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A CONTROLLED METRIC TYPE SPACE AND SOME FIXED POINT RESULTS

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Abstract. In this article, we modify the controlled metric type spaces by using three control functions. Furthermore, we provide some fixed point results for Kannan type contractions in the setting of a generalized controlled metric type space.

Keywords: Kannan contraction; controlled metric type; b-metric.

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1. Introduction

Fixed point theory has numerous applications in science and Engineering [4, 5]. The proof of existence and uniqueness theorems for the solutions of ordinary and boundary value problems depends on applying different fixed point theorems. The most applied fixed point theorem is the Banach contraction principle, which has been generalized by modifying the contractive type condition or the finding generalizations of the underlying metric type space, [6, 8, 7].
Recently, Kamran et al. initiated the concept of an extended b-metric space, [3].

Definition 1.1. *Let X be a nonempty set, $d : X \times X \to [0,\infty)$, $\mu : X \times X \to [1,\infty)$ be mappings such that*

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(i) $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq \mu(x,z)d(x,z) + \mu(z,y)d(z,y)$,

for all $x,y,z \in X$.

Then d *is a controlled metric type and* (X,d) *is a controlled metric type space.*

Example 1.2. [1] *Let* $X = \{1,2,3,\cdots\}$. *Define* d *by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd} \\ \frac{1}{y}, & \text{if } x \text{ is odd and } y \text{ is even} \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

and the control function $\mu : X \times X \to [1,\infty)$

$$\mu(x,y) = \begin{cases} x, & \text{if } x \text{ is even and } y \text{ is odd} \\ y, & \text{if } x \text{ is odd and } y \text{ is even} \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

Then d *is a controlled type metric and* (X,d) *is a controlled metric type space.*

In a recent article the authors Abdeljawad et al. introduced the concept of a double controlled metric type, [2].

Definition 1.3. *Let* X *be a nonempty set,* $d : X \times X \to [0,\infty)$, $\mu, \nu : X \times X \to [1,\infty)$ *be mappings such that*

(i) $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq \mu(x,z)d(x,z) + \nu(z,y)d(z,y)$,

for all $x,y,z \in X$.

Then d *is a double controlled metric type and* (X,d) *is a double controlled metric type space.*

Example 1.4. [2] *Let $X = [0,\infty)$. Define d by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{1}{x}, & \text{if } x \geq 1 \text{ and } y \in [0,1) \\ \frac{1}{y}, & \text{if } y \geq 1 \text{ and } x \in [0,1) \\ 1, & \text{otherwise} \end{cases}$$

and the control functions $\mu, \nu : X \times X \to [1,\infty)$ by

$$\mu(x,y) = \begin{cases} x, & \text{if } x,y \geq 1 \\ 1, & \text{otherwise} \end{cases}$$

and

$$\nu(x,y) = \begin{cases} 1, & \text{if } x,y < 1 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

Then d is a double controlled metric type and (X,d) is a double controlled metric type space.

2. Main Results

We introduce a generalization of the controlled metric type by introducing three control functions.

Definition 2.1. *Let X be a non-empty set $\xi_1, \xi_2, \xi_3 : X \times X \times X \to [1,\infty)$ and $d : X \times X \times X \to [0,\infty)$ be mappings satisfying the following properties:*

(i) $d(x,y,z) = 0$ if at least two of the three points are the same.

(ii) For $x,y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

(iii) For $x,y,z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) For $x,y,z,t \in X$:

$$d(x,y,z) \leq \xi_1(x,y,t)d(x,y,t) + \xi_2(y,x,t)d(y,z,t) + \xi_3(z,x,t)d(z,x,t)$$

Then d is a controlled metric type and (X,d) is a controlled metric type space.

Example 2.2. *Let $X=(0,1)$ and define*

$$d(x,y,z)=\begin{cases} 0, & \text{if at least two of the three points are the same} \\ \alpha(x,y,z)e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}}, & \text{otherwise} \end{cases}$$

where $\xi \geq 1$ and $\alpha : X\times X\times X \to [0,\infty)$ is a continuous function such that $d(x,y,z)$ is symmetric with respect to x,y,z. It suffices to only verify property (iv) of definition 2.1:
For $x,y,z,t \in X$ and using Jensen's inequality, we get

$$\begin{aligned}
& d(x,y,z) \\
& = \alpha(x,y,z)e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-z|^{\xi}+\frac{1}{6}|z-x|^{\xi}} \\
& \leq \alpha(x,y,z)\left[\frac{1}{2}e^{|x-y|^{\xi}}+\frac{1}{3}e^{|y-z|^{\xi}}+\frac{1}{6}e^{|z-x|^{\xi}}\right] \\
& = \alpha(x,y,z)\left[\frac{1}{2}e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{2}|x-y|^{\xi}}+\frac{1}{3}e^{\frac{1}{2}|y-z|^{\xi}+\frac{1}{2}|y-z|^{\xi}}+\frac{1}{6}e^{\frac{1}{2}|z-x|^{\xi}+\frac{1}{2}|z-x|^{\xi}}\right] \\
& \leq \left[\frac{1}{2}e^{\frac{1}{2}|x-y|^{\xi}}+1\right]\alpha(x,y,z)e^{\frac{1}{2}|x-y|^{\xi}+\frac{1}{3}|y-t|^{\xi}+\frac{1}{6}|t-x|^{\xi}}+ \\
& + \left[\frac{1}{3}e^{\frac{1}{2}|z-y|^{\xi}}+1\right]\alpha(x,y,z)e^{\frac{1}{2}|z-y|^{\xi}+\frac{1}{3}|y-t|^{\xi}+\frac{1}{6}|t-z|^{\xi}} \\
& + \left[\frac{1}{6}e^{\frac{1}{2}|z-x|^{\xi}}+1\right]\alpha(x,y,z)e^{|z-x|^{\xi}+|x-t|^{\xi}+|t-z|^{\xi}} \\
& = \xi_1(x,y,t)d(x,y,t)+\xi_2(z,y,t)d(z,y,t)+\xi_3(z,x,t)d(z,x,t)
\end{aligned}$$

where $\xi_1(x,y,t)=\frac{1}{2}e^{\frac{1}{2}|x-y|^{\xi}}+1\geq 1$, $\xi_2(z,y,t)=\frac{1}{3}e^{\frac{1}{2}|y-z|^{\xi}}+1\geq 1$ and $\xi_3(z,x,t)=\frac{1}{6}e^{\frac{1}{2}|z-x|^{\xi}}+1\geq 1$

The topological concepts as continuity, convergence and Cauchy is given as follows.

Definition 2.3. *Let (X,d) be a controlled metric type space then*

$$B_{\varepsilon}(a,b)=\{z\in X : d(a,b,z)<\varepsilon\}$$

is an open ball with radius $\varepsilon>0$ and center (a,b). A mapping $T : X\to X$ is continuous at $a,b\in X$, if for all $\varepsilon>0$, there exists $r>0$ such that $T(B_r(a,b))\subset B_{\varepsilon}(a,b)$.

Definition 2.4. *Let (X,d) be a controlled metric type space.*

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x\in X$, if for each $\varepsilon>0$, there exist some integer

$N \in \mathbb{N}$ such that $d(x_n, x, z) < \varepsilon$ for each $n \geq \mathbb{N}$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m, z) < \varepsilon$ for $n, m > N$.

(iii) The space (X, d) is complete, if every Cauchy sequence is convergent.

In order to simplify results, we define the following products as:

Define $P_0(z, x_n, x_{n+1}) = \xi_1$

$P_1(x_m, z, x_{n+1}, x_{n+2}) = \xi_2\xi_3$

$P_2(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}) = \xi_2\xi_1\xi_3$

$P_3(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}) = \xi_2\xi_1\xi_1\xi_3$

in general, we get

$$P_{m-n-1}(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \cdots, x_{m-1}) = \xi_2 \underbrace{\xi_1 \cdots \xi_1}_{m-n-2} \xi_3$$

Define $Q_0(z, x_n, x_m) = \xi_3$

$Q_1(x_m, z, x_{n+1}, x_{n+2}) = \xi_2\xi_2$

$Q_2(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}) = \xi_2\xi_1\xi_2$

$Q_3(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}) = \xi_2\xi_1\xi_1\xi_2$

in general, we get

$$Q_{m-n-1}(x_m, z, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, \cdots, x_{m-1}) = \xi_2 \underbrace{\xi_1 \cdots \xi_1}_{m-n-2} \xi_2.$$

Theorem 2.5. *Let X be a nonempty set and (X, d) be a complete controlled metric type space. Let $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty, z) \leq kd(x, y, z) \tag{3}$$

for $x, y, z \in X$ and $0 < k < 1$. Assume that

$$\sup_{m\geq 1} \lim_{j\to\infty} \left| \frac{kP_{j+1}(x_m, z, x_{n+1}, \cdots, x_{j+2})}{P_j(x_m, z, x_{n+1}, \cdots, x_{j+1})} \right| < 1 \tag{4}$$

and

$$\sup_{m\geq 1} \lim_{j\to\infty} \left| \frac{kQ_{j+1}(x_m, z, x_{n+1}, \cdots, x_{j+2})}{Q_j(x_m, z, x_{n+1}, \cdots, x_{j+1})} \right| < 1 \tag{5}$$

and for each $x \in X$ assume that

$$\lim_{n\to\infty} \xi_1(x, x_n, z), \tag{6}$$

exists,

$$\lim_{n\to\infty} \xi_2(x, x_n, z) \tag{7}$$

exists and

$$\lim_{n\to\infty} \xi_3(x, x_n, z) \tag{8}$$

exists. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Then define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned}
& d(x_n, x_{n+1}, z) \\
& = d(Tx_{n-1}, Tx_n, z) \\
& \leq kd(x_{n-1}, x_n, z) \\
& = kd(Tx_{n-2}, Tx_{n-1}, z) \\
& \leq k^2 d(x_{n-2}, x_{n-1}, z) \\
& \vdots \\
& \leq k^n d(x_1, x_0, z) \qquad (9)
\end{aligned}$$

For $m, n \in \mathbb{N}$, $m \geq n$, we obtain

$$\begin{aligned}
& d(x_n, x_m, z) \\
& \leq \xi_1(x_n, x_m, x_{n+1}) d(x_n, x_m, x_{n+1}) + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_1(x_m, z, x_{n+3}) \\
& \quad \xi_1(x_m, z, x_{n+4}) d(x_m, z, x_{n+4}) \\
& + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_1(x_m, z, x_{n+3}) \xi_2(z, x_{n+3}, x_{n+4}) d(z, x_{n+3}, x_{n+4}) \\
& + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_1(x_m, z, x_{n+3}) \xi_3(x_{n+3}, x_m, x_{n+4}) d(x_{n+3}, x_m, x_{n+4}) \\
& + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_2(z, x_{n+2}, x_{n+3}) d(z, x_{n+2}, x_{n+3})
\end{aligned}$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_3(x_{n+2},x_m,x_{n+3})d(x_{n+2},x_m,x_{n+3})$$

$$+\xi_2(x_m,z,x_{n+1})\xi_2(z,x_{n+1},x_{n+2})d(z,x_{n+1},x_{n+2})$$

$$+\xi_2(x_m,z,x_{n+1})\xi_3(x_{n+1},x_m,x_{n+2})d(x_{n+1},x_m,x_{n+2})$$

$$+\xi_3(z,x_m,x_{n+1})d(z,x_n,x_{n+1}) \tag{10}$$

Using inequality 9, we get

$$d(x_n,x_m,z)$$

$$\leq \xi_1(x_n,x_m,x_{n+1})k^n d(x_0,x_1,x_m)+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_1(x_m,z,x_{n+3})$$

$$\xi_1(x_m,z,x_{n+4})d(x_m,z,x_{n+4})$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_1(x_m,z,x_{n+3})\xi_2(z,x_{n+3},x_{n+4})k^{n+3}d(x_0,x_1,z)$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_1(x_m,z,x_{n+3})\xi_3(x_{n+3},x_m,x_{n+4})k^{n+3}d(x_0,x_1,x_m)$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_2(z,x_{n+2},x_{n+3})k^{n+2}d(x_0,x_1,z)$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_3(x_{n+2},x_m,x_{n+3})k^{n+2}d(x_0,x_1,x_m)$$

$$+\xi_2(x_m,z,x_{n+1})\xi_2(z,x_{n+1},x_{n+2})k^{n+1}d(x_0,x_1,z)$$

$$+\xi_2(x_m,z,x_{n+1})\xi_3(x_{n+1},x_m,x_{n+2})k^{n+1}d(x_0,x_1,x_m)$$

$$+\xi_3(z,x_m,x_{n+1})k^n d(x_0,x_1,z) \tag{11}$$

It follows that inequality 11, can be written as

$$d(x_n,x_m,z)$$

$$\leq k^n d(x_0,x_1,x_m)\left[P_0(x_n,x_m,x_{n+1})+kP_1(x_m,z,x_{n+1},x_{n+2})\right.$$

$$\left.+k^2P_2(x_m,z,x_{n+1},x_{n+2},x_{n+3})+\cdots+k^{m-n-1}P_{m-n-1}(x_m,z,x_{n+1},x_{n+2},x_{n+3},\cdots,x_{m-1})+\cdots\right]+$$

$$k^n d(x_0,x_1,z)\left[Q_0(x_n,x_m,x_{n+1})+kQ_1(x_m,z,x_{n+1},x_{n+2})\right.$$

$$\left.+k^2Q_2(x_m,z,x_{n+1},x_{n+2},x_{n+3})+\cdots+k^{m-n-1}Q_{m-n-1}(x_m,z,x_{n+1},x_{n+2},x_{n+3},\cdots,x_{m-1})+\cdots\right] \tag{12}$$

The ratio test together with 4 and 5, implies that the series converges and thus the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete controlled metric type space there exist $x^*\in X$ such that $d(x_n,x^*,z)\to 0$ as $n\to\infty$.

We claim that x^* is a fixed point of T:

Using 9, we get

$$
\begin{aligned}
&d(x^*, Tx^*, z)\\
&\leq \xi_1(x^*, Tx^*, x_{n+1})d(x^*, Tx^*, x_{n+1}) + \xi_2(Tx^*, z, x_{n+1})d(Tx^*, z, x_{n+1}) + \xi_3(z, x^*, x_{n+1})d(z, x^*, x_{n+1})\\
&\leq \xi_1(x^*, Tx^*, x_{n+1})d(x^*, Tx^*, x_{n+1}) + \xi_2(Tx^*, z, x_{n+1})d(Tx^*, z, Tx_n)\\
&+ \xi_3(z, x^*, x_{n+1})d(z, x^*, x_{n+1})\\
&\leq \xi_1(x^*, Tx^*, x_{n+1})d(x^*, Tx^*, x_{n+1}) + \xi_2(Tx^*, z, x_{n+1})kd(x^*, z, x_n)\\
&+ \xi_3(z, x^*, x_{n+1})d(z, x^*, x_{n+1})
\end{aligned}
$$

Taking the limit as $n \to \infty$ and using 19, we get

$$d(x^*, Tx^*, z) \leq 0.$$

It follows that $Tx^* = x^*$.

To prove uniqueness, we assume that there exist $x' \in X$ such that $Tx' = x'$. Then

$$d(x^*, x', z) = d(Tx^*, Tx', z) \leq kd(x^*, x', z), \tag{13}$$

since $k < 1$, this will lead to a contradiction, unless $d(x^*, x', z) = 0$. Hence, x^* is the unique fixed point of T. □

In the theorem that follows we propose the related fixed point results based on the Kannan contraction in a three dimensional controlled metric type space.

Theorem 2.6. *Let (X, d) be a complete controlled metric type space. Let $T : X \to X$ be a mapping subject to the Kannan type contraction*

$$d(Tx, Ty, z) \leq \alpha[d(x, Tx, z) + d(y, Ty, z)] \tag{14}$$

for $x, y, z \in X$ and $0 < \alpha < \frac{1}{2}$. Futhermore, we assume that

$$\sup_{m\geq 1} \lim_{j\to\infty} \left| \left(\frac{\alpha}{1-\alpha}\right) \frac{P_{j+1}(x_m, z, x_{n+1}, \cdots, x_{j+2})}{P_j(x_m, z, x_{n+1}, \cdots, x_{j+1})} \right| < 1 \tag{15}$$

and

$$\sup_{m\geq 1} \lim_{j\to\infty} \left| \left(\frac{\alpha}{1-\alpha}\right) \frac{Q_{j+1}(x_m, z, x_{n+1}, \cdots, x_{j+2})}{Q_j(x_m, z, x_{n+1}, \cdots, x_{j+1})} \right| < 1 \tag{16}$$

and for each $x \in X$ assume that

$$\lim_{n\to\infty} \xi_1(x, x_n, z), \tag{17}$$

exists,

$$\lim_{n\to\infty} \xi_2(x, x_n, z) \tag{18}$$

exists and

$$\lim_{n\to\infty} \xi_3(x, x_n, z) \tag{19}$$

exists.

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned} & d(x_n, x_{n+1}, z) \\ & = d(Tx_{n-1}, Tx_n, z) \\ & \leq \alpha\,[d(x_{n-1}, Tx_{n-1}, z) + d(x_n, Tx_n, z)] \end{aligned} \tag{20}$$

It follows that

$$d(x_n, x_{n+1}, z) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n, z) \tag{21}$$

since $0 < \alpha < \frac{1}{2}$, we get $\frac{\alpha}{1-\alpha} < 1$. Inductively, we can show that

$$d(x_n, x_{n+1}, z) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, x_1, z) \tag{22}$$

Next, we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Repeated use of the triangle inequality, we get

$$\begin{aligned} & d(x_n, x_m, z) \\ & \leq \xi_1(x_n, x_m, x_{n+1}) d(x_n, x_m, x_{n+1}) + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_1(x_m, z, x_{n+3}) \\ & \quad \xi_1(x_m, z, x_{n+4}) d(x_m, z, x_{n+4}) \\ & + \xi_2(x_m, z, x_{n+1}) \xi_1(x_m, z, x_{n+2}) \xi_1(x_m, z, x_{n+3}) \xi_2(z, x_{n+3}, x_{n+4}) d(z, x_{n+3}, x_{n+4}) \end{aligned}$$

$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_1(x_m,z,x_{n+3})\xi_3(x_{n+3},x_m,x_{n+4})d(x_{n+3},x_m,x_{n+4})$$
$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_2(z,x_{n+2},x_{n+3})d(z,x_{n+2},x_{n+3})$$
$$+\xi_2(x_m,z,x_{n+1})\xi_1(x_m,z,x_{n+2})\xi_3(x_{n+2},x_m,x_{n+3})d(x_{n+2},x_m,x_{n+3})$$
$$+\xi_2(x_m,z,x_{n+1})\xi_2(z,x_{n+1},x_{n+2})d(z,x_{n+1},x_{n+2})$$
$$+\xi_2(x_m,z,x_{n+1})\xi_3(x_{n+1},x_m,x_{n+2})d(x_{n+1},x_m,x_{n+2})$$
$$+\xi_3(z,x_m,x_{n+1})d(z,x_n,x_{n+1}) \tag{23}$$

$$\begin{aligned}
&d(x_n,x_m,z)\\
&\leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0,x_1,x_m)\Bigg[P_0(x_n,x_m,x_{n+1})+\left(\frac{\alpha}{1-\alpha}\right)P_1(x_m,z,x_{n+1},x_{n+2})\\
&+\left(\frac{\alpha}{1-\alpha}\right)^2 P_2(x_m,z,x_{n+1},x_{n+2},x_{n+3})+\cdots+\left(\frac{\alpha}{1-\alpha}\right)^{m-n-1}P_{m-n-1}(x_m,z,x_{n+1},x_{n+2},x_{n+3},\cdots,x_{m-1})+\cdots\Bigg]\\
&+\left(\frac{\alpha}{1-\alpha}\right)^n d(x_0,x_1,z)\Bigg[Q_0(x_n,x_m,x_{n+1})+\left(\frac{\alpha}{1-\alpha}\right)Q_1(x_m,z,x_{n+1},x_{n+2})\\
&+\left(\frac{\alpha}{1-\alpha}\right)^2 Q_2(x_m,z,x_{n+1},x_{n+2},x_{n+3})+\cdots+\left(\frac{\alpha}{1-\alpha}\right)^{m-n-1}Q_{m-n-1}(x_m,z,x_{n+1},x_{n+2},x_{n+3},\cdots,x_{m-1})+\cdots\Bigg]
\end{aligned} \tag{24}$$

In an argument similar to that in theorem 2.5, we can conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete metric type space, thus there exist $x^*\in X$, such that $d(x_n,x^*,z)\to 0$ as $n\to\infty$. Next, we prove that x^* is a fixed point for T:

$$\begin{aligned}
&d(x^*,Tx^*,z)\\
&\leq \xi_1(x^*,Tx^*,x_{n+1})d(x^*,Tx^*,x_{n+1})+\xi_2(Tx^*,z,x_{n+1})d(Tx^*,z,x_{n+1})\\
&+\xi_3(z,x^*,x_{n+1})d(z,x^*,x_{n+1})\\
&\leq \xi_1(x^*,Tx^*,x_{n+1})d(x^*,Tx^*,x_{n+1})+\xi_2(Tx^*,z,x_{n+1})\alpha d(x^*,z,x_{n+1})\\
&+\xi_2(Tx^*,z,x_{n+1})\alpha d(x_n,Tx_n,z)+\xi_3(z,x^*,x_{n+1})d(z,x^*,x_{n+1})\\
&\leq \xi_1(x^*,Tx^*,x_{n+1})d(x^*,Tx^*,x_{n+1})+\xi_2(Tx^*,z,x_{n+1})\alpha d(x^*,z,x_{n+1})\\
&+\xi_2(Tx^*,z,x_{n+1})\alpha\left(\frac{\alpha}{1-\alpha}\right)^n d(x_1,x_0,z)+\xi_3(z,x^*,x_{n+1})d(z,x^*,x_{n+1})
\end{aligned}$$

Taking the limit as $n \to \infty$, we deduce that $d(x^*, Tx^*, z) = 0$, which implies that $Tx^* = x^*$. To prove the uniqueness of the fixed point for T, we assume that $x' \in X$ is another fixed point of T:

$$\begin{aligned} & d(x^*, x', z) \\ & = d(Tx^*, Tx', z) \\ & \leq \alpha d(x^*, Tx^*, z) + \alpha d(x', Tx', z) \\ & = \alpha d(x^*, x^*, z) + \alpha d(x', x', z) = 0 \end{aligned}$$

It follows that $d(x^*, x', z) = 0$ which implies that $x^* = x$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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