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SOME FIXED POINT RESULTS IN DISLOCATED METRIC SPACES UNDER MINIMUM CONDITIONS

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Abstract. The purposes of this paper are three; the first one is to prove the existence and uniqueness of a common fixed point for two pairs of occasionally weakly biased maps of type $(\mathscr{A})$ in a dislocated metric space, the second is to furnish an example to support our main result, and the third purpose is to present an application of this result to an integral equation.

Keywords: metric domain; partial metric space; metric-like space; dislocated metric space; occasionally weakly biased maps of type $(\mathscr{A})$; common fixed point theorems.

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1. Introduction and Preliminary Notes

In his thesis submitted for the degree of PhD at the university of Warwick in United Kingdom, Matthews suggested the concept of metric domain as a generalisation of metric spaces:

Definition 1.1. *([9]) A metric domain is a pair* $<\mathscr{D},\mathfrak{D}>$ *where* $\mathscr{D}$ *is a non-empty set, and* $\mathfrak{D}$ *is a function from* $\mathscr{D}\times\mathscr{D}$ *to* $\mathbb{R}_+$ *such that*

(C_1*)* $\forall\, a_1,\, a_2\in\mathscr{D}$, $\mathfrak{D}(a_1,a_2)=0\Rightarrow a_1=a_2$,

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(C_2) $\forall\, a_1,\, a_2 \in \mathscr{D},\, \mathfrak{D}(a_1,a_2) = \mathfrak{D}(a_2,a_1)$,

(C_3) $\forall\, a_1,\, a_2,\, a_3 \in \mathscr{D},\, \mathfrak{D}(a_1,a_2) \leq \mathfrak{D}(a_1,a_3) + \mathfrak{D}(a_3,a_2)$.

Again, the same author introduced another generalisation of metric spaces which is the notion of partial metric spaces.

Definition 1.2. *(*[10]*) A partial metric is a function $\mathfrak{D} : \mathscr{D} \times \mathscr{D} \to \mathbb{R}$, such that*

(C_1): $\forall\, a_1,\, a_2 \in \mathscr{D},\, a_1 = a_2 \Leftrightarrow \mathfrak{D}(a_1,a_1) = \mathfrak{D}(a_1,a_2) = \mathfrak{D}(a_2,a_2)$,

(C_2): $\forall\, a_1,\, a_2 \in \mathscr{D},\, \mathfrak{D}(a_1,a_1) \leq \mathfrak{D}(a_1,a_2)$,

(C_3): $\forall\, a_1,\, a_2 \in \mathscr{D},\, \mathfrak{D}(a_1,a_2) = \mathfrak{D}(a_2,a_1)$,

(C_4): $\forall\, a_1,\, a_2,\, a_3 \in \mathscr{D},\, \mathfrak{D}(a_1,a_3) \leq \mathfrak{D}(a_1,a_2) + \mathfrak{D}(a_2,a_3) - \mathfrak{D}(a_2,a_2)$.

In 2012, Amini-Harandi generalised the concept of partial metric spaces by proposing the concept of metric-like spaces.

Definition 1.3. *(*[2]*) A map $\mathfrak{D} : \mathscr{D} \times \mathscr{D} \to \mathbb{R}_+$, where $\mathscr{D}$ is a nonempty set, is said to be metric-like on $\mathscr{D}$ if for any a_1, a_2, $a_3 \in \mathscr{D}$, the following three conditions hold true:*

(C_1): $\mathfrak{D}(a_1,a_2) = 0 \Rightarrow a_1 = a_2$,

(C_2): $\mathfrak{D}(a_1,a_2) = \mathfrak{D}(a_2,a_1)$,

(C_3): $\mathfrak{D}(a_1,a_2) \leq \mathfrak{D}(a_1,a_3) + \mathfrak{D}(a_3,a_2)$.

The pair $(\mathscr{D},\mathfrak{D})$ is then called a metric-like space. Then a metric-like on $\mathscr{D}$ satisfies all of the conditions of a metric except that $\mathfrak{D}(l,l)$ may be positive for $l \in \mathscr{D}$.

Indeed, metric domains and metric-like spaces are the same and they also called d-metric or dislocated metric spaces.

In their paper [11], Mirkov et al. illustrated the relationships between the above mentioned notions as follows:

- Metric space $\to$ Partial metric space $\to$ Metric-like space.

On the other hand, recently, we generalised the well known concepts of weakly compatible [8], occasionally weakly compatible [1], weakly biased [7] and weakly biased maps of type $(\mathscr{A})$ [12] by proposing the concept of occasionally weakly biased maps of type $(\mathscr{A})$.

Definition 1.4. *([5]) Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be self-maps of a dislocated metric space $(\mathscr{D}, \mathfrak{D})$. The pair $(\mathscr{B}_1, \mathscr{B}_2)$ is said to be occasionally weakly $\mathscr{B}_1$-biased of type $(\mathscr{A})$ and occasionally weakly $\mathscr{B}_2$-biased of type $(\mathscr{A})$, respectively, if and only if, there exists a point τ in $\mathscr{D}$ such that $\mathscr{B}_1\tau = \mathscr{B}_2\tau$ implies*

$$\mathfrak{D}(\mathscr{B}_1\mathscr{B}_1\tau, \mathscr{B}_2\tau) \leq \mathfrak{D}(\mathscr{B}_2\mathscr{B}_1\tau, \mathscr{B}_1\tau),$$

$$\mathfrak{D}(\mathscr{B}_2\mathscr{B}_2\tau, \mathscr{B}_1\tau) \leq \mathfrak{D}(\mathscr{B}_1\mathscr{B}_2\tau, \mathscr{B}_2\tau),$$

respectively.

Now, recently in 2017, Bairagi et al. [4] discussed the existence and uniqueness of common fixed point and some new common fixed point theorems for two pairs of weakly compatible maps in a dislocated metric space.

Theorem 1.1. *Let $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_4 : \mathscr{D} \to \mathscr{D}$ be four self-maps of a complete dislocated metric space $(\mathscr{D}, \mathfrak{D})$ such that*

(1) $\mathscr{A}_3\mathscr{D} \subset \mathscr{A}_1\mathscr{D}$ and $\mathscr{A}_4\mathscr{D} \subset \mathscr{A}_2\mathscr{D}$,

(2) The pairs $(\mathscr{A}_1, \mathscr{A}_4)$ and $(\mathscr{A}_2, \mathscr{A}_3)$ are weakly compatible,

(3) for all x, $y \in \mathscr{D}$

$$\begin{aligned} \mathfrak{D}(\mathscr{A}_4 x, \mathscr{A}_3 y) \leq\ & c_1\mathfrak{D}(\mathscr{A}_1 x, \mathscr{A}_3 y) + c_2\mathfrak{D}(\mathscr{A}_2 y, \mathscr{A}_4 x) + c_3\mathfrak{D}(\mathscr{A}_1 x, \mathscr{A}_2 y) \\ & + c_4\mathfrak{D}(\mathscr{A}_2 y, \mathscr{A}_3 y), \end{aligned}$$

where $c_i \geq 0$ $(i = 1, \ldots, 4)$ satisfying $c_1 + c_2 + c_3 + c_4 < \frac{1}{2}$ or $(c_1 + c_2 + c_3 + c_4 \leq \frac{1}{2})$,

(4) the range of one of the maps $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ or $\mathscr{A}_4$ is a complete subspace of $\mathscr{D}$.

Then $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_4$ have a unique common fixed point in $\mathscr{D}$.

In this work, we will improve the above result by removing some conditions, of course using our new concept of occasionally weakly biased maps of type $(\mathscr{A})$. Moreover, we will furnish an example to support our theorem, and we will apply this result to an integral equation.

2. Discussion of the Existence and Uniqueness of Common Fixed Points

Theorem 2.1. *Let $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ be self-maps of a complete dislocated metric space $(\mathscr{D},\mathfrak{D})$ such that*

$$(2.1)\mathfrak{D}(\mathscr{P}x,\mathscr{Q}y) \leq a_1\mathfrak{D}(\mathscr{R}x,\mathscr{Q}y)+a_2\mathfrak{D}(\mathscr{S}y,\mathscr{P}x)+a_3\mathfrak{D}(\mathscr{R}x,\mathscr{S}y)+a_4\mathfrak{D}(\mathscr{S}y,\mathscr{Q}y)$$

for all x, $y \in \mathscr{D}$, where $a_i \geq 0$ for $i = 1,2,3,4$ satisfying $a_1+a_2+a_3+2a_4 < 1$. If $\mathscr{P}$ and $\mathscr{R}$ are occasionally weakly $\mathscr{R}$-biased of type $(\mathscr{A})$ and $\mathscr{Q}$ and $\mathscr{S}$ are occasionally weakly $\mathscr{S}$-biased of type $(\mathscr{A})$, then, the four maps have a unique common fixed point.

Proof. Since maps $\mathscr{P}$ and $\mathscr{R}$ as well as $\mathscr{Q}$ and $\mathscr{S}$ are occasionally weakly $\mathscr{R}$-biased and $\mathscr{S}$-biased of type $(\mathscr{A})$, then, there exist two elements w and z such that

$$\begin{aligned} \mathscr{P}w &= \mathscr{R}w \text{ implies } \mathfrak{D}(\mathscr{R}\mathscr{R}w,\mathscr{P}w) \leq \mathfrak{D}(\mathscr{P}\mathscr{R}w,\mathscr{R}w) \text{ and} \\ \mathscr{Q}z &= \mathscr{S}z \text{ implies } \mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}z) \leq \mathfrak{D}(\mathscr{Q}\mathscr{S}z,\mathscr{S}z). \end{aligned}$$

The existence and uniqueness of the common fixed point require four steps:

First step: Suppose that $\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)$ is positive, using inequality (2.1) we get

$$\begin{aligned} \mathfrak{D}(\mathscr{P}w,\mathscr{Q}z) &\leq a_1\mathfrak{D}(\mathscr{R}w,\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{S}z,\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{R}w,\mathscr{S}z)+a_4\mathfrak{D}(\mathscr{S}z,\mathscr{Q}z) \\ &= a_1\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{Q}z,\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)+a_4\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}z) \\ &= [a_1+a_2+a_3]\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)+a_4\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}z) \\ &\leq [a_1+a_2+a_3]\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)+a_4[\mathfrak{D}(\mathscr{Q}z,\mathscr{P}w)+\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z)] \\ &= [a_1+a_2+a_3+2a_4]\mathfrak{D}(\mathscr{P}w,\mathscr{Q}z) \\ &< \mathfrak{D}(\mathscr{P}w,\mathscr{Q}z) \end{aligned}$$

a contradiction unless $\mathscr{P}w = \mathscr{Q}z$.

Second step: Assume that $\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w) > 0$, then, the use of condition (2.1) gives

$$\begin{aligned} \mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{Q}z) &\leq a_1\mathfrak{D}(\mathscr{R}\mathscr{P}w,\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{S}z,\mathscr{P}\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{R}\mathscr{P}w,\mathscr{S}z) \\ &\quad +a_4\mathfrak{D}(\mathscr{S}z,\mathscr{Q}z); \end{aligned}$$

i.e.,

$$\begin{aligned}\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w) &\leq a_1\mathfrak{D}(\mathscr{R}\mathscr{R}w,\mathscr{P}w)+a_2\mathfrak{D}(\mathscr{P}w,\mathscr{P}\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{R}\mathscr{R}w,\mathscr{P}w)\\ &\quad +a_4\mathfrak{D}(\mathscr{P}w,\mathscr{P}w);\end{aligned}$$

Since $\mathscr{P}$ and $\mathscr{R}$ are occasionally weakly $\mathscr{R}$-biased of type $(\mathscr{A})$, we get

$$\begin{aligned}\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w) &\leq a_1\mathfrak{D}(\mathscr{P}\mathscr{R}w,\mathscr{R}w)+a_2\mathfrak{D}(\mathscr{P}w,\mathscr{P}\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{P}\mathscr{R}w,\mathscr{R}w)\\ &\quad +a_4\mathfrak{D}(\mathscr{P}w,\mathscr{P}w)\\ &= [a_1+a_2+a_3]\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)+a_4\mathfrak{D}(\mathscr{P}w,\mathscr{P}w)\\ &\leq [a_1+a_2+a_3]\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)\\ &\quad +a_4[\mathfrak{D}(\mathscr{P}w,\mathscr{P}\mathscr{P}w)+\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)]\\ &= [a_1+a_2+a_3+2a_4]\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)\\ &< \mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)\end{aligned}$$

a contradiction, hence $\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)\leq 0$ which implies $\mathfrak{D}(\mathscr{P}\mathscr{P}w,\mathscr{P}w)=0\Rightarrow\mathscr{P}\mathscr{P}w=\mathscr{P}w$, consequently $\mathscr{R}\mathscr{P}w=\mathscr{P}w$.

Third step: Now, consider that $\mathscr{Q}\mathscr{Q}z\neq\mathscr{Q}z$, then, inequality (2.1) gives

$$\begin{aligned}\mathfrak{D}(\mathscr{P}w,\mathscr{Q}\mathscr{Q}z) &\leq a_1\mathfrak{D}(\mathscr{R}w,\mathscr{Q}\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{S}\mathscr{Q}z,\mathscr{P}w)+a_3\mathfrak{D}(\mathscr{R}w,\mathscr{S}\mathscr{Q}z)\\ &\quad +a_4\mathfrak{D}(\mathscr{S}\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z);\end{aligned}$$

i.e.,

$$\begin{aligned}\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z) &\leq a_1\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}z)+a_3\mathfrak{D}(\mathscr{Q}z,\mathscr{S}\mathscr{S}z)\\ &\quad +a_4\mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}\mathscr{Q}z),\end{aligned}$$

using the relationship between $\mathscr{Q}$ and $\mathscr{S}$, we find

$$\begin{aligned}\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z) &\leq a_1\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z)+a_2\mathfrak{D}(\mathscr{Q}\mathscr{S}z,\mathscr{S}z)+a_3\mathfrak{D}(\mathscr{S}z,\mathscr{Q}\mathscr{S}z)\\ &\quad +a_4\mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}\mathscr{Q}z)\\ &= [a_1+a_2+a_3]\mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)+a_4\mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}\mathscr{Q}z)\end{aligned}$$

$$
\begin{aligned}
&\leq [a_1+a_2+a_3]\mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)+a_4[\mathfrak{D}(\mathscr{S}\mathscr{S}z,\mathscr{Q}z)+\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z)]\\
&\leq [a_1+a_2+a_3]\mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)+a_4[\mathfrak{D}(\mathscr{Q}\mathscr{S}z,\mathscr{S}z)+\mathfrak{D}(\mathscr{Q}z,\mathscr{Q}\mathscr{Q}z)]\\
&= [a_1+a_2+a_3+2a_4]\mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)\\
&< \mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)
\end{aligned}
$$

this contradiction confirms that $\mathfrak{D}(\mathscr{Q}\mathscr{Q}z,\mathscr{Q}z)=0 \Rightarrow \mathscr{Q}\mathscr{Q}z=\mathscr{Q}z$, in consequence, $\mathscr{S}\mathscr{Q}z=\mathscr{Q}z$. Therefore $\mathscr{P}w=\mathscr{R}w=\mathscr{Q}z=\mathscr{S}z=\theta$ is a common fixed point of maps $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$.

Fourth step: Let us envisage the existence of another common fixed point σ of the four maps, then, by condition (2.1) we obtain

$$\mathfrak{D}(\mathscr{P}\theta,\mathscr{Q}\sigma)\leq a_1\mathfrak{D}(\mathscr{R}\theta,\mathscr{Q}\sigma)+a_2\mathfrak{D}(\mathscr{S}\sigma,\mathscr{P}\theta)+a_3\mathfrak{D}(\mathscr{R}\theta,\mathscr{S}\sigma)+a_4\mathfrak{D}(\mathscr{S}\sigma,\mathscr{Q}\sigma);$$

i.e.,

$$
\begin{aligned}
\mathfrak{D}(\theta,\sigma) &\leq a_1\mathfrak{D}(\theta,\sigma)+a_2\mathfrak{D}(\sigma,\theta)+a_3\mathfrak{D}(\theta,\sigma)+a_4\mathfrak{D}(\sigma,\sigma)\\
&= [a_1+a_2+a_3]\mathfrak{D}(\theta,\sigma)+a_4\mathfrak{D}(\sigma,\sigma)\\
&\leq [a_1+a_2+a_3+2a_4]\mathfrak{D}(\theta,\sigma)
\end{aligned}
$$

a contradiction, except if $\sigma=\theta$; i.e., the common fixed point is unique and this completes the proof. □

Now, we give an illustrative example which supports our result.

Example 2.1. *Let $\mathscr{D}=[-1,+\infty)$ be endowed with the dislocated metric $\mathfrak{D}(x,y)=\max\{|x|,|y|\}$. Taking $a_1=a_2=a_4=\frac{1}{17}$, $a_3=\frac{3}{4}$, and define*

$$\mathscr{P}x=\begin{cases} x \text{ if } x\in[-1,0]\\ \frac{1}{10} \text{ if } x\in(0,+\infty),\end{cases}\quad \mathscr{Q}x=\begin{cases} x \text{ if } x\in[-1,0]\\ \frac{1}{20} \text{ if } x\in(0,+\infty),\end{cases}$$

$$\mathscr{R}x=\begin{cases} -10x \text{ if } x\in[-1,0]\\ x+10 \text{ if } x\in(0,+\infty),\end{cases}\quad \mathscr{S}x=\begin{cases} -20x \text{ if } x\in[-1,0]\\ x+20 \text{ if } x\in(0,+\infty).\end{cases}$$

We have the following table:

	$x,\ y\in[-1,0]$	$x,\ y\in(0,+\infty)$	$x\in[-1,0],\ y\in(0,+\infty)$	$y\in[-1,0],\ x\in(0,+\infty)$
$\mathfrak{D}(\mathscr{P}x,\mathscr{Q}y)$	$\max\{\lvert x\rvert,\lvert y\rvert\}$	$\frac{1}{10}$	$\max\left\{\lvert x\rvert,\frac{1}{20}\right\}$	$\max\left\{\frac{1}{10},\lvert y\rvert\right\}$
$\mathfrak{D}(\mathscr{R}x,\mathscr{Q}y)$	$\max\{-10x,\lvert y\rvert\}$	$\max\{x+10,\frac{1}{20}\}$	$\max\left\{-10x,\frac{1}{20}\right\}$	$\max\{x+10,\lvert y\rvert\}$
$\mathfrak{D}(\mathscr{S}y,\mathscr{P}x)$	$\max\{-20y,\lvert x\rvert\}$	$\max\{y+20,\frac{1}{10}\}$	$\max\{y+20,\lvert x\rvert\}$	$\max\left\{-20y,\frac{1}{10}\right\}$
$\mathfrak{D}(\mathscr{R}x,\mathscr{S}y)$	$\max\{-10x,-20y\}$	$\max\{x+10,y+20\}$	$\max\{-10x,y+20\}$	$\max\{x+10,-20y\}$
$\mathfrak{D}(\mathscr{S}y,\mathscr{Q}y)$	$\max\{-20y,\lvert y\rvert\}$	$\max\{y+20,\frac{1}{20}\}$	$\max\{y+20,\frac{1}{20}\}$	$\max\{-20y,\lvert y\rvert\}$

so,

(1) *first case: for $-1\le x,y\le 0$, we have $\mathscr{P}x=x$, $\mathscr{Q}y=y$, $\mathscr{R}x=-10x$, $\mathscr{S}y=-20y$ and*

$$\begin{aligned}\mathfrak{D}(\mathscr{P}x,\mathscr{Q}y) &= \max\{|x|,|y|\}\\ &\le \frac{1}{17}\max\{-10x,|y|\}+\frac{1}{17}\max\{-20y,|x|\}\\ &\quad+\frac{3}{4}\max\{-10x,-20y\}+\frac{1}{17}\max\{-20y,|y|\}\\ &= a_1\mathfrak{D}(\mathscr{R}x,\mathscr{Q}y)+a_2\mathfrak{D}(\mathscr{S}y,\mathscr{P}x)+a_3\mathfrak{D}(\mathscr{R}x,\mathscr{S}y)\\ &\quad+a_4\mathfrak{D}(\mathscr{S}y,\mathscr{Q}y),\end{aligned}$$

(2) *second case: for $0<x,y<+\infty$, we have $\mathscr{P}x=\frac{1}{10}$, $\mathscr{Q}y=\frac{1}{20}$, $\mathscr{R}x=x+10$, $\mathscr{S}y=y+20$ and*

$$\begin{aligned}\mathfrak{D}(\mathscr{P}x,\mathscr{Q}y) &= \frac{1}{10}\\ &\le \frac{x+10}{17}+\frac{59(y+20)}{68}\\ &= \frac{4x+59y+1220}{68}\\ &= a_1\mathfrak{D}(\mathscr{R}x,\mathscr{Q}y)+a_2\mathfrak{D}(\mathscr{S}y,\mathscr{P}x)+a_3\mathfrak{D}(\mathscr{R}x,\mathscr{S}y)\\ &\quad+a_4\mathfrak{D}(\mathscr{S}y,\mathscr{Q}y),\end{aligned}$$

(3) *third case: for $-1\le x\le 0<y<+\infty$, we have $\mathscr{P}x=x$, $\mathscr{Q}y=\frac{1}{20}$, $\mathscr{R}x=-10x$, $\mathscr{S}y=y+20$ and*

$$\begin{aligned}\mathfrak{D}(\mathscr{P}x,\mathscr{Q}y) &= \max\left\{|x|,\frac{1}{20}\right\}\\ &\le \frac{1}{17}\max\left\{-10x,\frac{1}{20}\right\}+\frac{59(y+20)}{68}\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{17}\max\left\{-10x, \frac{1}{20}\right\} + \frac{59y+1180}{68} \\ &= a_1\mathfrak{D}(\mathscr{R}x, \mathscr{Q}y) + a_2\mathfrak{D}(\mathscr{S}y, \mathscr{P}x) + a_3\mathfrak{D}(\mathscr{R}x, \mathscr{S}y) \\ &\quad + a_4\mathfrak{D}(\mathscr{S}y, \mathscr{Q}y), \end{aligned}$$

(4) fourth case: for $-1 \leq y \leq 0 < x < +\infty$, we have $\mathscr{P}x = \frac{1}{10}$, $\mathscr{Q}y = y$, $\mathscr{R}x = x+10$, $\mathscr{S}y = -20y$ and

$$\begin{aligned} \mathfrak{D}(\mathscr{P}x, \mathscr{Q}y) &= \max\left\{\frac{1}{10}, |y|\right\} \\ &\leq \frac{x+10}{17} + \frac{1}{17}\max\left\{-20y, \frac{1}{10}\right\} \\ &\quad + \frac{3}{4}\max\{x+10, -20y\} + \frac{1}{17}\max\{-20y, |y|\} \\ &= a_1\mathfrak{D}(\mathscr{R}x, \mathscr{Q}y) + a_2\mathfrak{D}(\mathscr{S}y, \mathscr{P}x) + a_3\mathfrak{D}(\mathscr{R}x, \mathscr{S}y) \\ &\quad + a_4\mathfrak{D}(\mathscr{S}y, \mathscr{Q}y). \end{aligned}$$

Note that $\mathscr{P}$ and $\mathscr{R}$ are occasionally weakly $\mathscr{R}$-biased of type $(\mathscr{A})$ and $\mathscr{Q}$ and $\mathscr{S}$ are occasionally weakly $\mathscr{S}$-biased of type $(\mathscr{A})$. So, all hypotheses of Theorem 2.1 are satisfied and 0 *is the unique common fixed point of the four maps.*

Remark 2.1. *Note that Theorem 2.1 and Theorem 2.2 of* [4] *are not applicable because we have $\mathscr{P}\mathscr{D} = [-1,0] \cup \{\frac{1}{10}\} \nsubseteq \mathscr{S}\mathscr{D} = [0,+\infty)$ and $\mathscr{Q}\mathscr{D} = [-1,0] \cup \{\frac{1}{20}\} \nsubseteq \mathscr{R}\mathscr{D} = [0,+\infty)$.*

3. Application to an Integral Equation

Consider the following integral equation:

$$\eta(x) = f_i(\eta(x)) + \int_0^x y(x,t)j_i(t,\eta(t))dt + \int_0^1 z(x,t)l_i(t,\eta(t))dt \tag{3.1}$$

for all $x \in [0,1]$, where

(1) $f_i : [0,1] \to \mathbb{R}$, $i = 1,2$ are continuous,

(2) $y(x,t)$, $z(x,t) : [0,1] \times [0,1] \to \mathbb{R}_+$ are continuous functions,

(3) j_i, $l_i : [0,1] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous functions.

Let $\mathscr{D} = C([0,1])$ be the set of real continuous functions on $[0,1]$, endowed with the dislocated metric

$$\begin{aligned}\mathfrak{D}(\eta,\theta) &= \|\eta\|_\infty + \|\theta\|_\infty \\ &= \max_{x\in[0,1]} \eta(x) + \max_{x\in[0,1]} \theta(x)\end{aligned}$$

for all η, $\theta \in \mathscr{D}$. It is clear to see that $(\mathscr{D},\mathfrak{D})$ is a dislocated metric space.

Theorem 3.1. *The integral equation* (3.1) *has a unique solution in* $\mathscr{D}$ *for* $\rho\xi_2 < 1$ *and* $\frac{\varpi+\rho\xi_1}{1-\rho\xi_2} = a_3 < 1$ *if the following conditions hold:*

(1) $\int_0^1 \max_{x\in[0,1]} |y(x,t)|dt = \xi_1 < +\infty$*,*
(2) $\int_0^1 \max_{x\in[0,1]} |z(x,t)|dt = \xi_2 < +\infty$*,*
(3) the functions commute at their each coincidence points,
(4) there is $0 < \rho < 1$ *such that for all* $t \in [0,1]$ *and* $\eta \in \mathscr{D}$*,* $|j_i(t,\eta(t))| \leq \rho|\eta(t)|$ *for* $i = 1,2$*,*
(5) there is $0 < \rho < 1$ *such that for all* $t \in [0,1]$ *and* $\eta \in \mathscr{D}$*,* $|l_i(t,\eta(t))| \leq \rho|\eta(t)|$ *for* $i = 1,2$*,*
(6) there is $0 < \varpi < 1$ *such that for all* $t \in [0,1]$*,* $|f_i(t)| \leq \varpi|t|$*.*

Proof. Define $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{M}$, $\mathscr{N}$, $\mathscr{R}$, $\mathscr{S} : \mathscr{D} \to \mathscr{D}$ by

$$\begin{aligned}\mathscr{P}\eta(x) &= f_1(\eta(x)) + \int_0^x y(x,t)j_1(t,\eta(t))dt \\ \mathscr{Q}\eta(x) &= f_2(\eta(x)) + \int_0^x y(x,t)j_2(t,\eta(t))dt \\ \mathscr{M}\eta(x) &= \int_0^1 z(x,t)l_1(t,\eta(t))dt \\ \mathscr{N}\eta(x) &= \int_0^1 z(x,t)l_2(t,\eta(t))dt \\ \mathscr{R}\eta(x) &= (\mathscr{I} - \mathscr{M})\eta(x) \\ \mathscr{S}\eta(x) &= (\mathscr{I} - \mathscr{N})\eta(x)\end{aligned}$$

where $\mathscr{I}$ is the identity function on $\mathscr{D}$.

By the virtue of condition 3, we can see that $\mathscr{P}$ and $\mathscr{R}$ as well as $\mathscr{Q}$ and $\mathscr{S}$ are occasionally weakly $\mathscr{R}$-biased (respectively $\mathscr{S}$-biased) of type $(\mathscr{A})$.

Now, we prove that condition (2.1) of Theorem 2.1 is satisfied.

$$\begin{aligned}
|\mathscr{P}\eta(x)| &= \left| f_1(\eta(x)) + \int_0^x y(x,t) j_1(t,\eta(t))dt \right| \\
&\leq |f_1(\eta(x))| + \left| \int_0^x y(x,t) j_1(t,\eta(t))dt \right| \\
&\leq |f_1(\eta(x))| + \int_0^x |y(x,t)||j_1(t,\eta(t))|dt \\
&\leq |f_1(\eta(x))| + \rho \int_0^x |y(x,t)||\eta(t)|dt \\
&\leq \varpi \max_{x\in[0,1]} |\eta(x)| + \rho \int_0^1 |y(x,t)| \max_{t\in[0,1]} |\eta(t)|dt \\
&\leq \varpi ||\eta||_\infty + \rho \|\eta\|_\infty \int_0^1 \max_{x\in[0,1]} |y(x,t)|dt
\end{aligned}$$

implies that

$$\|\mathscr{P}\eta\|_\infty \leq (\varpi + \rho\xi_1)\|\eta\|_\infty.$$

It follows that, for all η, $\theta \in \mathscr{D}$

$$\mathfrak{D}(\mathscr{P}\eta, \mathscr{Q}\theta) \leq (\varpi + \rho\xi_1)\mathfrak{D}(\eta,\theta). \tag{3.2}$$

Similarly, we have

$$\begin{aligned}
|\mathscr{M}\eta(x)| &= \left| \int_0^1 z(x,t) l_1(t,\eta(t))dt \right| \\
&\leq \int_0^1 |z(x,t)||l_1(t,\eta(t))|dt \\
&\leq \rho \int_0^1 |z(x,t)||\eta(t)|dt \\
&\leq \rho \int_0^1 |z(x,t)| \max_{t\in[0,1]} |\eta(t)|dt \\
&\leq \rho \|\eta\|_\infty \int_0^1 \max_{x\in[0,1]} |z(x,t)|dt
\end{aligned}$$

implies that

$$\|\mathscr{M}\eta\|_\infty \leq \rho\xi_2\|\eta\|_\infty.$$

It follows that, for all η, $\theta \in \mathscr{D}$

$$\mathfrak{D}(\mathscr{M}\eta, \mathscr{N}\theta) \leq \rho\xi_2\mathfrak{D}(\eta,\theta).$$

Hence, we have

$$\begin{aligned}
\mathfrak{D}(\mathscr{R}\eta,\mathscr{S}\theta) &= \|\mathscr{R}\eta\|_\infty + \|\mathscr{S}\theta\|_\infty \\
&= \max_{x\in[0,1]} \mathscr{R}\eta(x) + \max_{x\in[0,1]} \mathscr{S}\theta(x) \\
&= \max_{x\in[0,1]} [\mathscr{R}\eta(x) + \mathscr{S}\theta(x)] \\
&= \max_{x\in[0,1]} [(\mathscr{I}-\mathscr{M})\eta(x) + (\mathscr{I}-\mathscr{N})\theta(x)] \\
&= \max_{x\in[0,1]} [\eta(x)+\theta(x)] - \max_{x\in[0,1]} [\mathscr{M}\eta(x) + \mathscr{N}\theta(x)] \\
&= \mathfrak{D}(\eta,\theta) - \mathfrak{D}(\mathscr{M}\eta,\mathscr{N}\theta) \\
&\geq \mathfrak{D}(\eta,\theta) - \rho\xi_2\mathfrak{D}(\eta,\theta) \\
&= (1-\rho\xi_2)\mathfrak{D}(\eta,\theta)
\end{aligned}$$

which implies that

$$\mathfrak{D}(\eta,\theta) \leq \left(\frac{1}{1-\rho\xi_2}\right)\mathfrak{D}(\mathscr{R}\eta,\mathscr{S}\theta). \tag{3.3}$$

From (3.2) and (3.3), we get

$$\begin{aligned}
\mathfrak{D}(\mathscr{P}\eta,\mathscr{Q}\theta) &\leq \left(\frac{\varpi+\rho\xi_1}{1-\rho\xi_2}\right)\mathfrak{D}(\mathscr{R}\eta,\mathscr{S}\theta) \\
&= a_3\mathfrak{D}(\mathscr{R}\eta,\mathscr{S}\theta) \\
&\leq a_1\mathfrak{D}(\mathscr{R}\eta,\mathscr{Q}\theta) + a_2\mathfrak{D}(\mathscr{S}\theta,\mathscr{P}\eta) + a_3\mathfrak{D}(\mathscr{R}\eta,\mathscr{S}\theta) + a_4\mathfrak{D}(\mathscr{S}\theta,\mathscr{Q}\theta),
\end{aligned}$$

thus, all the conditions of Theorem 2.1 are satisfied. Therefore, there is a unique point $\eta' \in \mathscr{D}$ such that $\mathscr{P}\eta' = \mathscr{Q}\eta' = \mathscr{R}\eta' = \mathscr{S}\eta'$, consequently, η' is a unique solution of 3.1. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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