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APPROXIMATE FIXED POINT THEOREMS FOR VARIOUS CONTRACTIONS IN MULTIPLICATIVE METRIC SPACES

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Abstract. In this manuscript, we have used several contraction mappings to provide approximate fixed point results in multiplicative metric spaces. We established the new results in multiplicative metric spaces to get a more accurate approximation of the fixed point. These findings are an extension of several results on metric spaces.

Keywords: approximate fixed point; diameter; Kannan-type Contraction; Chatterjea contraction; Zamfirescu contraction; Bianchini contraction; n-convex contraction; multiplicative metric space.

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1. INTRODUCTION

Mathematics study on the topic of fixed points is one of the most interesting area. Several researchers have applied various kinds of contraction mappings and spaces to advance fixed point theory in the realm of mathematics. Furthermore, it is beneficial in demonstrating the existence theorems for integral and nonlinear differential equations. The fixed point theorem for continuous mapping on finite dimensional spaces was established in the early 1900s by mathematician Brouwer [7], who is regarded as the father of fixed point theory. Banach [1] established the well-known Banach contraction principle in 1922. Through the use of multiplicative calculus,

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Bashirov established the definition of multiplicative metric space and proved the basic principle of multiplicative calculus. A few fixed point theorems for contraction mappings of multiplicative metric spaces were established by Ozavsar and Cevikel [21].

In numerous real-world problems, the necessary condition outlined in fixed point theorems are too strong, resulting an inability to guarantee the existence of a fixed point. In such cases, one may settle for nearly fixed points, which we refer to as approximate fixed points. By "near to" u, we mean that Tu is an approximate fixed point of a function T. A point that is almost exactly located at its corresponding fixed point is called an approximation fixed point. This distance, $d(Tu_0,u_0) < \varepsilon$, indicates that it is smaller than ε. For a contractive mapping T in a multiplicative metric space, the distance between points in the set of all approximate fixed points decreases iteratively under T, leading to a reduction in the diameter of the set. The reduction in the diameter of the set in multiplicative metric spaces ensures that successive iterations of the mapping bring points closer together. This makes it possible to approximate the fixed point with increasingly higher accuracy.

Berinde [3] formulated essential approximate fixed point theorems in metric space, inspired by the findings of Theiva [24] as detailed in the article. The goal of this research is to use contraction mappings, such as Kannan contraction [15], Bianchini contraction [6], Chatterjea contraction, Zamfirescu contraction and n-convex contraction [11] to generate approximate fixed point results in a multiplicative metric space(not necessarily complete). We examine the fundamental ideas and definitions that are required throughout the work in Section 2. We demonstrate the main concept behind approximation fixed point results in Section 3 by applying various contraction mapping in multiplicative metric spaces.

2. Preliminaries

This section comprises definitions and lemmas that will be utilized in subsequent sections. Let $(\check{X},d)$ be a metric space.

Definition 2.1. [3] *Let $T:\check{X}\to\check{X}$, $\varepsilon>0$. Then $u\in T$ is said to be an ε-fixed point (approximate fixed point) of T if*

$$d(u,Tu) < \varepsilon.$$

Definition 2.2. [3] *A mapping $T : \check{X} \to \check{X}$. Then T has an approximate fixed point property(a.f.p.p) if for every $\varepsilon > 0$,*

$$F_\varepsilon(T) \neq \emptyset.$$

Lemma 2.1. [3] *Let $T : \check{X} \to \check{X}$ such that T is asymptotic regular, i.e., $d(T^n(u), T^{n+1}(u)) \to 0$ as $n \to \infty$, for all $u \in T$. Then, $F_\varepsilon(T) \neq \emptyset$, for every $\varepsilon > 0$.*

Definition 2.3. [3] *Let M be a closed subset and $T : M \to X$ be a compact map. Then T has a fixed point if and only if it has an approximate fixed point property.*

Definition 2.4. [3] *Let $T : \check{X} \to \check{X}$ a operator and $\varepsilon > 0$. We define the diameter of the set $F_\varepsilon(T)$, i.e.,*

$\delta(F_\varepsilon(T)) = sup\{d(u,v) : u,v \in F_\varepsilon(T)\}$

Lemma 2.2. [3] *Let T:$\check{X} \to \check{X}$ an operator and $\varepsilon > 0$. We assume that: (i) $F_\varepsilon(T) \neq \emptyset$; (ii) for all $\gamma > 0$, there exist $\phi(\gamma) > 0$ such that $d(u,v) - d(Tu,Tv) \leq \gamma$ implies $d(u,v) \leq \phi(\gamma)$, for all $u,v \in F_\varepsilon(T)$.*

Then:

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon).$$

Definition 2.5. [6] *A selfmap $T : \check{X} \to \check{X}$ is said to be a Bianchini contraction if there exists $k \in (0,1)$ such that*

$$d(Tu,Tv) \leq kB(u,v),$$

where $B(u,v) = max\{d(u,Tu), d(v,Tv)\}$, for all $u,v \in \check{X}$.

Definition 2.6. [11] *Let $T : \check{X} \to \check{X}$ be a continuous map. Then T is said to be n-convex contraction if there exists $k_0, k_1, ..., k_{n-1} \in (0,1)$ such that the following conditions hold:*

(i) $k_0 + k_1 + ... + k_{n-1} < 1$; and

(ii) $d(T^n u, T^n v) \leq k_0 d(u,v) + k_1 d(Tu,Tv) + ... + k_{n-1} d(T^{n-1}u, T^{n-1}v)$, for all $u,v \in \check{X}$.

Let (χ, d^*) be a multiplicative metric space.

Definition 2.7. [2] *Let χ be a nonempty set. A mapping $T : \chi \times \chi \to R^+$ is called multiplicative metric if for all $u,v,w \in \chi$*

(i) $d^*(u,v) < 1$ *and* $d^*(u,v) = 1$ *if and only if* $u = v$,

(ii) $d^*(u,v) = d^*(v,u)$,

(iii) $d^*(u,w) \leq d^*(u,v).\, d^*(v,w)$ *(Multiplicative triangle inequality)*

Definition 2.8. [21] *(Multiplicative characterization of supremum) Let A be a nonempty subset of* $\mathbb{R}^+$*. Then s = sup A if and only if*

(i) $a \leq s$ *for all* $a \in A$

(ii) there exists at least a point $a \in A$ *such that* $|\frac{s}{a}|^* < \varepsilon$ *for all* $\varepsilon > 1$.

Definition 2.9. [21] *A mapping* $T : \chi \to \chi$ *is a Kannan contraction if there exists* $k \in [0, \frac{1}{2})$ *such that*

$d^*(Tu,Tv) \leq [d^*(u,Tu).\, d^*(v,Tv)]^k$*, for all* $u,v \in \chi$.

Definition 2.10. [21] *Let* $T : \check{X} \to \check{X}$ *is a Chatterjea operator if there exist* $k \in (0, \frac{1}{2})$ *such that* $d^*(Tu,Tv) \leq [d^*(u,Tv) + d^*(v,Tu)]^k$*, for all* $u,v \in \check{X}$.

Definition 2.11. [23] *A mapping* $T : \check{X} \to \check{X}$ *is a Zamfirescu operator if* $\exists \lambda, \mu, \nu \in \mathbb{R}$, $\lambda \in [0,1), \mu \in [0, \frac{1}{2}), \nu \in [0, \frac{1}{2})$ *such that* $\forall\ u,v \in \check{X}$*, at least one of the following is true.* *(i)* $d^*(Tu,Tv) \leq d(u,v)^\lambda$; *(ii)*$d^*(Tu,Tv) \leq [d(u,Tu).d(v,Tv)]^\mu$; *(iii)*$d^*(Tu,Tv) \leq [d^*(u,Tv).d^*(v,Tu)]^\nu$.

3. MAIN RESULTS

In the following, we introduce the concept of approximate fixed point and ensuring that an operator on a multiplicative metric space has ε-fixed points.
Let (χ, d^*) be a multiplicative metric spaces(MMS).

Definition 3.1. *Let* $T : \chi \to \chi$*,* $\varepsilon > 1$*. Then* $u \in T$ *is said to be an* ε*-fixed point (approximate fixed point) of T if*

$$d^*(u,Tu) < \varepsilon.$$

Definition 3.2. *Consider a mapping* $T : \chi \to \chi$*. Then T has an approximate fixed point property if for every* $\varepsilon > 1$,

$$F_\varepsilon(T) \neq \emptyset.$$

Definition 3.3. *Let M be a closed subset and $T : M \to \chi$ be a compact map. Then T has a fixed point if and only if it has an approximate fixed point property.*

Definition 3.4. *Let $T : \chi \to \chi$ a operator and $\varepsilon > 1$. We define the diameter of the set $F_\varepsilon(T)$, i.e.,*
$\delta(F_\varepsilon(T)) = sup\{d^*(u,v) : u,v \in F_\varepsilon(T)\}$.

Definition 3.5. *A selfmap $T : \chi \to \chi$ is said to be a Bianchini contraction if there exists $k \in (0,1)$ such that*

$$d^*(Tu,Tv) \leq B(u,v)^k,$$

where $B(u,v) = max\{d^*(u,Tu), d^*(v,Tv)\}$, for all $u,v \in \chi$.

Definition 3.6. *Let $T : \chi \to \chi$ be a continuous map. Then T is said to be n-convex contraction if there exists $k_0, k_1, ..., k_{n-1} \in (0,1)$ such that the following conditions hold:*
(i) $k_0 + k_1 + ... + k_{n-1} < 1$; and
(ii) $d^*(T^n u, T^n v) \leq d^*(u,v)^{k_0} . d^*(Tu,Tv)^{k_1} . .... d^*(T^{n-1}u, T^{n-1}v)^{k_{n-1}}$, for all $u,v \in \chi$.

The following results ensuring the existence of ε-fixed points for an operator in a multiplicative metric spaces.

Lemma 3.1. *Let (χ, d^*) be a MMS, T: $\chi \to \chi$ such that T is asymptotic regular i.e.,*
$d^*(T^n u_0, T^{n+1} u_0) \to 1$ as $n \to \infty$, $\forall u \in \chi$. Then T has the approximate fixed point property.
Proof:
Let $u_0 \in \chi$. Then
$d^*(T^n u_0, T^{n+1} u_0) \to 1$ as $n \to \infty \Leftrightarrow \forall \varepsilon > 1$, $\exists\, n_0(\varepsilon) \in \mathbb{N}$ such that $\forall n \geq n_0(\varepsilon)$,
$d^*(T^n u_0, T^{n+1} u_0) < \varepsilon \Leftrightarrow \forall \varepsilon > 1$, $\exists\, n_0(\varepsilon) \in \mathbb{N}$ such that $\forall n \geq n_0(\varepsilon)$,
$d^*(T^n u_0, T(T^n u_0)) < \varepsilon$
Denoting $v_0 = T^n u$,
it follows that: $\forall \varepsilon > 1, \exists v_0 \in \chi$ such that $d^*(v_0, Tv_0) < \varepsilon$, so for each $\varepsilon > 1$ there exists an ε-fixed point of T in χ, namely v_0. This means exactly that T has the approximate fixed point property.

Lemma 3.2. *Let (χ, d^*) be a MMS, T: $\chi \to \chi$ an operator and $\varepsilon > 1$. We assume that:*

(i) $F_\varepsilon(T) \neq \emptyset$;

(ii) for all $\gamma > 1$, there exist $\phi(\gamma) > 1$ such that $\frac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$ implies $d^*(u,v) \leq \phi(\gamma)$, for all $u,v \in F_\varepsilon(T)$. Then:

$\delta(F_\varepsilon(T)) \leq \phi(\varepsilon^2)$.

Proof:

Let $\varepsilon > 1$ and $u,v \in F_\varepsilon(T)$. Then:

$d^*(u,Tu) < \varepsilon$, $d^*(v,Tv) < \varepsilon$.

we can write:

$$\begin{aligned} d^*(u,v) &\leq d^*(u,Tu)\ .\ d^*(Tu,Tv)\ .\ d^*(v,Tv) \\ &\leq d^*(Tu,Tv).\ \varepsilon.\varepsilon \\ &\leq d^*(Tu,Tv).\ \varepsilon^2 \end{aligned}$$

$$\frac{d^*(u,v)}{d^*(Tu,Tv)} \leq \varepsilon^2$$

Now by (ii) it follows that $d^*(u,v) \leq \phi(\varepsilon^2)$

so $\delta(F_\varepsilon(T)) \leq \phi(\varepsilon^2)$.

Theorem 3.1. *Let (χ, d^*) be a multiplicative metric space and T:$\chi \to \chi$ be a contraction mapping. Then T has an approximate fixed point (ε-fixed point).*

Proof. Fix $u_0 \in \chi$ and a sequence $\{u_n\}$ is defined by $u_{n+1} = Tu_n$, for all $n \geq 0$. Which implies that $\{u_n\}$ is a Cauchy sequence. That is, for every $\varepsilon > 1$, there exists $h_0 \in \mathbb{N}$ such that for every $s,t \geq h_0$ implies $d^*(u_s,u_t) < \varepsilon$. In particular, if $n \geq h_0$, $d^*(u_n,u_{n+1}) < \varepsilon$. That is, $d^*(u_n,Tu_n) < \varepsilon$. Therefore, $u_n \in F_\varepsilon(T) \neq \emptyset$, for all $\varepsilon > 1$. Hence T has an approximate fixed point(ε-fixed point). □

Theorem 3.2. *Let T:$\chi \to \chi$ be a Kannan Type contraction on a multiplicative metric space (χ, d^*). Then T possesses an ε-fixed point and $\delta(F_\varepsilon(T)) \leq \varepsilon^{2(k+1)}$, for all $\varepsilon > 1$.*

Proof. Let $\varepsilon > 1$ and $u_0 \in T$.

$$\begin{aligned} d^*(T^n u, T^{n+1}u) &= d^*(T(T^{n-1}(u)), T(T^n(u))) \\ &\leq \left[d^*(T^{n-1}u, T(T^{n-1}(u))).d^*(T^n u, T(T^n u))\right]^k \\ &= \left[d^*(T^{n-1}u, T^n u)\right]^k.\left[d^*(T^n u, T^{n+1}u)\right]^k \end{aligned}$$

$$d^*(T^n u, T^{n+1}u)^{1-k} \leq \left[d^*(T^{n-1}u, T^n u)\right]^k$$

$$d^*(T^nu,T^{n+1}u) \leq \left[d^*(T^{n-1}u,T^nu)\right]^{\frac{k}{1-k}} \qquad \text{(Denote } \eta = \tfrac{k}{1-k}\text{)}$$
$$\leq d^*(T^{n-1}u,T^nu)^{\eta}$$
$$\leq d^*(T^{n-1}u,T^nu)^{\eta^2}$$
$$\vdots$$
$$\leq d^*(u,Tu)^{\eta^n}$$

But $k \in [0,\frac{1}{2}) \implies \frac{k}{1-k} \in (0,1) \implies$

$d^*(T^nu,T^{n+1}u) \to 1$ as $n \to \infty$ $\forall u \in T$

Now by Lemma 3.1, $F_\varepsilon(T) \neq \emptyset$, $\forall \varepsilon > 1$. Here, T has an ε-fixed point.

Let $\gamma > 1$ and $u,v \in F_\varepsilon(T)$ and assume that $\dfrac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$.

Then

$d^*(u,v) \leq [d^*(u,Tu).d^*(v,Tv)]^k.\gamma$

As $u,v \in F_\varepsilon(T)$, we know that $d^*(u,Tu) < \varepsilon$ and $d^*(v,Tv) < \varepsilon$

$\Rightarrow d^*(u,v) \leq \varepsilon^{2k}.\gamma$

So $\forall \gamma > 1$, there exists $\phi(\gamma) = \varepsilon^{2k}.\gamma > 1$ such that
$\dfrac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$ implies $d^*(u,v) \leq \phi(\gamma)$.
Now by Lemma 3.2,

$\delta(F_\varepsilon(\mathrm{T})) \leq \phi(\varepsilon^2)$

which means exactly that

$$\delta(F_\varepsilon(\mathrm{T})) \leq \varepsilon^{2k}.\varepsilon^2$$
$$\leq \varepsilon^{2k+2}$$
$$\leq \varepsilon^{2(k+1)}, \text{ for all } \varepsilon > 1.$$

□

Theorem 3.3. *Let T:$\chi \to \chi$ be a Chatterjea operator on a multiplicative metric space (χ, d^*). Then T possesses an ε-fixed point and $\delta(F_\varepsilon(T)) \leq \dfrac{\varepsilon^{2(1+k)}}{1-2k}$, for all $\varepsilon > 1$.*

Proof. Let $\varepsilon > 1$ and $u \in \chi$.

$$d^*(T^nu,T^{n+1}u) = d^*(T(T^{n-1}(u)),T(T^n(u)))$$
$$\leq \left[d^*(T^{n-1}u,T(T^n(u))).d^*(T^nu,T(T^{n-1}u))\right]^k$$
$$= \left[d^*(T^{n-1}u,T^{n+1}u).d^*(T^nu,T^nu)\right]^k$$
$$\leq d^*(T^{n-1}u,T^{n+1}u)^k$$

On the other hand

$d^*(T^{n-1}u, T^{n+1}u) = d^*(T^{n-1}(u), T^n(u)).d^*(T^n(u), T^{n+1}(u)$

$\Rightarrow d^*\left[T^n u, T^{n+1}(u)\right]^{1-k} \leq d^*\left[T^{n-1}u, T^n(u)\right]^k$

$\Rightarrow d^*\left[T^n u, T^{n+1}(u)\right] \leq d^*\left[T^{n-1}u, T^n(u)\right]^{\frac{k}{1-k}}$

$\Rightarrow d^*(T^n u, T^{n+1}u) \to 1 asn \to \infty, \forall u \in \chi$

Now, by lemma 3.1 it follows that $F_\varepsilon \neq \phi, \forall \varepsilon > 1$.

Let $\varepsilon > 1$. We will once again demonstrate that condition ii) in Lemma 3.2 is satisfied.

Let $\gamma > 1$ and u,v $\in F_\varepsilon(T)$ and assume that

$\dfrac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$

Then $d^*(u,v) \leq [d^*(u,Tv).d^*(v,Tu)]^k.\gamma$

$\leq d^*(u,Tv)^k.d^*(v,Tu)^k.\gamma$

$\leq [d^*(u,v).d^*(v,Tv)]^k.[d^*(v,u).d^*(u,Tu)]^k.\gamma$

As u,v $\in F_\varepsilon(T)$, it follows that

$d^*(u,v) \leq d^*(u,v)^{2k}.\varepsilon^{2k}.\gamma$

$d^*(u,v)^(1-2k) \leq \dfrac{\varepsilon^{2k}.\gamma}{1-2k}$

So $\forall \gamma > 1, \exists \phi(\gamma) = \dfrac{\varepsilon^{2k}.\gamma}{1-2k} > 1$ such that

$\dfrac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma \Rightarrow d^*(u,v) \leq \phi(\gamma)$

Now by lemma 3.2 it follows that

$\delta(F_\varepsilon(T)) \leq \phi(\varepsilon^2)\ \forall \varepsilon > 1$

which means exactly that

$\delta(F_\varepsilon(T)) \leq \dfrac{\varepsilon^{2(1+k)}}{1-2k}, \forall \varepsilon > 1.$ □

Theorem 3.4. *Let T:$\chi \to \chi$ be a Zamfirescu contraction on a multiplicative metric space (X, d^*). Then T Possesses an ε-fixed point and $\delta(F_\varepsilon(T)) \leq \dfrac{\varepsilon^{2(1+k)}}{1-2k}$, for all $\varepsilon > 1$.*

Proof. Let u,v$\in \chi$, Supposing (ii) holds, we have that:

$d^*(Tu,Tv) \leq [d^*(u,Tu).d^*(v,Tv)]^\mu$

$\leq [d^*(u,Tu)^\mu.[d^*(v,u).d^*(u,Tu).d(Tu,Tv)]^\mu$

$= d^*(u,Tu)^{2\mu}.d^*(u,v)^\mu.d^*(Tu,Tv)^\mu$

$d^*(Tu,Tv) \leq d^*(u,Tu)^{\frac{2\mu}{1-\mu}}.d^*(u,v)^{\frac{\mu}{1-\mu}} \longrightarrow (1)$

Suppose (iii) holds, we have that

$d^*(Tu,Tv) \leq [d^*(u,Tu).d^*(v,Tu)]^{\nu}$

$\leq [d^*(u,v).d^*(v,Tv)^{\nu}.[d^*(v,Tv).d^*(Tv,Tu)]^{\nu}$

$= d^*(Tu,Tv)^{\nu}.d^*(v,Tv)^{2\nu}.d^*(u,v)^{\nu}$

$d^*(Tu,Tv) \leq d(v,Tv)^{\frac{2\nu}{1-\nu}}.d^*(u,v)^{\frac{\nu}{1-\nu}}$ $\longrightarrow$ (2a)

Similarly $d^*(Tu,Tv) \leq [d^*(u,Tv).d^*(v,Tu)]^{\nu}$

$\leq [d^*(u,Tu).d^*(Tu,Tv)]^{\nu}.[d^*(v,u).d^*(u,Tu)]^{\nu}$

$= d^*(Tu,Tv)^{\nu}.d^*(u,Tu)^{2\nu}.d^*(u,v)^{\nu}$

$d^*(Tu,Tv) \leq d(u,Tu)^{\frac{2\nu}{1-\nu}}.d^*(u,v)^{\frac{\nu}{1-\nu}}$ $\longrightarrow$ (2b)

By (i),(1),(2a),(2b) we can denote:

$\delta = \max\left\{\lambda, \dfrac{\mu}{1-\mu}, \dfrac{\nu}{1-\nu}\right\}$,

and it is easy to see that $\delta \in [0,1)$

For T satisfying at least one of the condition (i), (ii), (iii)

We have that

$d^*(Tu,Tv) \leq d(u,Tu)^{2\delta}.d^*(u,v)^{\delta}$ $\longrightarrow$ (3a)

and $d^*(Tu,Tv) \leq d(v,Tv)^{2\delta}.d^*(u,v)^{\delta}$ $\longrightarrow$ (3b)

Using these conditions implied by (i)-(iii) and taking $u \in \chi$, we have:

$d^*(T^n u, T^{n+1}u) = d^*(T(T^{n-1}u, T(T^n u)))$

$\leq d^*(T^{n-1}u, T(T^{n-1}u))^{2\delta}.d^*(T^{n-1}u, T^n u)^{\delta}$

$= d^*(T^{n-1}u, T^n u)^{3\delta}$

$= d^*(T^n u, T^{n+1}u) \leq ... \leq d^*(u,Tu)^{3\delta^n}$

$= d^*(T^n u, T^{n+1}u) \longrightarrow 1$ as $n \to \infty \forall u \in \chi$.

Now by lemma 3.1 it follows that $F_{\varepsilon}(T) \neq \phi, \forall \varepsilon > 1$.

In the Proof of 3.4 we have already shown that if f satisfies at least one of the conditions (i), (ii), (iii) from definition 2.11, then

$d^*(Tu,Tv) \leq d^*(u,Tu)^{2\rho}.d^*(u,v)^{\rho}$ and

$d^*(Tu,Tv) \leq d^*(v,Tv)^{2\rho}.d^*(u,v)^{\rho}$ hold.

Let $\varepsilon > 1$, Again we will only show that condition (ii) in lemma 3.2 is satisfied, as (i) holds, see the proof of theorem 3.4.

Let $\eta > 1$ and $u, v \in F_\varepsilon(T)$, and assume that

$$\frac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$$

Then $d^*(u,v) \leq d^*(Tu,Tv).\gamma$

$$d^*(u,v) \leq d^*(u,Tu)^{2\rho}.d^*(u,v)^{\rho}.\gamma$$

$$d^*(u,v)^{1-\rho} \leq \varepsilon^{2\rho}.\gamma$$

$$d^*(u,v) \leq \frac{\varepsilon^{2\rho}.\gamma}{1-\rho}$$

So $\forall > 1, \exists \phi(\gamma) = \dfrac{\gamma.\varepsilon^{2\rho}}{1-\rho} > 1$ such that

$$\frac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$$

$$\implies d^*(u,v) \leq \gamma$$

Now by lemma 3.2 it follows that

$$\delta(F_\varepsilon(T)) \leq \phi(\varepsilon^2), \forall \varepsilon > 1$$

which means exactly that

$$\delta(F_\varepsilon(T)) \leq \frac{\varepsilon^2.\varepsilon^{2\rho}}{1-\rho}$$

$$\delta(F_\varepsilon(T)) \leq \frac{\varepsilon^{2(1-\rho)}}{1-\rho}, \forall \varepsilon > 1$$

□

Theorem 3.5. *Let T:$\chi \to \chi$ be a Bianchini contraction on a multiplicative metric space (χ, d^*). Then T has an ε-fixed point and $\delta(F_\varepsilon(T)) \leq \varepsilon^{k+2}$, for all $\varepsilon > 1$.*

Proof. Given T is Bianchini contraction. Let $\varepsilon > 1$ and $u_0 \in T$. Define a sequence $\{u_n\}$ such that $u_{n+1} = Tu_n$, for all $n \geq 0$.

Case 1. If $B(u,v) = d^*(u,Tu)$. Then, Definition 3.5 becomes:

$$d^*(Tu,Tv) \leq d^*(u,Tu)^k$$

Substitute $v = Tu$, $d^*(Tu,T^2u) \leq d^*(u,Tu)^k$

Again substitute $u = Tu$, $d^*(T^2u,T^3u) \leq d^*(Tu,T^2u)^k$

$$\leq d^*(u,Tu)^{k^2}$$

$$\vdots$$

$$d^*(T^nu,T^{n+1}u) \leq d^*(u,Tu)^{k^n}$$

Case 2. If $B(u,v) = d^*(v,Tv)$. Then, Definition 3.5 becomes:

$$d^*(Tu,Tv) \leq d^*(v,Tv)^k$$

Substitute v=Tu, $d^*(Tu,T^2u) \leq d^*(Tu,T^2u)^k$

which is impossible because $k \in (0,1)$. Therefore, Case 2 does not exists. Now by Case 1, $d^*(T^n u, T^{n+1}u) \to 1$ as $n \to +\infty$, for all $u,v \in \chi$. Thus, $\{u_n\}$ is a Cauchy sequence, by Theorem 3.1, $F_\varepsilon(T) \neq \emptyset$ for all $\varepsilon > 1$. That is, T has an ε-fixed point. Consider,

$$
\begin{aligned}
d^*(u,v) &\leq d^*(Tu,Tv).\gamma \\
&\leq B(u,v)^k.\gamma \\
&= d^*(u,Tu)^k.\gamma \\
&= \varepsilon^k.\gamma
\end{aligned}
$$

So, for every $\gamma > 1$, there exists $\phi(\gamma) = \varepsilon^k.\gamma > 1$ such that

$\dfrac{d^*(u,v)}{d^*(Tu,Tv)} \leq \gamma$ implies $d^*(u,v) \leq \phi(\gamma)$

By Lemma 3.2, $\delta(F_\varepsilon(\mathrm{T})) \leq \phi(\varepsilon^2)$, for all $\varepsilon > 1$. Hence,

$\delta(F_\varepsilon(T)) \leq \varepsilon^{k+2}$, for all $\varepsilon > 1$. □

Corollary 3.1. *Let (χ, d^*) be a multiplicative metric space and $T:\chi \to \chi$. Then there exists $k \in (0,1)$ such that $d^*(Tu,Tv) \leq d^*(v,Tv)^k$, for all $u,v \in \chi$. Then T possesses an ε-fixed point and $\delta(F_\varepsilon(T)) \leq \varepsilon^{k+2}$, for all $\varepsilon > 1$.*

Proof:

Substituting $B(u,v) = d^*(v,Tv)$ in Theorem 3.5 completes this corollary.

Theorem 3.6. *Let (χ, d^*)be a multiplicative metric space. Suppose a self-map $T : \chi \to \chi$ is a n-convex contraction. Prove that for every $\varepsilon > 1$, $F_\varepsilon(T) \neq \emptyset$.*

Proof. Let $u_0 \in \chi$ and define $u_{n+1} = Tu_n$, for all $n \in \mathbb{N}$. Consider,

k = max $\{d^*(u_0,u_1), d^*(u_1,u_2), ..., d^*(u_{n-1},u_n)\}$

Now,

$$
\begin{aligned}
d^*(u_n,u_{n+1}) &= d^*(T^n u_0, T^n u_1) \\
&\leq d^*(u_0,u_1)^{k_0}.d^*(u_1,u_2)^{k_1}.\dots.d^*(u_{n-1},u_n)^{k_{n-1}} \\
&\leq k^{k_0}.k^{k_1}.\dots.k^{k_{n-1}} \\
&\leq k^{k_0+k_1+k_2+\dots+k_{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
d^*(u_{n+1},u_{n+2}) &= d^*(T^n u_1, T^n u_2) \\
&\leq d^*(u_1,u_2)^{k_0}.d^*(u_2,u_3)^{k_1}.\dots.d^*(u_{n-1},u_n)^{k_{n-2}}.d^*(u_n,u_{n+1})^{k_{n-1}}
\end{aligned}
$$

$$\leq k^{k_0}.k^{k_1}.\dots.k^{k_{n-2}}.k(k_0+k_1+...+k_{n-1})^{k_{n-1}}$$
$$\leq k^{k_0+k_1+k_2+...+k_{n-1}}$$

Similarly,

$$d^*(u_{n+2},u_{n+3}) \leq k^{k_0+k_1+k_2+...+k_{n-1}}$$
$$\vdots$$
$$d^*(u_{2n-1},u_{2n}) \leq k^{k_0+k_1+k_2+...+k_{n-1}}$$
$$d^*(u_{2n},u_{2n+1}) \leq d^*(u_n,u_{n+1})^{k_0}.d^*(u_{n+1},u_{n+2})^{k_1}.\ :\dots.d^*(u_{2n-1},u_{2n})^{k_{n-1}}$$
$$\leq k^{(k_0+k_1+...+k_{n-1})k_0}.\dots.k^{(k_0+k_1+...+k_{n-1})k_{n-1}}$$
$$\leq k^{k_0(k_0+k_1+...+k_{n-1})+...+k_{n-1}(k_0+k_1+...+k_{n-1})}$$
$$\leq k^{(k_0+k_1+...+k_{n-1})^2}$$

Again

$$d^*(u_{3n},u_{3n+1}) \leq k^{(k_0+k_1+...+k_{n-1})^3}$$

In general,

$$d^*(u_{n^2},u_{n^2+1}) \leq k^{(k_0+k_1+...+k_{n-1})^n}$$
$$\Pi d^*(u_{n^2},u_{n^2+1}) \leq k^{\Sigma(k_0+k_1+...+k_{n-1})^n}$$

That is $d^*(u_{n^2},u_{n^2+1}) \to 1$ as $n \to +\infty$. Therefore, $u_{n^2} \in F_\varepsilon(T)$, for all $\varepsilon > 1$ provides that $F_\varepsilon(T) \neq \emptyset$, for all $\varepsilon > 1$. Hence, χ has an approximate fixed point (ε-fixed point). □

Corollary 3.2. *Let (χ,d^*) be a multiplicative metric spaces. Suppose a selfmap $T:\chi\to\chi$ is a 2-convex contraction. Prove that for every $\varepsilon > 1$, $F_\varepsilon(T) \neq \emptyset$.*

Proof:

Substituting n=2 in Theorem 3.6 completes this corollary.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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