_1 \int_{\phi^*(t)}^{\phi(t)} (b|x^*(s)| + M) ds \\
&\leq k_1 b T ||x - x^*||_C + k_1 (b||x^*||_C + M) |\phi(t) - \phi^*(t)|.
\end{aligned}$$

Thus

$$||x - x^*||_C \leq k_1 b T ||x - x^*||_C + k_1 \delta (b||x^*||_C + M).$$

Hence

$$||x - x^*||_C \leq \frac{k_1 \delta (b||x^*||_C + M)}{1 - k_1 b T} = \varepsilon.$$

□

Definition 5. *The solution $x \in C[0,T]$ of (4) depends continuously on the function $k(t,s)$, if $\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|k(t,s) - k^*(t,s)| \leq \delta\ \rightarrow\ ||x - x^*||_C \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation

$$x^*(t) = g(t) + \int_0^{\phi(t)} k^*(t,s) f(s, x^*(s)) ds,\ t \in [0,T]. \tag{8}$$

Theorem 6. *Let the assumptions of Theorem 1 be satisfied, then the solution $x \in C[0,T]$ of (4) depends continuously on the function $k(t,s)$.*

Proof. Let x and x^* be the two solutions of (4) and (8), then

$$\begin{aligned}
& |x(t) - x^*(t)| \\
= & \left| g(t) + \int_0^{\phi(t)} k(t,s) f(s,x(s)) ds - g(t) - \int_0^{\phi(t)} k^*(t,s) f(s,x^*(s)) ds \right| \\
\leq & \int_0^{\phi(t)} |k(t,s) f(s,x(s)) - k^*(t,s) f(s,x^*(s))| ds \\
= & \int_0^{\phi(t)} |k(t,s) f(s,x(s)) - k^*(t,s) f(s,x(s)) + k^*(t,s) f(s,x(s)) - k^*(t,s) f(s,x^*(s))| ds \\
\leq & \int_0^{\phi(t)} |k(t,s) - k^*(t,s)| |f(s,x(s))| ds + \int_0^{\phi(t)} |k^*(t,s)| |f(s,x(s)) - f(s,x^*(s))| ds \\
\leq & \delta T (b||x||_C + M) + k_1 b T ||x - x^*||_C.
\end{aligned}$$

Hence

$$||x - x^*||_C \leq \frac{\delta T (b||x||_C + M)}{1 - k_1 b T} = \varepsilon.$$

□

Definition 6. *The solution $x \in C[0,T]$ of (4) depends continuously on the function $f(t,x)$, if $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that*

$$|f(t,x) - f^*(t,x)| \leq \delta\ \rightarrow\ ||x - x^*||_C \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation

$$x^*(t) = g(t) + \int_0^{\phi(t)} k(t,s) f^*(s, x^*(s)) ds,\ t \in [0,T]. \tag{9}$$

Theorem 7. *Let the assumptions of Theorem 1 be satisfied, then the solution $x \in C[0,T]$ of (4) depends continuously on the function $f(t,x)$.*

Proof. Let x and x^* be the two solutions of (4) and (9), then

$$\begin{aligned}
|x(t)-x^*(t)| &= \left|g(t)+\int_0^{\phi(t)} k(t,s)f(s,x(s))ds - g(t) - \int_0^{\phi(t)} k(t,s)f^*(s,x^*(s))ds\right| \\
&\leq \int_0^{\phi(t)} |k(t,s)||f(s,x(s))-f^*(s,x^*(s))|ds \\
&= \int_0^{\phi(t)} |k(t,s)||f(s,x(s))-f^*(s,x(s))+f^*(s,x(s))-f^*(s,x^*(s))|ds \\
&\leq k_1\int_0^{\phi(t)} |f(s,x(s))-f^*(s,x(s))|ds + k_1\int_0^{\phi(t)} |f^*(s,x(s))-f^*(s,x^*(s))|ds \\
&\leq k_1\delta T + k_1 b\int_0^{\phi(t)} |x(s)-x^*(s)|ds \\
&\leq k_1\delta T + k_1 bT||x-x^*||_C.
\end{aligned}$$

Hence

$$||x-x^*||_C \leq \frac{k_1\delta T}{1-k_1 bT} = \varepsilon.$$

□

Corollary 2. *Let the assumptions of Theorem 1 be satisfied, then the solution $u \in C^1[0,T]$ of (1)-(2) depends continuously on the the functions g, k, and f, and on the delay function $\phi(t)$.*

3. Study of an Absolutely Continuous Solution

In this section, we extend our investigation to the space $AC[0,T]$, where we relax the requirement for equation (1) to hold almost everywhere for $t \in (0,T]$,

$$\frac{du(t)}{dt} = g(t) + \int_0^{\phi(t)} k(t,s) f\big(s, \frac{du(s)}{ds}\big) ds, \ a.e.\ t \in (0,T]. \tag{10}$$

Let $\frac{du(t)}{dt} = x \in L^1[0,T]$, then the problem (10) with the initial condition (2) is equivalent to the integral equation (3) with the delay integro-functional equation (4).

We will prove the existence of a unique absolutely continuous solution $u \in AC[0,T]$ of the problem (10) with the initial condition (2), for this aim, we assume that:

- $(i)'$ $g : [0,T] \to \mathbb{R}$ is integrable.
- $(ii)'$ $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition with $L > 0$ such that

$$|f(t,x) - f(t,u)| \leq L|x-u|.$$

$(iii)^{'}$ $f(t,0) \in L^1[0,T]$ such that $\int_0^T |f(s,0)|ds \leq A$.

$(iv)^{'}$ $k_1LT < 1$.

Theorem 8. *Let the assumptions $(iii),(iv)$ and $(i)^{'} - (iv)^{'}$ be satisfied, then the problem (10) with (2) has a unique solution $u \in AC[0,T]$.*

Proof. From assumption $(ii)^{'}$, we have

$$|f(t,x)| \leq L|x| + |f(t,0)|.$$

Define the operator F_2 associated with (4) by

$$F_2x(t) = g(t) + \int_0^{\phi(t)} k(t,s)f(s,x(s))ds.$$

Let $x \in L^1[0,T]$, then

$$\begin{aligned}
|F_2x(t)| &= \left|g(t) + \int_0^{\phi(t)} k(t,s)f(s,x(s))ds\right| \\
&\leq |g(t)| + \int_0^{\phi(t)} |k(t,s)||f(s,x(s))|ds \\
&\leq |g(t)| + k_1 \int_0^{t} (L|x| + |f(s,0)|)ds \\
&\leq |g(t)| + k_1L||x||_{L^1} + k_1 \int_0^{T} |f(s,0)|ds \\
&\leq |g(t)| + k_1L||x||_{L^1} + k_1A.
\end{aligned}$$

Then

$$||F_2x||_{L^1} \leq ||g||_{L^1} + k_1LT||x||_{L^1} + k_1AT.$$

This prove that $F_2 : L^1[0,T] \to L^1[0,T]$.

Now, Let $x,z \in L^1[0,T]$, then

$$\begin{aligned}
|F_2x(t) - F_2z(t)| &= \left|g(t) + \int_0^{\phi(t)} k(t,s)f(s,x(s))ds - g(t) - \int_0^{\phi(t)} k(t,s)f(s,z(s))ds\right| \\
&\leq \int_0^{\phi(t)} |k(t,s)||f(s,x(s)) - f(s,z(s))|ds \\
&\leq k_1L \int_0^{\phi(t)} |x(s) - z(s)|ds \\
&\leq k_1L \int_0^{t} |x(s) - z(s)|ds
\end{aligned}$$

$$= \quad k_1 L ||x-z||_{L^1}.$$

Then

$$||F_2 x - F_2 z||_{L^1} \leq k_1 L T ||x-z||_{L^1},$$

since $k_1 LT < 1$, then F_2 is contraction. Then by using the Banach fixed point Theorem, there exists a unique solution $x \in L^1[0,T]$ of (4) and therefore (3) also has a unique solution $u \in C[0,T]$. Consequently, $\frac{du}{dt} = x(t) \in L^1[0,T]$, which implies $u \in AC[0,T]$. □

3.1. Hyers-Ulam stability.

Definition 7. [4, 5, 8] *Let the solution $x \in L^1[0,T]$ of (4) be exists. The delay integro-functional equation (4) is Hyers-Ulam stable, if $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that for any solution $x_s \in L^1[0,T]$ of (4) satisfying*

$$\Big|x_s(t) - g(t) - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta\Big| \leq \delta,$$

then

$$||x - x_s||_{L^1} \leq \varepsilon.$$

Theorem 9. *Let the assumptions of Theorem (8) be satisfied, then (4) is Hyers-Ulam stable.*

Proof. Let $-\delta \leq x_s(t) - g(t) - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta \leq \delta$, consider

$$\begin{aligned}
|x(t) - x_s(t)| &= \left| g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x(\theta)) d\theta - x_s(t) \right| \\
&= \left| g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x(\theta)) d\theta - \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta \right. \\
&+ \left. \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta - x_s(t) \right| \\
&\leq \left| \int_0^{\phi(t)} k(t,\theta) [f(\theta, x(\theta)) - f(\theta, x_s(\theta))] d\theta \right| \\
&+ \left| -x_s(t) + g(t) + \int_0^{\phi(t)} k(t,\theta) f(\theta, x_s(\theta)) d\theta \right| \\
&\leq \int_0^{\phi(t)} |k(t,\theta)| |f(\theta, x(\theta)) - f(\theta, x_s(\theta))| d\theta + \delta \\
&\leq \int_0^{\phi(t)} k_1 L |x(\theta) - x_s(\theta)| d\theta + \delta \\
&\leq k_1 L ||x - x_s||_{L^1} + \delta.
\end{aligned}$$

Then

$$||x-x_s||_{L^1} \leq k_1LT||x-x_s||_{L^1} + \delta T.$$

Hence

$$||x-x_s||_{L^1} \leq \frac{\delta T}{1-k_1LT} = \varepsilon.$$

Since $k_1LT < 1$, then (4) is Hyers-Ulam stable. □

Corollary 3. *Let the assumptions of Theorem (9) be satisfied, then the problem (10) with (2) is Hyers-Ulam stable.*

Proof. Consider

$$\begin{aligned} |u(t)-u_s(t)| &= \left| u_0 + \int_0^t x(\theta)d\theta - u_0 - \int_0^t x_s(\theta)d\theta \right| \\ &\leq \int_0^T |x(\theta)-x_s(\theta)|d\theta \\ &= ||x-x_s||_{L^1} \leq \varepsilon. \end{aligned}$$

□

3.2. Continuous Dependence.

In this section, we study the continuous dependence of the unique solution on the initial data u_0 and the functions g, k, and f, and on the delay function $\phi(t)$.

Definition 8. *The solution $u \in AC[0,T]$ of (10) with (2) depends continuously on the initial data u_0 and the function $x \in L^1[0,T]$, $if \ \forall \ \varepsilon > 0, \ \exists \ \delta(\varepsilon) > 0$ such that*

$$\max\left\{|u_0-u_0^*|, ||x-x^*||_{L^1}\right\} \leq \delta \ \rightarrow \ ||u-u^*||_C \leq \varepsilon,$$

where u^* is the unique solution of the integral equation (5).

Theorem 10. *Let the assumptions of Theorem 8 be satisfied, then the solution $u \in AC[0,T]$ of (10) with (2) depends continuously on the initial data u_0 and the function $x \in L^1[0,T]$.*

Proof. Let u and u^* be the two solutions of (3) and (5), then

$$\begin{aligned} |u(t)-u^*(t)| &\leq |u_0-u_0^*| + \int_0^t |x(s)-x^*(s)|ds \\ &\leq \delta + ||x-x^*||_{L^1} = 2\delta. \end{aligned}$$

Hence

$$||u-u^*||_{L^1} \leq 2\delta T = \varepsilon.$$

□

Definition 9. *The solution $x \in L^1[0,T]$ of (4) depends continuously on the function g, $if\ \forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|g(t)-g^*(t)| \leq \delta\ \rightarrow\ ||x-x^*||_{L^1} \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation (6).

Theorem 11. *Let the assumptions of Theorem 8 be satisfied, then the solution $x \in L^1[0,T]$ of (4) depends continuously on the function g.*

Proof. Let x and x^* be the two solutions of (4) and (6), then

$$\begin{aligned} |x(t)-x^*(t)| &\leq |g(t)-g^*(t)| + \int_0^{\phi(t)} |k(t,s)||f(s,x(s))-f(s,x^*(s))|ds \\ &\leq \delta + k_1 L \int_0^{\phi(t)} |x(s)-x^*(s)|ds \\ &\leq \delta + k_1 L \int_0^{t} |x(s)-x^*(s)|ds \\ &\leq \delta + k_1 L ||x-x^*||_{L^1}. \end{aligned}$$

Thus

$$||x-x^*||_{L^1} \leq \delta T + k_1 L T ||x-x^*||_{L^1}.$$

Hence

$$||x-x^*||_{L^1} \leq \frac{\delta T}{1-k_1 L T} = \varepsilon.$$

□

Definition 10. *The solution $x \in L^1[0,T]$ of (4) depends continuously on the delay function $\phi(t)$, $if\ \forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0$ such that*

$$|\phi(t)-\phi^*(t)| \leq \delta\ \rightarrow\ ||x-x^*||_{L^1} \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation (7).

Theorem 12. *Let the assumptions of Theorem 8 be satisfied, then the solution $x \in L^1[0,T]$ of (4) depends continuously on the delay function $\phi(t)$.*

Proof. Let x and x^* be the two solutions of (4) and (7), then by using the calculations of Theorem (5), we have

$$\begin{aligned}
|x(t)-x^*(t)| &\leq \int_0^{\phi(t)} |k(t,s)||f(s,x(s))-f(s,x^*(s))|ds + \int_{\phi^*(t)}^{\phi(t)} |k(t,s)||f(s,x^*(s))|ds \\
&\leq k_1 L \int_0^{\phi(t)} |x(s)-x^*(s)|ds + k_1 \int_{\phi^*(t)}^{\phi(t)} (L|x^*(s)|+|f(s,0)|)ds \\
&\leq k_1 L \int_0^{t} |x(s)-x^*(s)|ds + k_1 L \int_{\phi^*(t)}^{\phi(t)} |x^*(s)|ds + k_1 \int_{\phi^*(t)}^{\phi(t)} |f(s,0)|)ds.
\end{aligned}$$

But from the integrability of x^* and $f(t,0)$, we have

$$|\phi(t)-\phi^*(t)| \leq \delta \rightarrow \int_{\phi^*(t)}^{\phi(t)} |x^*(s)|ds \leq \varepsilon_1, \ \int_{\phi^*(t)}^{\phi(t)} |f(s,0)|ds \leq \varepsilon_2.$$

Then

$$|x(t)-x^*(t)| \leq k_1 L ||x-x^*||_{L^1} + k_1 L \varepsilon_1 + k_1 \varepsilon_2.$$

Thus

$$||x-x^*||_{L^1} \leq k_1 L T ||x-x^*||_{L^1} + k_1 L T \varepsilon_1 + k_1 T \varepsilon_2.$$

Hence

$$||x-x^*||_{L^1} \leq \frac{k_1 T (L\varepsilon_1 + \varepsilon_2)}{1 - k_1 L T} = \varepsilon.$$

□

Definition 11. *The solution $x \in L^1[0,T]$ of (4) depends continuously on the function $k(t,s)$, if $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that*

$$|k(t,s)-k^*(t,s)| \leq \delta \ \rightarrow \ ||x-x^*||_{L^1} \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation (8).

Theorem 13. *Let the assumptions of Theorem 8 be satisfied, then the solution $x \in L^1[0,T]$ of (4) depends continuously on the function $k(t,s)$.*

Proof. Let x and x^* be the two solutions of (4) and (8), then by using the calculations of Theorem (6), we have

$$\begin{aligned} |x(t)-x^*(t)| &\leq \int_0^{\phi(t)} |k(t,s)-k^*(t,s)||f(s,x(s))|ds+\int_0^{\phi(t)} |k^*(t,s)||f(s,x(s))-f(s,x^*(s))|ds \\ &\leq \int_0^t \delta(L|x(s)|+|f(s,0)|)ds+k_1L\int_0^t |x(s)-x^*(s)|ds \\ &\leq \delta L||x||_{L^1}+\delta\int_0^T |f(s,0)|ds+k_1L||x-x^*||_{L^1} \\ &\leq \delta L||x||_{L^1}+\delta A+k_1L||x-x^*||_{L^1}. \end{aligned}$$

Then

$$||x-x^*||_{L^1} \leq \delta LT||x||_{L^1}+\delta AT+k_1LT||x-x^*||_{L^1}.$$

Hence

$$||x-x^*||_{L^1} \leq \frac{\delta T(L||x||_{L^1}+A)}{1-k_1LT} = \varepsilon.$$

□

Definition 12. *The solution $x \in L^1[0,T]$ of (4) depends continuously on the function $f(t,x)$, if $\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0$ such that*

$$|f(t,x)-f^*(t,x)| \leq \delta \ \to \ ||x-x^*||_{L^1} \leq \varepsilon,$$

where x^* is the unique solution of the delay integro-functional equation (9).

Theorem 14. *Let the assumptions of Theorem 8 be satisfied, then the solution $x \in L^1[0,T]$ of (4) depends continuously on the function $f(t,x)$.*

Proof. Let x and x^* be the two solutions of (4) and (9), then by using the calculations of Theorem (7), we have

$$\begin{aligned} |x(t)-x^*(t)| &\leq k_1\int_0^{\phi(t)} |f(s,x(s))-f^*(s,x(s))|ds+k_1\int_0^{\phi(t)} |f^*(s,x(s))-f^*(s,x^*(s))|ds \\ &\leq k_1\delta\int_0^{\phi(t)} ds+k_1L\int_0^{\phi(t)} |x(s)-x^*(s)|ds \\ &\leq k_1\delta T+k_1L||x-x^*||_{L^1}. \end{aligned}$$

Then

$$||x-x^*||_{L^1} \leq k_1\delta T^2+k_1LT||x-x^*||_{L^1}.$$

Hence

$$||x - x^*||_{L^1} \leq \frac{k_1 \delta T^2}{1 - k_1 LT}.$$

□

Corollary 4. *Let the assumptions of Theorem 8 be satisfied, then the solution $u \in AC[0,T]$ of (10) with (2) depends continuously on the the functions g, k, and f, and on the delay function $\phi(t)$.*

4. Conclusion

In this study, we have explored the existence of a unique solution to the initial value problem (1)-(2), and (10) with (2) in two spaces $C^1[0,T]$ and $L^1[0,T]$. Additionally, we apply the Hyers-Ulam stability of the problem, demonstrating that small change in the problem lead to correspondingly small deviations in the solution. Furthermore, we proved the continuous dependence of the unique solution on all functions and initial data involved in the problem, ensuring that slight changes in inputs produce correspondingly minor effects on the solution of the problem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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