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Advances in Fixed Point Theory, 3 (2013), No. 1, 306-314

ISSN: 1927-6303

COMMON FIXED POINTS FOR NONEXPANSIVE TYPE SINGLE-VALUED MAPS

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Abstract. In the present paper, we establish a result for the existence of common fixed points for pair of Weakly Reciprocally Continuous (w.r.c) maps satisfying nonexpansive type condition with some weaker form of commutativity. Our result extends the result of pant et.al [6] and also generalizes several well known results available in the literature. Moreover, our result complements the result of Pant et. al[7] by extending the scope of applications of w.r.c. to nonexpansive type conditions (see [5], [4]) in metric spaces.

Keywords: weakly reciprocally continuous, compatible maps, nonexpansive maps

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

In 1982, Sessa[9] introduced the weak commutativity condition for a pair of single valued maps. Jungck[4] generalized the concept of weak commutativity condition by introducing compatibility of maps. The study of noncompatibilty was initiated by Pant[7] by introducing point wise R-weakly commutativity of maps. Recently Al- Thagafi and Shahzad [1] introduced the notion of occasionally weakly compatible maps and employed the new notion to prove fixed point theorem under new condition. Here it seems important to mention that weak commutativity implies compatibility but the converse is not true. Weak commutativity implies R- weak commutativity but R- weak commutativity implies weak commutativity only when $R \leq 1$. Self-mappings f and g of a metric space (X, d) are called R-weakly commuting of type A_g [3] if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, two

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Received March 2, 2013

selfmappings f and g of a metric space (X, d) are called R-weakly commuting of type A_f [3] if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X . It is to be noted that pointwise R-weakly commuting maps commute at their coincidence points and pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Compatible and noncompatible maps can be R-weakly commuting of type A_g or A_f.

In 1998 pant [8] introduced reciprocal continuity (r.c.) for the pair of single-valued maps which states that maps f and g are r.c. if and only if $lim_n\, gfx_n = gt$ and $lim_n\, fgx_n = ft$ whenever $\{x_n\}$ is a sequence in X such that $lim_n\, fx_n = lim_n\, gx_n = t$ for some t in X. They also established some common fixed point theorems for reciprocally continuous maps. It is also proved that a pair of maps which is reciprocally continuous need not to be continuous even on their common fixed point [see example [8]]

2. Preliminaries

Recently Pant et.al. [6], generalized reciprocal continuity and introduced Weakly Reciprocal Continuity (w.r.c.) for a pair of single-valued maps as follows:

Definition1[6]. Two self mappings f and g of a metric space (X, d) are called weakly reciprocally continuous if $lim_n\, fgx_n = ft$ or $lim_n\, gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $lim_n\, fx_n = lim_n\, gx_n = t$ for some t in X.

It is to be noted that that reciprocal continuity implies weak reciprocal continuity but the converse is not true as shown in the following example.

Example1[6]. Let $X = [2, 20]$ and d be a usual metric in X. Define $f, g : X \to X$ as follows,

$fx = 2$ if $x = 2$ or $x > 5$, $fx = 6$ if $2 < x \leq 5$,

$g2 = 2$, $gx = 12$ if $2 < x \leq 5$, $gx = (x + 1) / 3$ if $x > 5$.

Here f and g are clearly weakly reciprocally continuous but not reciprocally continuous. Some common fixed point theorems for the w.r.c pair of maps was also obtained by Pant et al. [6]. Here it is to be noted that only w.r.c. does not guarantee the existence of common fixed point or even coincidence point. This fact is illustrated in following example.

Example2. Let X = [0, ∞) be endowed with the usual metric and

$$T(x) = \begin{Bmatrix} 0 & if\ x \leq 2 \\ \frac{3}{2} & if\ 2 < x \leq 3 \\ 3 & if\ 3 < x \leq 4 \\ x & if\ x > 4 \end{Bmatrix}, \qquad f(x) = \begin{Bmatrix} \frac{3}{2} & if\ x \leq 2 \\ x+1 & if\ 2 < x < 3 \\ x & if\ 3 \leq x < 4 \\ x+6 & if\ x \geq 4 \end{Bmatrix},$$

If we take $\{x_n\} = (3 + \frac{1}{n})$

$$lim_n\, T[\,3 + \tfrac{1}{n}] = 3,$$

$$lim_n\, f[\,3 + \tfrac{1}{n}] = \lim_n \{3 + \tfrac{1}{n}\} = 3$$

$$lim_n f\,T[\,3 + \tfrac{1}{n}] = \lim_n f\{3\} = 3 = f(3)$$

$$lim_n\, Tf[\,3 + \tfrac{1}{n}] = \lim_n T\{3 + \tfrac{1}{n}\} = 3 \neq T(3)$$

Since $lim_n\, fTx_n = ft$ but $lim_n\, Tfx_n \neq Tt$, the pair *(T, f)* is not reciprocally continuous but weakly reciprocally continuous and compatible. It is to be noted that the *T* and *f* do not have any coincidence point.

A map $T: X \rightarrow X$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all *x*, *y* in *X*. Ciric [4] investigated a class of nonexpansive type self maps *T* of X and established some interesting result for the existence of fixed points for such mapping.

Chandra et.al [5] generalized the condition given by Ciric [4] and gave the following nonexapansive type condition. Let $T, f: X \rightarrow X$.

$$d(Tx, Ty) \leq a(x, y)\, d(fx, fy) + b(x, y) \max\{d(fx, Tx), d(fy, Ty)\}$$

$$+ c(x, y)\,[d(fx, Ty) + d(fy, Tx)],$$

Where $a(x, y) \geq 0$, $\beta : inf_{x, y \in X}\, b(x, y) > 0$, $\gamma = \inf_{x, y \in X} c(x, y) > 0$, and

$$sup_{x, y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = 1. \qquad (1)$$

Taking f as an identity map and $a(x, y)= a$, $b(x, y) = b$, $c(x, y) = c$ then above condition is generalization of Ciric [3] as well as many other contractive conditions (see [2]).

It is to be noted that the condition of Ciric [3] is contained in (1) (see [5]). Chandra et.al [5] proved that a compatible pair of maps on the complete metric space satisfying the above condition will have a coincidence point if f is surjective or continuous. Relaxing the condition of continuity by w.r.c. condition, we extend the scope of study of nonexapansive type condition to the class of mappings which include both continuous and discontinuous maps and establish following result for the existence of common fixed points.

3. Main result

Theorem 3.1. Let T and f be weakly reciprocally continuous self maps of a complete metric space (X, d) satisfying (1) with $T(X) \subseteq f(X)$ then T and f have a common fixed point in X if either (a) T and f are compatible or (b) T and f are R weakly commuting of type A_f or (c) T and f are R weakly commuting of type A_T,

Proof. Case (a). Suppose T and f are Compatible.

Pick $x_o \in X$. We construct a sequence $\{x_n\}$ in X such that $fx_1 = Tx_0$ .In general, choose x_{n+1} such that $fx_{n+1}=Tx_n$. As proved in Theorem 2.1 of [2], we get $\{fx_n\}$, $\{Tx_n\}$, as Cauchy sequences and completeness of the space implies $lim_n fx_{n+1} = lim_n Tx_n = t$ for some t in X. Since f and T are weakly reciprocally continuous hence either $lim_n fTx_n \to ft$ or $lim_n Tfx_n \to Tt$. Let $lim_n fTx_n \to ft$. Compatibility of f and T shows $\lim_n d(fTx_n, Tfx_n) = 0$. Letting $n \to \infty$ we get $lim_n Tfx_n \to ft$ and $Tfx_{n+1} = TTx_n \to ft$.

Now using (1)

$$d(Tt, TTx_n) \leq a(x, y)\, d(ft, fTx_n)$$

$$+b(x, y) \max\{ d(ft, Tt), d(fTx_n, TTx_n)\}$$

$$+ c(x, y)[d(ft, TTx_n) + d(fTx_n, Tt)]$$

On letting $n \to \infty$ we get

$d(Tt, ft) \leq \{b(x, y) + c(x, y)\}\, d(Tt, ft)$

Since $\sup_{x, y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = 1$ and inf $c(x, y) > 0$ implies $\{b(x, y) + c(x, y)\} < 1$. Hence $ft = Tt$. Compatibility of f and T implies commutativity at coincidence point, hence $fTt = Tft = TTt = fft$. Again using (1)

$$d(Tt, TTt) \leq a(x, y)\, d(ft, fTt)$$

$$+ b(x, y) \max\{d(ft, Tt), d(fTt, TTt)\}$$

$$+ c(x, y)\{ d(ft, TTt) + d(fTt, Tt)\}$$

$$= [a(x, y) + 2c(x, y)]\, d(Tt, TTt)$$

Since $\beta > 0 \Rightarrow \sup[a(x, y) + 2c(x, y)] < 1$. Hence

$Tt = TTt = fTt$, i.e. Tt is a common fixed point of f and T.

Next suppose that for $\lim_n Tfx_n \rightarrow Tt$. Since $T(X) \subseteq f(X)$ implies that $Tt = fz$ for some $z \in X$ and $lim_n Tfx_n \rightarrow fz$. Compatibility of f and T implies, $\lim_n fTx_n \rightarrow fz$. Since $Tfx_{n+1} = TTx_n$ and $Tfx_{n+1} \rightarrow fz$, it follows $TTxn \rightarrow fz$. Now using (1)

$$d(Tz, TTx_n) \leq a(x, y)\, d(fz, fTx_n)$$

$$+ b(x, y)\max\{ d(fz, Tz), d(fTx_n, TTx_n)\}$$

$$+ c(x, y)\{ d(fz, TTx_n) + d(fTx_n, Tz)\},$$ On letting $n \rightarrow \infty$ we get

$d(fz, Tz) \leq [b(x, y) + c(x, y)]\, d(fz, Tz)$ which implies $fz = Tz$. Compatibility of f and T implies commutativity at coincidence point, hence $fTz = Tfz = TTz = ffz$. Again using (1)

$$d(Tz, TTz) \leq a(x, y)\, d(fz, fTz)$$

$$+ b(x, y)\max\{ d(fz, Tz), d(fTz, TTz)\}$$

$$+ c(x, y)\{ d(fz, TTz) + d(fTz, Tz))\}$$

$$= [a(x, y) + 2\, c(x, y)]\, d(Tz, TTz).$$

This implies $Tz = TTz = fTz$ *i.e.* Tz is common fixed point of f and T.

Case (b). Now suppose that T and f are R – weakly commuting of type (A_f).

Since f and T are weakly reciprocally continuous hence either $lim_n fTx_n \to ft$ or $lim_n Tfx_n \to Tt$. Let $lim_n fTx_n \to ft$. Then R- weak commutativity of type A_f of f and T gives $d(TTx_n, fTx_n) \leq Rd(Tx_n, fx_n)$. Making $n \to \infty$ we get $TTx_n \to ft$. Now using (1) we get

$$d(Tt, TTx_n) \leq a(x, y)\, d(ft, fTx_n)$$

$$+b(x, y) \max\{ d(ft, Tt), d(fTx_n, TTx_n)\}$$

$$+ c(x, y)[d(ft, TTx_n) + d(fTx_n, Tt)]$$

On letting $n \to \infty$ we get

$$d(Tt, ft) \leq \{b(x, y) + c(x, y)\}\, d(Tt, ft)$$

i.e. $ft = Tt$. Again by R- weak commutativity of type A_f , $d(TTt, fTt) \leq Rd(ft, Tt)$. This gives $TTt = fTt$ or $TTt = Tft = fTt = fft$. Again using (1),

$$d(Tt, TTt) \leq a(x, y)\, d(ft, fTt)$$

$$+ b(x, y) \max\{d(ft, Tt), d(fTt, TTt)\}$$

$$+ c(x, y)\{ d(ft, TTt) + d(fTt, Tt)\}$$

$$= [a(x, y) + 2c(x, y)]\, d(Tt, Tt).$$

This implies $Tt = TTt = fTt$, i.e. Tt is a common fixed point of f and T.

Now suppose that $\lim_n Tfx_n \to Tt$. Since $T(X) \subseteq f(X)$ implies that $Tt = fz$ for some $z \in X$ and $lim_n Tfx_n \to fz$. Since $Tfx_{n+1} = TTx_n$ and $Tfx_{n+1} \to fz$, it follows that $TTx_n \to fz$. Then R weak commutativity of type (A_f) of f and T gives $d(TTx_n, fTx_n) \leq Rd(Tx_n, fx_n)$ on letting $n \to \infty$ we get $fTx_n \to fz$. Now using (1)

$$d(Tz, TTx_n) \leq a(x, y)\, d(fz, fTx_n)$$

$$+ b(x, y) \max\{ d(fz, Tz), d(fTx_n, TTx_n)\}$$

$+ c(x, y)\{ d(fz, TTx_n) + d(fTx_n, Tz)\}$, On letting $n \to \infty$ we get

$d(fz, Tz) \leq [b(x, y) + c(x, y)]\, d(fz, Tz)$ which implies $fz = Tz$. Again by R- weak commutativity of type A_f implies $d(TTz, fTz) \leq Rd(fz, Tz)$. This gives $TTz = fTz$ or $TTz = Tfz = fTz = ffz$. Again using (1),

$d(Tz, TTz) \leq a(x, y)\, d(fz, fTz)$

$+ b(x, y) max\{ d(fz, Tz), d(fTz, TTz)\}$

$+ c(x, y)\{ d(fz, TTz) + d(fTz, Tz))\}$

$= [a(x, y) + 2\, c(x, y)]\, d(Tz, TTz)$.

This implies $Tz = TTz = fTz$ i.e. Tz i.e. Tz is common fixed point of f and T.

Case (c). Let T and f are R – weakly commuting of type (A_T). Since f and T are weakly reciprocally continuous hence either $lim_n fTx_n \to ft$ or $lim_n Tfx_n \to Tt$. Let $lim_n fTx_n \to ft$. Then R- weak commutativity of type A_T of f and T gives $d(Tfx_n, ffx_n) \leq Rd(Tx_n, fx_n)$. Making $n \to \infty$ we get $Tfx_n \to ft$. Now using (1)

$d(Tt, TTx_n) \leq a(x, y)\, d(ft, fTx_n)$

$+b(x, y) max\{ d(ft, Tt), d(fTx_n, TTx_n)\}$

$+ c(x, y)[d(ft, TTx_n) + d(fTx_n, Tt)]$

On letting $n \to \infty$ we get

$d(Tt, ft) \leq \{b(x, y) + c(x, y)\}\, d(Tt, ft)$

i.e. $ft = Tt$. Again R- weak commutativity of type A_T implies $d(Tft, fft) \leq Rd(Tt, ft)$. This gives $Tft = fft$ and $TTt = Tft = fTt = fft$. Again Using (1)

$d(Tt, TTt) \leq a(x, y)\, d(ft, fTt)$

$+ b(x, y) max\{d(ft, Tt), d(fTt, TTt)\}$

$+ c(x, y)\{ d(ft, TTt) + d(fTt, Tt)\}$

$= [a(x, y) + 2c(x, y)]\, d(Tt, TTt).$

This implies $Tt = TTt = fTt$, i.e. Tt is a common fixed point of f and T.

Now suppose $\lim_n Tfx_n \to Tt$. Since $T(X) \subseteq f(X)$ we get $Tt = fz$ for some $z \in X$ and $lim_n Tfx_n \to fz$. Since $Tfx_{n+1} = TTx_n$ and $Tfx_{n+1} \to fz$, it follows that $TTx_n \to fz$. R weak commutativity of type (A_T) implies $d(Tfx_n, ffx_n) \leq Rd\,(Tx_n, fx_n)$ on letting $n \to \infty$ we get $ffx_n \to fz$. Now using (1) we get,

$d(Tz, TTx_n) \leq a(x, y)\, d(fz, fTx_n)$

$+ b(x, y) max\{ d(fz, Tz), d(fTx_n, TTx_n)\}$

$+ c(x, y)\{ d(fz, TTx_n) + d(fTx_n, Tz)\}$, On letting $n \to \infty$ we get

$d(fz, Tz) \leq [b(x, y) + c(x, y)]\, d(fz, Tz)$ which implies $fz = Tz$. R- weak commutativity of type A_T implies $d(Tfz, ffz) \leq Rd\,(Tz, fz)$, this gives $Tfz = ffz$ and $TTz = Tfz = fTz = ffz$. Again using (1)

$d(Tz, TTz) \leq a(x, y)\, d(fz, fTz)$

$+ b(x, y) max\{ d(fz, Tz), d(fTz, TTz)\}$

$+ c(x, y)\{ d(fz, TTz) + d(fTz, Tz))\}$

$= [a(x, y) + 2\, c(x, y)]\, d\,(Tz, TTz).$

This implies $Tz = TTz = fTz$ i.e. Tz i.e. Tz is common fixed point of f and T.

REFERENCES

[1] Al- Thagafi, N. Shahzad, Generalized I- nonexapansive self maps and invariant approximations. Acta. Math. Sin. 24(2008), 867-876.

[2] G. Jungck, Compatible mappings and common fixed point, Internat. J. Math. Math Sci. 9 (1986), No. 4, 771-779.

[3] H.K. Pathak, Y.J. Cho, S.M. Kang, Remark of R-weak commuting mappings and common fixed point theorems, Bull. Korean Math. Soc. 34, 247-257(1997).

[4] Lj. B. Ciric, On some nonexpansive type mapping and fixed points , Indian j. pure appl. Math. 24(1993), 145-149.

[5] M.Chandra, , S.N.Mishra, S.L. Singh, And B.E. Rhoades, Coincidence and fixed points of nonexapansive type multi-valued and single-valued maps, Indian J. pure appl. Math., 26(5)(1995), 393-401.

[6] R.P Pant, R.K Bisht, D. ARORA, Weak reciprocal continuity and fixed points theorems, Ann. Univ. Ferrara 57(2011) 181-190.

[7] R.P Pant., Common fixed points of non-commuting mappings, J. Math. Anal. Appl., 188(1994), 436-440.

[8] R.P. Pant, Common fixed point theorems for contractive maps, J. Math. Anal. Appl., 1226(1998), 251-258.

[9] S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math., 32(1982), 129-153.