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MODELING THE DYNAMICS OF BUSINESS CYCLE WITH MEMORY IN GOODS AND MONEY MARKETS VIA THE GENERALIZED HATTAF FRACTIONAL DERIVATIVE

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Abstract. In this paper, we propose a new IS-LM model that describes the dynamics of business cycle by taking into account the memory effect in both goods and money markets involving the generalized Hattaf fractional (GHF) derivative. By mains of the fixed point theory, we prove the existence and uniqueness of solutions of our proposed fractional model. Moreover, the existence of the economic equilibrium and its local stability are rigorously established. Finally, numerical simulations are presented to illustrate the effect of memory on the dynamical behavior of the proposed model.

Keywords: business cycle, memory; fixed point theory; Hattaf fractional derivative; stability.

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1. INTRODUCTION

Memory means the existence of a response or endogenous variable at the current moment that is dependent on historical changes in the input or exogenous variable over a finite or infinite

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period of time. In other words, a system's memory is the ability of its current state to be influenced by past changes in its input conditions. This characteristic permits the system to retain, retrieve and remember past information and use it to modify future responses. In this sense, memory is a process that allows the system to incorporate effects from the past and use them to shape its current response.

The concept of memory can be seen in many different domains. In biology, memory concerns the ability of living organisms to store and recall information, often linked to learning which allows behaviors to be adjusted according to experienced responses and to adaptation which ensures the evolution of organisms in the long term. In materials science, materials with shape memory exhibit this property by returning to their original shape when exposed to certain conditions, such as heat, after being deformed. In psychology, memory helps explain why certain information, such as items at the beginning or end of a list, is easier to remember, known as precedence and repetition effects. Furthermore, memory in economics refers to the set of knowledge, experiences and information that a society has accumulated over time. This includes economic events, policies and social and cultural norms that influence economic behavior. This memory is essential for consumer and investor decisions because past events, such as financial crises, affect their risk management and future choices. Therefore, memory plays a key role in economic stability, growth and development.

On the other hand, to model the memory effect, the fractional order derivative proves to be a particularly powerful and suitable tool. It captures and accurately represents the complex dependency between the current state of a system and its history, offering an advanced mathematical method for modeling economic phenomena where the influence of the past plays a significant role in future behavior. For instance, Xie et al. [1] studied a new delayed fractional-order model for business cycle with a general liquidity preference function and an investment function and they obtained some conditions of stability and Hopf bifurcation. In 2021, Wang et al. [2] demonstrated the influence of the fractional order on the bifurcation threshold on a Kaldorian business cycle model with investment and money supply time delay using Caputo fractional derivative.

The aim of this work is to extend and generalize the main results of [3] without time delay to the fractional framework using the new generalized Hattaf fractional (GHF) derivative [4]. This new fractional derivative generalizes the most fractional derivatives with non-singular kernels like the Caputo-Fabrizio fractional derivative [5], the Atangana-Baleanu fractional derivative [6] and the recent weighted Atangana–Baleanu fractional derivative presented [7]. Moreover, many researchers have used the GHF derivative to model the dynamics of various scientific and engineering fields [8, 9, 10].

The rest of our paper is organized as follows. The next section is devoted with some basic definitions and results needed for this work. In section 3, we reconstruct the fractional IS-LM model with the new GHF derivative. After, we prove the main results of this work through the existence and uniqueness of the solution, the existence of economic equilibrium and its stability analysis. Numerical simulations are presented in Section 4 to illustrate our theoretical results. We end up our paper with a conclusion in Section 5.

2. PRELIMINARIES

We start this section by introducing some fundamental definitions and results from fractional calculus that are essential for this study.

Definition 2.1. [4] *Let $p \in [0,1)$, $q,\gamma > 0$ and $f \in H^1(a,b)$. The GHF derivative of order p in the Caputo sense of the function $f(t)$ with respect to the weight function $\omega(t)$ is defined as follows:*

$$D^{p,q,\gamma}_{a,t,\omega} f(t) = \frac{N(p)}{1-p}\frac{1}{\omega(t)}\int_a^t E_q[-\mu_p(t-\tau)^{\gamma}]\frac{d}{d\tau}(\omega f)(\tau)d\tau, \tag{1}$$

where $\omega \in C^1(a,b)$, $\omega > 0$ on $[a,b]$, $N(p)$ is a normalization function such that $N(0) = N(1) = 1$, $\mu_p = \frac{p}{1-p}$ and $E_q(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(qk+1)}$ is the Mittag-Leffler function of parameter q.

The GHF derivative introduced in the above definition generalizes and extends many special cases. In the fact, when $\omega(t) = 1$ and $q = \gamma = 1$, we get the Caputo-Fabrizio fractional derivative [5], which is given by

$$^{C}D^{p,1,1}_{a,t,1} f(t) = \frac{N(p)}{1-p}\int_a^t \exp[-\mu_p(t-\tau)]f'(\tau)d\tau.$$

We obtain the Atangana-Baleanu fractional derivative [6] when $\omega(t)=1$ and $q=\gamma=p$, which is given by

$$^{C}D_{a,t,1}^{p,p,p}f(t)=\frac{N(p)}{1-p}\int_a^t E_p[-\mu_p(t-\tau)^p]f'(\tau)d\tau.$$

For $q=\gamma=p$, we get the weighted Atangana-Baleanu fractional derivative [7], which is given by

$$^{C}D_{a,t,\omega}^{p,p,p}f(t)=\frac{N(p)}{1-p}\frac{1}{\omega(t)}\int_a^t E_p[-\mu_p(t-\tau)^p]\frac{d}{d\tau}(\omega f)(\tau)d\tau.$$

For simplicity, we denote $^{C}D_{a,t,\omega}^{p,q,q}$ by $\mathscr{D}_{a,\omega}^{p,q}$. According to [4], the generalized Hattaf fractional integral operator associated to $\mathscr{D}_{a,\omega}^{p,q}$ is defined by

$$\mathscr{I}_{a,\omega}^{p,q}f(t)=\frac{1-p}{N(p)}f(t)+\frac{p}{N(p)}\ {}^{RL}\mathscr{I}_{a,\omega}^{q}f(t), \tag{2}$$

where $^{RL}\mathscr{I}_{a,\omega}^{q}$ is the standard weighted Riemann-Liouville fractional integral of order q defined by

$$^{RL}\mathscr{I}_{a,\omega}^{q}f(t)=\frac{1}{\Gamma(q)}\frac{1}{\omega(t)}\int_a^t (t-\tau)^{q-1}\omega(\tau)f(\tau)d\tau. \tag{3}$$

Lemma 2.2. [4] *Let $p\in[0,1]$, $q>0$ and $f\in H^1(a,b)$. Then we have the following property:*

$$\mathscr{I}_{a,\omega}^{p,q}(\mathscr{D}_{a,\omega}^{p,q}f)(t)=f(t)-\frac{\omega(a)f(a)}{\omega(t)}. \tag{4}$$

Lemma 2.3. [4] *The Laplace transform of $\omega(t)\mathscr{D}_{0,\omega}^{p,q}$ is given by*

$$\mathscr{L}\{\omega(t)\mathscr{D}_{0,\omega}^{p,q}f(t)\}(s)=\frac{N(p)}{1-p}\frac{s^q\mathscr{L}\{\omega(t)f(t)\}(s)-s^{q-1}\omega(0)f(0)}{s^q+\mu_p}.$$

Lemma 2.4. [11] *Let $q>0$, $x(t)$, $u(t)$ be nonnegative functions and $v(t)=M\geq 0$ with $N(p)-(1-p)M>0$. If*

$$x(t)\leq u(t)+M\mathscr{I}_{0,\omega}^{p,q}x(t),$$

then

$$x(t)\leq\frac{N(p)}{N(p)-(1-p)M}\left[u(t)+\int_0^t\sum_{n=1}^{+\infty}\frac{(pM)^n(t-\tau)^{nq-1}u(\tau)}{\Gamma(nq)\,[N(p)-(1-p)M]^n}d\tau\right].$$

Furthermore, if in addition $u(t)$ is a nondecreasing function on $[0,T]$, we have

$$x(t)\leq\frac{N(p)u(t)}{N(p)-(1-p)M}E_q\left(\frac{pMT^q}{N(p)-(1-p)M}\right).$$

For the existence and uniqueness of solution of our model, we need the following result.

Lemma 2.5. *(Krasnoselskii's fixed point theorem* [12, 13]*) Let E be a nonempty closed convex subset of a Banach space $(\mathscr{C}, ||.||)$. Suppose that $\mathscr{G}_1$ and $\mathscr{G}_2$ map E into $\mathscr{C}$ such that*

(i): *$\mathscr{G}_1\psi_1 + \mathscr{G}_2\psi_2 \in E$, for all $\psi_1, \psi_2 \in E$;*

(ii): *$\mathscr{G}_1$ is a contraction mapping;*

(iii): *$\mathscr{G}_2$ is continuous and $\mathscr{G}_2(E)$ is contained in a compact subset of $\mathscr{C}$.*

Then $\mathscr{G}_1 + \mathscr{G}_2$ has a fixed point $\psi \in E$.

3. Main Results

In this section, we first propose a fractional IS-LM model involving the GHF derivative. This model is governed by the following nonlinear system of fractional differential equations (FDEs),

$$\begin{cases} \mathscr{D}^{p,q}_{0,\omega}Y(t) = \alpha[I(Y(t),K(t),R(t)) - s_1Y(t) - s_2R(t)], \\ \mathscr{D}^{p,q}_{0,\omega}K(t) = I(Y(t),K(t),R(t)) - \delta K(t), \\ \mathscr{D}^{p,q}_{0,\omega}R(t) = \beta[L(Y(t),R(t)) - \overline{M}, \end{cases} \tag{5}$$

where $Y(t)$, $K(t)$ and $R(t)$ respectively represent the gross product, the capital stock and the interest rate at time t. The parameter α is the adjustment coefficient in the goods market while β is the coefficient of adjustment in the money market. The demand for money or liquidity preference function is labeled by $L(Y,R)$ while the investment is presented by $I(Y,K,R)$. The constant money supply is denoted by $\overline{M}$. The positive constants s_1 and s_2 are the propensities to save. Finally, δ is depreciation rate of the capital stock. In addition, we consider model (5) with the initial conditions:

$$\begin{cases} Y(0) = Y_0, \\ K(0) = K_0, \\ R(0) = R_0. \end{cases} \tag{6}$$

Next, we investigate the existence and uniqueness of solutions of system (5) by means of fixed point theory. As in [3], we assume that the liquidity preference function $L(Y,R)$ is of the form $L(Y,R) = \mathscr{L}(Y) - \gamma R$, where γ measures the variation of demand of liquidity related to interest rate.

Let $\mathscr{C} = C([0,b],\mathbb{R}^3)$ be the Banach space of continuous functions g from $[0,b]$ into $\mathbb{R}^3$ equipped with the sup-norm

$$\|g\| = \sup_{t\in[0,b]} |g(t)|.$$

The system (5) can be written as follows:

$$\begin{cases} \mathscr{D}_{0,\omega}^{p,q} Z(t) = F(t,Z(t)), \\ Z(0) = Z_0, \end{cases} \tag{7}$$

where $Z(t) = (Y(t),K(t),R(t))^T$, $Z_0 = (Y(0),K(0),R(0))^T$ and the vector function F is given by

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \alpha[I(Y,K,R) - s_1 Y - s_2 K] \\ I(Y,K,R) - \delta K \\ \beta[\mathscr{L}(Y) - \gamma R - \overline{M}] \end{pmatrix}.$$

Applying the Hattaf fractional integral to both sides of (5), we get

$$Z(t) = \frac{\omega(0)Z(0)}{\omega(t)} + \mathscr{I}_{0,\omega}^{p,q} F(t,Z(t)) \tag{8}$$

$$= \frac{\omega(0)Z_0}{\omega(t)} + \frac{1-p}{N(p)} F(t,Z(t)) + \frac{p}{N(p)} \frac{1}{\Gamma(q)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{q-1} \omega(\tau) F(\tau,Z(\tau)) d\tau.$$

We will now prove that F is Lipschitz in its second variable. This leads to the following lemma.

Lemma 3.1. *The vector F is Lipschtiz in its second variable.*

Proof. We have

$$\begin{aligned} |F(t,Z_1) - F(t,Z_2)| &= |F_1(t,Z_1(t)) - F_1(t,Z_2(t))| + |F_2(Z_1(t)) - F_2(Z_2(t))| \\ &\quad + |F_3(Z_1(t)) - F_3(Z_2(t))| \\ &= \alpha |I(Y_1(t),K_1(t),R_1(t)) - s_1 Y_1(t) - s_2 R_1(t) - I(Y_2(t),K_2(t),R_2(t)) \\ &\quad + s_1 Y_2(t) + s_2 R_2(t)| + |I(Y_1(t),K_1(t),R_1(t)) \\ &\quad - \delta K_1(t) - I(Y_2(t),K_2(t),R_2(t)) + \delta K_2(t)| \\ &\quad + \beta |\mathscr{L}(Y_1(t)) - \gamma R_1(t) - \overline{M} - \mathscr{L}(Y_2(t)) + \gamma R_2(t) + \overline{M}| \\ &\leq (\alpha+1) |I(Y_1(t),K_1(t),R_1(t)) - I(Y_2(t),K_2(t),R_2(t))| \end{aligned}$$

$$
\begin{aligned}
& +\alpha s_1|Y_1(t)-Y_2(t)|+\alpha s_2|R_1(t)-R_2(t)| \\
& +\delta|K_1(t)-K_2(t)|+\beta|\mathscr{L}(Y_1(t))-\mathscr{L}(Y_2(t))| \\
& +\beta\gamma|R_1(t)-R_2(t)| \\
\leq\ & (\alpha+1)(m_1|Y_1(t)-Y_2(t)|+m_2|K_1(t)-K_2(t)| \\
& +m_2|R_1(t)-R_2(t)|)+\alpha s_1|Y_1(t)-Y_2(t)| \\
& \alpha s_2|R_1(t)-R_2(t)|+\delta|K_1(t)-K_2(t)| \\
& +\beta\gamma|R_1(t)-R_2(t)|+\beta\mathscr{L}'(Y)|Y_1(t)-Y_2(t)|,
\end{aligned}
$$

where $m_1=\sup\limits_{t\in[0,b]}|\frac{\partial I(Y(t),K_2(t),R2(t))}{\partial Y}|$, $m_2=\sup\limits_{t\in[0,b]}|\frac{\partial I(Y_1(t),K(t),R_2(t))}{\partial K}|$, $m_3=\sup\limits_{t\in[0,b]}|\frac{\partial I(Y_1(t),K_1(t),R(t))}{\partial R}|$ and $m_4=\sup\limits_{t\in[0,b]}|\mathscr{L}'Y(t)|$. Hence, the Lipschitz condition holds and F satisfies

$$
|F(t,Z_1)-F(t,Z_2)|\leq D|Z_1-Z_2|, \tag{9}
$$

where $D=\max\{(\alpha+1)m_1+\alpha s_1+\beta m_4,(\alpha+1)m_2+\delta,(\alpha+1)m_3+\alpha s_2+\beta\gamma\}$. □

Next, we consider the following hypothesis:

(H_0): There exist positive constants ϕ_1 and ϕ_2 such that

$$
|F(t,Z(t))|\leq\phi_1||Z||+\phi_2.
$$

Further, we define the operators $\mathscr{G}_1$ and $\mathscr{G}_2$ such that:

$$
\begin{aligned}
\mathscr{G}_1Z(t) &= \frac{\omega(0)Z_0}{\omega(t)}+\frac{1-p}{N(p)}F(t,Z(t)),\\
\mathscr{G}_2Z(t) &= \frac{p}{N(p)\Gamma(q)\omega(t)}\int_0^t(t-\tau)^{q-1}\omega(\tau)F(\tau,Z(\tau))d\tau.
\end{aligned}
$$

Also, we put $\theta_1=\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)\phi_1$. Hence, we have the following result.

Theorem 3.2. *Assume that (H_0) holds. Then model (5) has at least one solution if $\theta_1<1$ and $\frac{D(1-p)}{N(p)}<1$.*

Proof. Consider $E_m=\{Z\in\mathscr{C}:||Z||\leq m\}$ is closed convex set with $m\geq\frac{\theta_2}{1-\theta_1}$, where $\theta_2=|Z_0|+\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)\phi_2$.

First, we prove that $\mathscr{G}_1\psi_1+\mathscr{G}_2\psi_2 \in E_m$, for all $\psi_1,\psi_2 \in E_m$. By hypothesis (H_0), we obtain

$$\begin{aligned}||\mathscr{G}_1\psi_1+\mathscr{G}_2\psi_2|| &= \max_{t\in[0,b]}\left|\frac{\omega(0)Z_0}{\omega(t)}+\frac{1-p}{N(p)}F(t,\psi_1(t))\right.\\ &\quad \left.+\frac{p}{N(p)\Gamma(q)\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)F(\tau,\psi_2(\tau))d\tau\right|\\ &\leq \max_{t\in[0,b]}\left\{\left|\frac{\omega(0)Z_0}{\omega(t)}\right|+\frac{1-p}{N(p)}|F(t,\psi_1(t))|\right.\\ &\quad \left.+\frac{p}{N(p)\Gamma(q)\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)|F(\tau,\psi_2(\tau))|d\tau\right\}.\end{aligned}$$

As $\omega(0)<\omega(t)$ for all $t\geq 0$, we have

$$\begin{aligned}||\mathscr{G}_1\psi_1+\mathscr{G}_2\psi_2|| &\leq |Z_0|+\frac{1-p}{N(p)}(\phi_1||\psi_1||+\phi_2)+\frac{pb^q}{N(p)\Gamma(q+1)}(\phi_1||\psi_2||+\phi_2)\\ &= |Z_0|+\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)\phi_2+\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)\phi_1 m\\ &= \theta_2+\theta_1 m\leq m.\end{aligned}$$

This confirms that $\mathscr{G}_1\psi_1+\mathscr{G}_2\psi_2 \in E_m$. Hence, the condition (i) of Lemma 2.5 is verified.

Now, we demonstrate that $\mathscr{G}_1$ is a contraction mapping. Let Z, $\tilde{Z}\in E_m$, we have

$$\begin{aligned}||\mathscr{G}_1 Z-\mathscr{G}_1\tilde{Z}|| &= \max_{t\in[0,b]}\frac{1-p}{N(p)}\left|F(t,Z(t))-F(t,\tilde{Z}(t))\right|\\ &\leq \frac{(1-p)}{N(p)}D||Z-\tilde{Z}||.\end{aligned}$$

Since $\frac{D(1-p)}{N(p)}<1$, we deduce that $\mathscr{G}_1$ is a contraction mapping. Thus, the condition (ii) of Lemma 2.5 is satisfied.

Finally, we show that the condition (iii) of Lemma 2.5 is satisfied. To do this, we prove that $\mathscr{G}_2$ is continuous, uniform bounded and equicontinuous. Obviously, the operator $\mathscr{G}_2$ is continuous because of the continuity of F.

Let $Z\in E_m$, we have

$$\begin{aligned}||\mathscr{G}_2 Z|| &= \max_{t\in[0,b]}\left|\frac{p}{N(p)\Gamma(q)\omega(t)}\int_0^t (t-\tau)^{q-1}\omega(\tau)F(\tau,Z(\tau))d\tau\right|\\ &\leq \frac{pb^q}{N(p)\Gamma(q+1)}[\phi_1||Z||+\phi_2]\\ &\leq \frac{pb^q}{N(p)\Gamma(q+1)}(\phi_1 m+\phi_2).\end{aligned}$$

Hence, $\mathscr{G}_2$ is uniformly bounded on E_m.

For equicontinuity, let $Z \in E_m$ and $t_1, t_2 \in [0,b]$ such that $t_1 < t_2$. Then

$$
\begin{aligned}
&|\mathscr{G}_2 Z(t_2) - \mathscr{G}_2 Z(t_1)| \\
&= \frac{p}{N(p)\Gamma(q)} \left| \int_0^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} F(\tau, Z(\tau)) d\tau - \int_0^{t_1} (t_1-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_1)} F(\tau, Z(\tau)) d\tau \right| \\
&= \frac{p}{N(p)\Gamma(q)} \left| \int_0^{t_1} \left[\frac{(t_2-\tau)^{q-1}}{\omega(t_2)} - \frac{(t_1-\tau)^{q-1}}{\omega(t_1)} \right] \omega(\tau) F(\tau, Z(\tau)) d\tau + \int_{t_1}^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} F(\tau, Z(\tau)) d\tau \right| \\
&\leq \frac{p}{N(p)\Gamma(q)} \left| \int_0^{t_1} \left[\frac{(t_2-\tau)^{q-1}}{\omega(t_2)} - \frac{(t_1-\tau)^{q-1}}{\omega(t_1)} \right] \omega(\tau) F(\tau, Z(\tau)) d\tau \right| \\
&+ \frac{p}{N(p)\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2-\tau)^{q-1} \frac{\omega(\tau)}{\omega(t_2)} F(\tau, Z(\tau)) d\tau \right| \\
&\leq \frac{p}{N(p)\Gamma(q+1)} (\phi_1 ||Z|| + \phi_2) \left[\frac{(t_2-t_1)^q}{\omega(t_2)} - \frac{t_2^q}{\omega(t_2)} + \frac{t_1^q}{\omega(t_1)} \right] \omega(t_1) \\
&+ \frac{p}{N(p)\Gamma(q+1)} (\phi_1 ||Z|| + \phi_2) \left[(t_2-t_1)^q \frac{\omega(t_1)}{\omega(t_2)} \right] \\
&\leq \frac{2p}{N(p)\Gamma(q+1)} (\phi_1 m + \phi_2) \left[(t_2-t_1)^q \frac{\omega(t_1)}{\omega(t_2)} \right].
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Consequently, $\mathscr{G}_2$ is equicontinuous. By Arzela-Ascoli theorem, we deduce that $\mathscr{G}_2$ is relatively compact and so completely continuous. As a result, the condition (iii) of Lemma 2.5 is proved. Therefore, we conclude that model (5) has at least one solution. □

Theorem 3.3. *Assume that $D < \frac{N(p)}{1-p}$. If Z and X are two solutions of (7), then $Z = X$. This implies the uniqueness of solution.*

Proof. Let X and Z are two solutions of (7), we get

$$Z(t) - X(t) = \mathscr{I}_{0,\omega}^{p,q} \left(F(t, Z(t) - F(t, X(t)) \right).$$

Using Lemma 3.1, we deduce that

$$|Z(t) - X(t)| \leq D \mathscr{I}_{0,\omega}^{p,q} |Z(t) - X(t)|.$$

According to Lemma 2.4, we get

$$|Z(t) - X(t)| \leq \frac{N(p) \times 0}{N(p) - (1-p)D} E_q \left(\frac{pDt^q}{N(p) - (1-p)D} \right).$$

This implies that $Z(t) = X(t)$ for all $t \in [0,b]$. □

Theorem 3.4. *If* $D\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)<1$*, then system (7) has a unique solution for any initial condition.*

Proof. We consider the operator $\Upsilon:\mathscr{C}\to\mathscr{C}$ as follows

$$(\Upsilon Z)(t)=\frac{\omega(0)Z(0)}{\omega(t)}+\mathscr{I}_{0,\omega}^{p,q}F(t,Z(t)),\ t\in[0,b].$$

It suffices to prove that the operator Υ has a unique fixed point. We first prove that Υ is well defined. We have

$$\begin{aligned}|(\Upsilon Z)(t)|&=|\frac{\omega(0)Z_0}{w(t)}+\mathscr{I}_{0,\omega}^{p,q}F(t,Z(t))|\\&\leq|Z_0|\frac{\omega(0)}{\omega(t)}+\mathscr{I}_{0,\omega}^{p,q}|F(t,Z(t))|.\end{aligned}$$

As $\omega(0)<\omega(t)$ for all $t\geq 0$, F is Lipschitz continuous and $t\leq b$, we deduce that F is bounded by constant ξ and

$$\begin{aligned}|(\Upsilon Z)(t)|&\leq|Z_0|+\xi\mathscr{I}_{0,w}^{p,q}(1)\\&\leq|Z_0|+\xi\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right),\end{aligned}$$

which implies that the operator is well defined. Therefore, for all $Z_1,Z_2\in\mathscr{C}$ and $t\in[0,b]$, we have

$$\begin{aligned}|\Upsilon Z_1(t)-\Upsilon Z_2(t)|&=|\mathscr{I}_{0,w}^{p,q}F(t,Z_1(t))-F(t,Z_2(t))|\\&\leq|\frac{1-p}{N(p)}(F(t,Z_1(t))-F(t,Z_2(t)))+\frac{p}{N(p)}\\&\quad{}^{RL}\mathscr{I}_{0,w}^{q}(F(t,Z_1(t))-F(t,Z_2(t)))|\\&\leq\frac{1-p}{N(p)}D|Z_1-Z_2|+\frac{p}{N(p)}D||Z_1-Z_2||\frac{t^q}{\Gamma(q+1)}.\end{aligned}$$

As a result,

$$||\Upsilon Z_1-\Upsilon Z_2||\leq D\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)||Z_1-Z_2||.$$

By applying the Banach contraction mapping principle, we deduce that Υ is a contraction mapping if $D\left(\frac{1-p}{N(p)}+\frac{pb^q}{N(p)\Gamma(q+1)}\right)<1$. Then the system (7) has a unique solution. □

In the order to investigate the existence of equilibria of (5), we consider the following hypotheses:

(H_1): There exists two constants $A > 0$ and $\bar{q} \geq 0$ such that $|I(Y,K,R) + \bar{q}K| \leq A$ for all $Y,K,R \in \mathbb{R}$.

(H_2): $\gamma I\left(0, \frac{s_2(\mathscr{L}(0)-\overline{M})}{\gamma\delta}, \frac{\mathscr{L}(0)-\overline{M}}{\gamma}\right) - s_2\left(\mathscr{L}(0) - \overline{M}\right) > 0$,

(H_3): $\gamma\frac{\partial I}{\partial Y} + \left[\frac{\gamma s_1}{\delta} + \frac{s_2}{\delta}\mathscr{L}'(Y)\right]\frac{\partial I}{\partial K} + \mathscr{L}'(Y)\frac{\partial I}{\partial R} - \gamma s_1 - s_2\mathscr{L}'(Y) < 0$.

Theorem 3.5. *If $(H_1)-(H_3)$ hold, then system (5) has a unique economic equilibrium defined by $E^* = \left(Y^*, \frac{\gamma s_1 Y^* + s_2(\mathscr{L}(Y^*)-\overline{M})}{\gamma\delta}, \frac{\mathscr{L}(Y^*)-\overline{M}}{\gamma}\right)$, such that Y^* is the unique solution of the following equation*

$$\gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathscr{L}(Y)-\overline{M})}{\gamma\delta}, \frac{\mathscr{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2(\mathscr{L}(Y)-\overline{M}) = 0.$$

Proof. Any equilibrium of (5) is a solution of the following equations

$$I(Y,K,R) - s_1 Y - s_2 R = 0, \tag{10}$$

$$I(Y,K,R) - \delta K = 0, \tag{11}$$

$$\mathscr{L}(Y) - \gamma R - \overline{M} = 0. \tag{12}$$

From (10)-(12), we have

$$R = \frac{\mathscr{L}(Y)-\overline{M}}{\gamma} \quad \text{and} \quad K = \frac{\gamma s_1 Y + s_2(\mathscr{L}(Y)-\overline{M})}{\gamma\delta}. \tag{13}$$

By replacing (13) in (10), we get

$$\gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathscr{L}(Y)-\bar{M})}{\gamma\delta}, \frac{\mathscr{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2\left(\mathscr{L}(Y)-\overline{M}\right) = 0.$$

Therefore, we consider a function ψ defined on interval $[0,+\infty)$ as follows

$$\psi(Y) = \gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathscr{L}(Y)-\bar{M})}{\gamma\delta}, \frac{\mathscr{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2\left(\mathscr{L}(Y)-\overline{M}\right).$$

From $(H_1)-(H_4)$, we obtain $\psi(0) > 0$, $\lim_{Y\to+\infty}\psi(Y) = -\infty$ and

$$\psi'(Y) = \gamma\frac{\partial I}{\partial Y} + \left[\frac{\gamma s_1}{\delta} + \frac{s_2}{\delta}\mathscr{L}'(Y)\right]\frac{\partial I}{\partial K} + \mathscr{L}'(Y)\frac{\partial I}{\partial R} - \gamma s_1 - s_2\mathscr{L}'(Y) < 0.$$

Consequently, there exists a unique $Y^* \in (0,+\infty)$ such that Y^* is the solution of the equation $\psi(Y) = 0$. This completes the proof. □

Next, we establish stability analysis of the economic equilibrium. Let $y = Y - Y^*$, $k = K - K^*$ and $r = R - R^*$. By substituting y, k and r into system (5) and linearizing, we get the following system

$$\begin{cases} D^{p,q}_{0,\omega} y(t) = \alpha[(a - s_1)y(t) + bk(t) + (c - s_2)r(t)], \\ D^{p,q}_{0,\omega} k(t) = ay(t) + (b - \delta)k(t) + cr(t), \\ D^{p,q}_{0,\omega} r(t) = \beta[l_1 y(t) - \gamma r(t)], \end{cases} \tag{14}$$

where $a = \frac{\partial I}{\partial Y}(Y^*, K^*, R^*)$, $b = \frac{\partial I}{\partial K}(Y^*, K^*, R^*)$, $l_1 = \mathscr{L}'(Y^*) > 0$ and $c = \frac{\partial I}{\partial R}(Y^*, K^*, R^*)$.
By applying the Laplace transform to system (14), we obtain

$$\Delta(s). \begin{pmatrix} \tilde{Y}(s) \\ \tilde{K}(s) \\ \tilde{R}(s) \end{pmatrix} = \begin{pmatrix} b_1(s) \\ b_2(s) \\ b_3(s) \end{pmatrix},$$

where $\tilde{Y}(s) = \mathscr{L}\{\omega(t)y(t)\}$, $\tilde{K}(s) = \mathscr{L}\{\omega(t)k(t)\}$, $\tilde{R}(s) = \mathscr{L}\{\omega(t)k(t)\}$,

$$\begin{cases} b_1(s) = s^{q-1} N(p)\omega(0)y(0), \\ b_2(s) = s^{q-1} N(p)\omega(0)k(0), \\ b_3(s) = s^{q-1} N(p)\omega(0)r(0), \end{cases}$$

and

$$\Delta(s) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix},$$

with

$$\begin{aligned}
x_1 &= s^q [N(p) - \alpha(a - s_1)(1 - p)] - \alpha\mu_p(a - s_1)(1 - p), \\
x_2 &= -[\alpha b s^q (1 - p) + \mu_p \alpha b(1 - p)], \\
x_3 &= -[s^q \alpha(1 - p)(c - s_2) + a\mu_p(1 - p)(c - s_2)], \\
x_4 &= -[as^q(1 - p) + a\mu_p(1 - p)], \\
x_5 &= s^q [N(p) - (b - \delta)(1 - p)] - \mu_p(b - \delta)(1 - p), \\
x_6 &= -[cs^q(1 - p) + c\mu_p(1 - p)],
\end{aligned}$$

$$\begin{aligned} x_7 &= -\left[\beta l_1 s^q(1-p)+\beta l_1 \mu_p(1-p)\right], \\ x_8 &= 0, \\ x_9 &= s^q\left[N(p)+\beta\gamma(1-p)\right]+\mu_p\beta\gamma(1-p). \end{aligned}$$

Thus, the characteristic equation about E^* is given by

$$a_0 s^{3q}+a_1 s^{2q}+a_2 s^q+a_3=0, \tag{15}$$

where

$$\begin{aligned} a_0 &= -\beta l_1\alpha bc(1-p)^3-\beta l_1\alpha(1-p)^2(c-s_2)[N(p)-(b-\delta)(1-p)]+[N(p)-\alpha(a-s_1)(1-p)] \\ &\quad [N(p)-(b-\delta)(1-p)][N(p)+\beta\gamma(1-p)], \\ a_1 &= \beta l_1\alpha\mu_p(1-p)^3[-3bc+(c-s_2)(b-\delta)]-\beta\alpha l_1(1-p)^2(c-s_2)(1+\mu_p)[N(p)-(b-\delta)(1-p)] \\ &\quad -(b-\delta)(1-p)\mu_p[N(p)+\beta\gamma(1-p)][N(p)-\alpha(a-s_1)(1-p)]+\beta\gamma\mu_p[N(p)-(b-\delta)(1-p)] \\ &\quad [N(p)-\alpha(a-s_1)(1-p)]-\alpha(a-s_1)(1-p)\mu_p[N(p)+\beta\gamma(1-p)][N(p)-(b-\delta)(1-p)], \\ a_2 &= -\beta l_1\alpha\mu_p^2(1-p)^2(c-s_2)[3bc(1-p)-2(1-p)(b-\delta)+[N(p)-(b-\delta)(1-p)]]+\alpha(a-s_1) \\ &\quad \mu_p(b-\delta)(1-p)^2[N(p)+\beta\gamma(1-p)]-\beta\gamma\mu_p^2(b-\delta)(1-p)^2[N(p)-\alpha(a-s_1)(1-p)] \\ &\quad -\beta\gamma\alpha\mu_p^2(a-s_1)(1-p)^2[N(p)-(b-\delta)(1-p)], \\ a_3 &= \beta l_1\alpha bc(1-p)^3+\beta l_1\alpha(1-p)^3(b-\delta)(c-s_2)+\beta\gamma\alpha\mu_p^3(1-p)^3(a-s_1)(b-\delta). \end{aligned}$$

Let $s^q=\lambda$ and substitute it into (15), we have

$$a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0. \tag{16}$$

Clearly, if $a<s_1$ and if the following conditions:

(A_1): $-\beta l_1\alpha(1-p)^2(c-s_2)[N(p)-(b-\delta)(1-p)]+[N(p)-\alpha(a-s_1)(1-p)][N(p)-(b-\delta)(1-p)][N(p)+\beta\gamma(1-p)] > -\beta l_1\alpha bc(1-p)^3$,

(A_2): $\beta l_1\alpha\mu_p(1-p)^3(c-s_2)(b-\delta)-\beta\alpha l_1(1-p)^2(c-s_2)(1+\mu_p)[N(p)-(b-\delta)(1-p)]-(b-\delta)(1-p)\mu_p[N(p)+\beta\gamma(1-p)][N(p)-\alpha(a-s_1)(1-p)]+\beta\gamma\mu_p[N(p)-(b-\delta)(1-p)][N(p)-\alpha(a-s_1)(1-p)]-\alpha(a-s_1)(1-p)\mu_p[N(p)+\beta\gamma(1-p)][N(p)-(b-\delta)(1-p)] > -3bc\beta l_1\alpha\mu_p(1-p)^3$,

(A_3): $\beta l_1\alpha(1-p)^3(b-\delta)(c-s_2)+\beta\gamma\alpha\mu_p^3(1-p)^3(a-s_1)(b-\delta) > \beta l_1\alpha bc(1-p)^3$, hold, then it not hard to see that the coefficients of the equation (16) satisfy:

$$a_0 > 0, \;\; a_1 > 0, \;\; a_3 > 0 \;\; and \;\; a_1a_2 > a_0a_3.$$

Based on Routh-Hurwitz criterion, all the roots of equation (15) have negative real parts In conclusion, we have the following results.

Theorem 3.6. *If $a < s_1$ and $(A_1) - (A_3)$ hold, Then the economic equilibrium E^* is locally asymptotically stable.*

4. Numerical Simulations

In this section, we present some numerical simulations to illustrate our theoretical results. Let $t_n = n\Delta t$, with $n \in \mathbb{N}$ and Δt be the time step. Based on the numerical method proposed in [14], we get the following discrete model

$$(17)\quad \begin{cases} Y(t_{n+1}) & = \frac{Y_0\omega(0)}{\omega(t_n)} + \frac{1-p}{N(p)}F_1(t_n, Z(t_n)) + \frac{p(\Delta t)^q}{N(p)\Gamma(q+2)\omega(t_n)}\sum_{k=0}^{n}[\omega(t_k)F_1(t_k, Z(t_k))\mathscr{A}_{n,k,q} \\ & \quad +\omega(t_{k-1})F_1(t_{k-1}, Z(t_{k-1}))\mathscr{B}_{n,k,q}], \\ K(t_{n+1}) & = \frac{K_0\omega(0)}{\omega(t_n)} + \frac{1-p}{N(p)}F_2(t_n, Z(t_n)) + \frac{p(\Delta t)^q}{N(p)\Gamma(q+2)\omega(t_n)}\sum_{k=0}^{n}[\omega(t_k)F_2(t_k, Z(t_k))\mathscr{A}_{n,k,q} \\ & \quad +\omega(t_{k-1})F_2(t_{k-1}, Z(t_{k-1}))\mathscr{B}_{n,k,q}], \\ R(t_{n+1}) & = \frac{W_0\omega(0)}{\omega(t_n)} + \frac{1-p}{N(p)}F_3(t_n, Z(t_n)) + \frac{p(\Delta t)^q}{N(p)\Gamma(q+2)\omega(t_n)}\sum_{k=0}^{n}[\omega(t_k)F_3(t_k, Z(t_k))\mathscr{A}_{n,k,q} \\ & \quad +\omega(t_{k-1})F_3(t_{k-1}, Z(t_{k-1}))\mathscr{B}_{n,k,q}], \end{cases}$$

where

$\mathscr{A}_{n,k,q} = (n-k+1)^q(n-k+2+q) - (n-k)^q(n-k+2+2q)$,

$\mathscr{B}_{n,k,q} = (n-k)^q(n-k+1+q) - (n-k+1)^{q+1}$.

For the simulation, we choose $N(p) = 1 - p + \frac{p}{\Gamma(p)}$ and we consider $I(Y,K,R) = I(Y) + \frac{q_1K}{\sqrt{1+\varepsilon K^2}} + q_2R$, where $q_1, q_2 < 0$, $\varepsilon \geq 0$ and $I(Y)$ is the Kaldor-type investment function defined by $\frac{e^Y}{1+e^Y}$. The liquidity preference function is chosen as $L(Y,R) = s_3Y - s_4R$, where $s_3, s_4 > 0$. We use the following parameter values: $\alpha = 3$, $q_1 = -0.3$, $q_2 = -0.2$, $\varepsilon = 0.01$, $\delta = 0.2$, $s_1 = 0.2$ $s_2 = 0.1$, $s_3 = 0.3$, $s_4 = 0.2$, $\bar{M} = 0.05$, $\beta = 0.2$ and $q = 0.9$. Then, by a simple calculation, our model has an economic equilibrium $E^*(0.4988, 0.7479, 0.4982)$. In this case, we fund that the economic equilibrium E^* is locally asymptotically stable if $a < s_1$ and the conditions in Theorem 3.6 are satisfied. Figures 1, 2 and 3 illustrate the impact of memory effect on the dynamical behaviors of our model for different values of the parameter p.

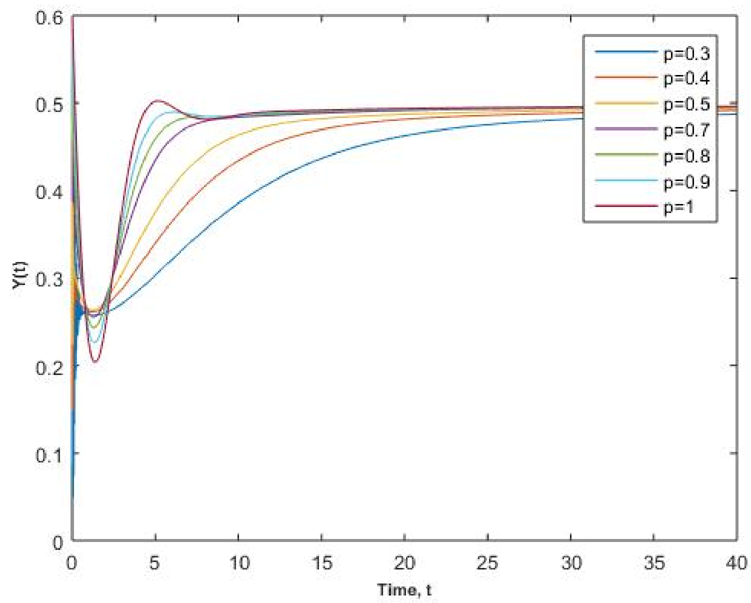

FIGURE 1. The curve of $Y(t)$ under different values of p.

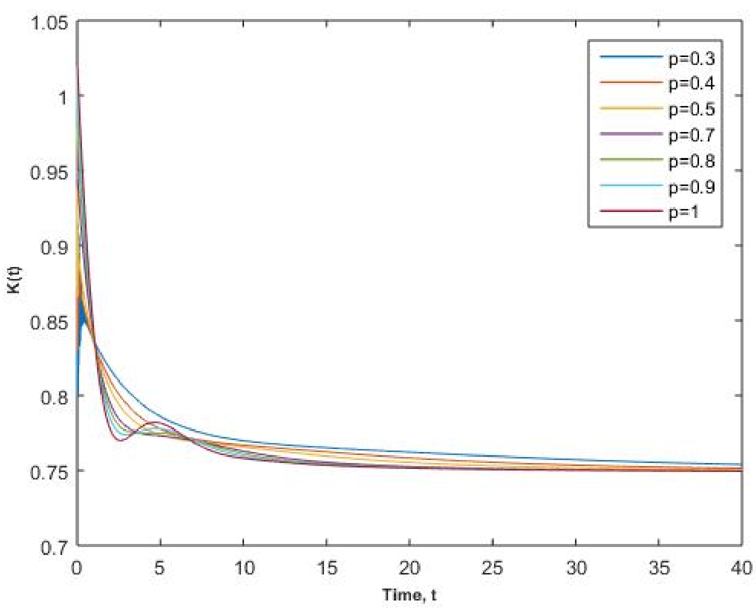

FIGURE 2. The curve of $K(t)$ under different values of p.

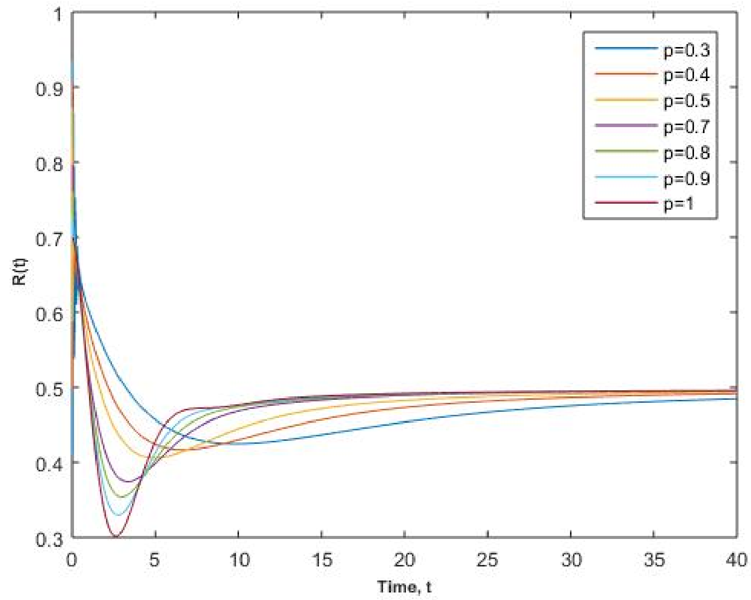

FIGURE 3. The curve of $R(t)$ under different values of p.

5. CONCLUSION

In this work, we have proposed and studied the dynamics of a fractional IS-LM business cycle model, considering the memory effect described by the generalized Hattaf fractional derivative. The well-posedness of the proposed model was proved through the existence and uniqueness of solutions. By analyzing the corresponding characteristic equation, the local stability of the economic equilibrium of our model was discussed. Numerical simulations showed the impact of memory effect on the dynamical behaviors of our model for different values of fractional order.

Though theoretical analysis and simulations, it is found that the order of the generalized Hattaf fractional derivative does not affect the stability of the economic equilibrium. On the other hand, it may have an impact on the time required to reach this equilibrium. In particular, increasing the order of the GHF derivative p leads to afaster convergence of the solution to the equilibrium point.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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