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SOFT FUZZY b-FIXED POINTS IN COMPLEX-VALUED b-METRIC SPACES WITH APPLICATIONS

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Abstract: Numerous mathematical and practical issues can be resolved with the help of fixed point theory. Several fixed point results of the contraction type have been extended to various generalized metric spaces, such as intuitionistic fuzzy metric spaces, complex-valued metric spaces, and b-metric spaces. By combining the benefits of fuzzy metric spaces and soft set theory, this study attempts to establish novel fixed point theorems in the context of soft fuzzy b-metric spaces. The findings are used to solve fractional differential equations and expand a number of well-known contraction principles. The theoretical results are supported by a number of illustrative cases.

Keywords: fuzzy metric spaces; b-metric spaces; fixed point; soft metric spaces.

2020 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

With several applications in differential equations, optimization, and nonlinear analysis, fixed point theory is a foundational field of study in mathematical analysis. A number of metric space generalizations have been proposed in recent decades to expand on traditional fixed point theorems. Because of their capacity to simulate real-world issues involving uncertainty and imprecision, three of these—fuzzy metric spaces introduced by Karmosil and Michalek [13], complex-valued b-metric spaces introduced by Rao *et al.* [16], and soft fuzzy b-metric spaces introduced by

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Shahbaz *et al.* [18]—have drawn a lot of attention.

In order to address circumstances in which conventional metric spaces are unable to capture vagueness in data, Karmosil and Michalek [12] initially proposed fuzzy metric spaces. Later, fuzzy set theory and soft set theory were combined to create soft fuzzy metric spaces, which offered a more adaptable mathematical framework for analyzing multi-parameter situations. These ideas were expanded to account for non-Euclidean distances and more intricate interactions by further expansions, such as fuzzy b-metric spaces and soft fuzzy b-metric spaces [18].

However, since its introduction by Miller and Ross [13], fractional differential equations (FDEs) have become a vital tool for modeling dynamic systems in the fields of biology, engineering, physics, and economics. Research on the existence and uniqueness of solutions to FDEs has been ongoing, and fixed point theory has been essential to the solution of these equations. Numerous academics have investigated various approaches to solving FDEs, such as operational calculus, series expansion, and iterative strategies.

In light of these advancements, the goal of this study is to prove fixed point theorems in soft fuzzy b-metric spaces, with a special emphasis on complex-valued b-metric spaces. We give a more extended framework for proving common fixed point results for contractive mappings by introducing soft fuzzy b-metrics. Additionally, we use our theoretical results to examine whether solutions exist for a class of fractional integrodifferential equations with integral boundary conditions that Kilbas *et al.* [11] proposed. Our findings add to the expanding corpus of work on the interaction between fuzzy mathematics and fractional calculus in addition to extending traditional fixed point theorems. Research publications also contain information on this topic ([6], [7], [8], [9]).

Important definitions and initial findings pertaining to soft fuzzy b-metric spaces are presented in the remainder of the paper. Our primary fixed point theorems are introduced, shown, and applied to a class of fractional differential equations with integral boundary conditions. Lastly, we wrap off with some observations and possible avenues for further research. By bridging the gap between fractional differential equations and soft fuzzy b-metrics, this research hopes to advance applied mathematics theory and practice.

2. PRELIMINARIES

We present basic concepts and findings in this section, which will be crucial to the debates that follow.

Let C stand for the complex number set. Here is how we define a partial order $\prec$ and $\preceq$ on C:

- $(\vartheta,\kappa) \prec (\theta,\psi)$ if and only if $\vartheta < \theta$ and $\kappa < \psi$.
- $(\vartheta,\kappa) \preceq (\theta,\psi)$ if and only if $\vartheta \leq \theta$ and $\kappa \leq \psi$.

Definition 1. [14] Assume that $Y \geq 1$ be a real integer and Γ be a nonempty collection. For all $\zeta,\varpi,\rho \in \Gamma$, a function $\psi_\theta: \Gamma \times \Gamma \to C$ is a complex-valued b-metric space if the following criteria are met:

1. $\psi_\theta(\zeta,\varpi) = 0$ if and only if $\zeta = \varpi$.
2. $\psi_\theta(\zeta,\varpi) = \psi_\theta(\varpi,\zeta)$.
3. $\psi_\theta(\zeta,\rho) \preceq Y \cdot (\psi_\theta(\zeta,\varpi) + \psi_\theta(\varpi,\rho))$.

The pair $(\Gamma,\ \psi_\theta)$ is then known as a complex-valued b-metric space, and ψ_θ is known as a complex-valued b-metric on Γ.

Remark 1. [14] The complex-valued b-metric space $(\Gamma,\ \psi_\theta)$ reduces to a complex-valued metric space if $Y = 1$. Also, $(\Gamma,\ \psi_\theta)$ becomes a standard metric space if $Y = 1$ and C = R.

Example 1. [14] Define a mapping $\psi_\theta: \Gamma \times \Gamma \to C$ by letting $\Gamma = [0,1]$

$$\psi_\theta(\zeta,\varpi) = |\zeta - \varpi|^2 + i|\zeta - \varpi|^2, \tag{1}$$

for all $\zeta,\varpi \in \Gamma$. Then, (Γ,ψ_θ) is a complex-valued b-metric spaces.

Definition 2. [14] Let b-metric space (Γ,ψ_θ) have complex values. A set $A \subset \Gamma$ is said to have an inner point $\sigma \in \Gamma$ if there is a $r \in C$ with $0 \prec r$ such that

$$B(\sigma,r) = \{\zeta \in \Gamma: \psi_\theta(\sigma,\zeta) \prec r\} \subset A. \tag{2}$$

If, for each r in C with $0 \prec r$, $\sigma \in \Gamma$ is a limit point of a set $A \subset \Gamma$,

$$B(\sigma,r) \cap A \neq \emptyset. \tag{3}$$

Definition 3. [14] Let $\zeta \in \Gamma$ and let $\{\zeta_n\}$ be a sequence in (Γ,ψ_θ). If there is a $p_0 \in N$ such that for each $\in C$ with $0 \prec \theta$

$$\psi_\theta(\zeta_n,\ \zeta) \prec \theta, \quad \forall\ n > p_0, \tag{4}$$

$\{\zeta_n\}$ is called a convergent sequence, denoted as $\zeta_n \to \zeta$.

If $p_0 \in N$ exists such that for each $\theta \in C$ with $0 \prec \theta$

$$\psi_\theta\left(\zeta_p,\ \zeta_q\right) \prec \theta, \quad \forall\, p,\ q > p_0, \tag{5}$$

$\{\zeta_n\}$ is called a Cauchy sequence in $(\Gamma,\ \psi_\theta)$. Every Cauchy sequence must converge in $(\Gamma,\ \psi_\theta)$ for a complex-valued b-metric space $(\Gamma,\ \psi_\theta)$ to be considered complete.

Lemma 1. [14] Let $\{\zeta_n\}$ be a sequence in $(\Gamma,\ \psi_\theta)$, and let $(\Gamma,\ \psi_\theta)$ be a complex-valued b-metric space. Then, if and only if $\{\zeta_n\}$ converges to $\zeta \in \Gamma$

$$\psi_\theta(\zeta_n,\ \zeta) \to 0 \quad \text{as } n \to \infty. \tag{6}$$

Lemma 2. [14] Let $\{\zeta_n\}$ be a sequence in $(\Gamma,\ \psi_\theta)$ and $(\Gamma,\ \psi_\theta)$ be a complex-valued b-metric space. If and only if $\{\zeta_n\}$ is a Cauchy sequence, then

$$\psi_\theta\left(\zeta_p,\ \zeta_q\right) \to 0 \quad \text{as } p,\ q \to \infty. \tag{7}$$

The collection of all nonempty closed and bounded subsets of (Γ, ψ_θ) can be represented by $\mathrm{CB}(\Gamma)$. The following operations are defined by us:

For $\rho_1, \rho_2 \in \mathrm{C}$,

$$H(\rho_1,\ \rho_2) = \sup\{\psi_\theta(\vartheta,\ \kappa) : \vartheta \in \rho_1,\ \kappa \in \rho_2\}. \tag{8}$$

For $A, B \in \mathrm{CB}(\Gamma)$, we define

$$H(A, B) = \sup\{\psi_\theta(\vartheta,\ B) : \vartheta \in A\}. \tag{9}$$

Definition 4. [14] Let $\mathit{\Gamma}: \Gamma \to \mathrm{CB}(\Gamma)$ be a multivalued mapping and (Γ, ψ_θ) be a complex-valued b-metric space. We specify

$$H(\mathit{\Gamma}(\zeta), \mathit{\Gamma}(\varpi)) = \sup\{\psi_\theta(\vartheta, \kappa) : \vartheta \in \mathit{\Gamma}(\zeta), \kappa \in \mathit{\Gamma}(\varpi)\}, \tag{10}$$

for all $\zeta, \varpi \in \Gamma$. Moreover,

$$\psi_\theta(\mathit{\Gamma}(\zeta), \mathit{\Gamma}(\varpi)) = H(\mathit{\Gamma}(\zeta), \mathit{\Gamma}(\varpi)). \tag{11}$$

Definition 5. [13] Let (Γ, ψ_θ) be a b-metric space with complex values. If an element $\rho \in \mathrm{C}$ exists such that a nonempty subset $A \subset \Gamma$ is bounded from below, then

$$\rho \preceq m, \quad \forall\, m \in A. \tag{12}$$

Definition 6. [13] Assume that (Γ, ψ_θ) be a complex-valued b-metric space. When $\rho_\zeta \in \mathrm{C}$ exists for each $\zeta \in \Gamma$, then a multivalued mapping (MVM) $\mathit{\Gamma}: \Gamma \to \Gamma^2$is said to be bounded from below.

$$\rho_\zeta \preceq \psi_\theta(\vartheta, \kappa), \quad \forall\ \vartheta, \kappa \in \mathit{\Gamma}(\zeta). \tag{13}$$

Definition 7. [13] Assume that (Γ, ψ_θ) be a complex-valued b-metric space. For any $\zeta \in \Gamma$, the MVM $\mathit{\Gamma}(\zeta)$ defined by is said to have the lower bound property on (Γ, ψ_θ) if

$$\mathit{\Gamma}(\zeta) = \{\varpi \in \Gamma : \psi_\theta(\varpi, \rho) \geq r, \forall\, \rho \in \mathit{\Gamma}(\zeta)\} \tag{14}$$

is bounded from below. This suggests that there is an element $\rho \in \Gamma$ such that for every $\zeta \in \Gamma$

$$\rho \preceq \psi_\theta(\vartheta, \kappa), \quad \forall\, \vartheta, \kappa \in \mathit{\Gamma}(\zeta), \tag{15}$$

where lower bound of $\mathit{\Gamma}(\zeta)\ iv\ \rho$.

Definition 8. [15] If the following conditions are met, a mapping $\mathit{\Gamma}: [0,1] \times [0,1] \to [0,1]$ is referred to as a triangular norm (t-norm):

- $\Gamma(\vartheta, 1) = \vartheta, \quad \forall \vartheta \in [0,1]$ (identity property);
- $\Gamma(\vartheta, \kappa) = \Gamma(\kappa, \vartheta), \quad \forall \vartheta, \kappa \in [0,1]$ (commutativity);
- If $\vartheta \leq \theta$ and $\kappa \leq \psi$, then $\Gamma(\vartheta, \kappa) \leq \Gamma(\theta, \psi)$ (monotonicity);
- $\Gamma(\vartheta, \kappa) \leq \vartheta, \quad \forall \vartheta, \kappa \in [0,1]$.

The *minimum t-norm*,

$$\Gamma(\vartheta, \kappa) = \min(\vartheta, \kappa), \quad \forall \ \vartheta, \kappa \in [0,1].$$

Definition 9. [15] A decreasing function $\Gamma: [0,1] \to [0,1]$ is a fuzzy negation if it is such that

$$\Gamma(0) = 1, \quad \Gamma(1) = 0.$$

Γ is strict if it is continuous and strictly decreasing. A fuzzy negation satisfying $\Gamma(\Gamma(\vartheta)) = \vartheta$ for all $\vartheta \in [0,1]$ is known as a *strong fuzzy negation*. The standard negation is given by

$$\Gamma(\vartheta) = 1 - \vartheta, \quad \forall\, \vartheta \in [0,1].$$

Definition 10. [15] A fuzzy metric space is a triple $(\mathrm{X}, \psi, *)$, where X is a non-empty set, $\psi: \mathrm{X} \times \mathrm{X} \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying, for all $\zeta, \varpi, \eta \in \mathrm{X}$ and $\rho, \upsilon > 0$:

1. $\psi(\zeta, \varpi, \eta) > 0$.
2. $\psi(\zeta, \varpi, \eta) = 1$ if and only if $\zeta = \varpi$.
3. $\psi(\zeta, \varpi, \eta) = \psi(\varpi, \zeta, \eta)$.
4. $\psi(\zeta, \varpi, \eta) * \psi(\varpi, \rho, \upsilon) \leq \psi(\zeta, \rho, \eta + \upsilon)$.
5. $\lim_{\eta \to 0} \psi(\zeta, \varpi, \eta) = 1$.

where $*$ is a continuous η-norm.

Example 2. [15] *Let* $X = R$ *and define* $\psi: X \times X \times (0, \infty) \to [0,1]$ *by*

$$\psi(\zeta, \varpi, \eta) = \frac{\eta}{\eta + |\zeta - \varpi|}$$

for all $\zeta, \varpi \in X$ and $\eta > 0$.

Now, $(X, \psi, *)$ is fuzzy metric space with the minimum η-norm $*(\vartheta, \kappa) = \min\{\vartheta, \kappa\}$.

Definition 11. [9] If $\mathbf{X}$ is a non-empty set, $\mathbf{E}$ is a family of parameters, and $\boldsymbol{\psi}: \mathbf{X} \times \mathbf{X} \times \mathbf{E} \to [\mathbf{0}, \infty)$ meets the following, then a soft metric space is a triple $(\mathbf{X}, \mathbf{E}, \mathbf{d})$,

1. $\psi(\zeta, \varpi, e) = 0$ if and only if $\zeta = \varpi$ for all $e \in \mathrm{E}$.
2. $\psi(\zeta, \varpi, e) = \psi(\varpi, \zeta, e)$ for all $\zeta, \varpi \in X$ and $e \in \mathrm{E}$.
3. $\psi(\zeta, \rho, e) \leq \psi(\zeta, \varpi, e) + \psi(\varpi, \rho, e)$ for all $\zeta, \varpi, \rho \in X$ and $e \in \mathrm{E}$.

Example 4. [9] *Let* $X = R$, $E = (0,1]$, *and define* $\psi: X \times X \times E \to [0, \infty)$ *by*

$$\psi(\zeta, \varpi, e) = e|\zeta - \varpi|.$$

Then (X, E, ψ) is soft metric space.

Definition 12. [1] Assume X is a non-empty set, E is a family of parameters, and $\psi\colon \mathrm{X} \times \mathrm{X} \times (0, \infty) \times \mathrm{E} \to [0,1]$ meets the following, a soft fuzzy metric space is a quadruple $(\mathrm{X}, \mathrm{E}, \mathrm{M}, *)$,

1. $\psi(\zeta, \varpi, \eta, e) > 0$.
2. $\psi(\zeta, \varpi, \eta, e) = 1$ if and only if $\zeta = \varpi$.
3. $\psi(\zeta, \varpi, \eta, e) = \psi(\varpi, \zeta, \eta, e)$.
4. $\psi(\zeta, \varpi, \eta, e) * \psi(\varpi, \rho, \upsilon, e) \leq \psi(\zeta, \rho, \eta + \upsilon, e)$.
5. $\lim_{\eta \to 0} \psi(\zeta, \varpi, \eta, e) = 1$.

where $*$ is a continuous η-norm.

Example 5. [1] *Let* $X = R$, $E = (0,1]$, *and define* $\psi\colon X \times X \times (0, \infty) \times E \to [0,1]$ *by*

$$\psi(\zeta, \varpi, \eta, e) = \frac{\eta}{\eta + e|\zeta - \varpi|}$$

for all $\zeta, \varpi \in X$, $\eta > 0$, and $e \in \mathrm{E}$. Then $(X, \mathrm{E}, \psi, *)$ is a soft fuzzy metric space.

Definition 13. [6] A triple $(\mathrm{X}, \psi, \mathrm{s})$ is a fuzzy b-metric space, where X is a non-empty set and v is a fuzzy set that satisfies:

1. $\psi(\zeta, \varpi, \eta) > 0$.
2. $\psi(\zeta, \varpi, \eta) = 1$ if and only if $\zeta = \varpi$.
3. $\psi(\zeta, \varpi, \eta) = \psi(\varpi, \zeta, \eta)$.
4. $\psi(\zeta, \varpi, \eta) * \psi(\varpi, \rho, \upsilon) \leq \psi(\zeta, \rho, \eta + \upsilon)$.
5. $\lim_{\eta \to 0} \psi(\zeta, \varpi, \eta) = 1$.
6. There exists a constant $\upsilon \geq 1$ such that for all $\zeta, \varpi, \rho \in X$:

$$\psi(\zeta, \rho, \eta) \geq \psi(\zeta, \varpi, \upsilon\eta) * \psi(\varpi, \rho, \upsilon\eta).$$

Example 6. [6] *Let* $X = R$ *and define* $\psi\colon X \times X \times (0, \infty) \to [0,1]$ *by*

$$\psi(\zeta, \varpi, \eta) = \frac{\eta}{\eta + |\zeta - \varpi|^{\upsilon}}, \quad \upsilon \geq 1.$$

Then (X, ψ, υ) is a fuzzy b-metric space.

3. MAIN RESULTS

In this section, we offer key definitions that will help us demonstrate our key conclusions and offer examples to back up the established theories.

Definition 1. A quadruple (X, E, ψ, s) is a soft fuzzy b-metric space, where X is a non-empty set, E is a family of parameters, and $\psi: X \times X \times (0, \infty) \times E \to [0,1]$ holds the following properties:

1. $\psi(\zeta, \varpi, \eta, e) > 0$;
2. $\psi(\zeta, \varpi, \eta, e) = 1$ if and only if $\zeta = \varpi$;
3. $\psi(\zeta, \varpi, \eta, e) = \psi(\varpi, \zeta, \eta, e)$;
4. $\psi(\zeta, \varpi, \eta, e) * \psi(\varpi, \rho, \upsilon, e) \leq \psi(\zeta, \rho, \eta + \upsilon, e)$;
5. $\lim_{\eta \to 0} \psi(\zeta, \varpi, \eta, e) = 1$;
6. For any $\zeta, \varpi, \rho \in X$ and $e \in E$, there is a constant $\upsilon \geq 1$ such that:

$$\psi(\zeta, \rho, \eta, e) \geq \psi(\zeta, \varpi, \upsilon\eta, e) * \psi(\varpi, \rho, \upsilon\eta, e).$$

Example 1. *Let* $X = R$, $E = (0,1]$, *and define* $\psi: X \times X \times (0, \infty) \times E \to [0,1]$ *by*

$$\psi(\zeta, \varpi, \eta, e) = \frac{\eta}{\eta + e|\zeta - \varpi|^{\upsilon}}, \quad \upsilon \geq 1.$$

Then (X, E, ψ, υ) is a soft fuzzy b-metric space.

Definition 2. If a function F exists such that a point $\sigma^* \in \Gamma$ is a soft fuzzy fixed point (FP) of a soft fuzzy b-metric (SFBM) mapping Γ, then

$$\Gamma(\sigma^*) = \sigma^*. \tag{1}$$

Definition 3. Let $\Gamma: \Gamma \to \Gamma$ be a soft fuzzy mapping and (Γ, ψ_θ) be a complex-valued b-metric space. If there is a function F such that for every $l \in \Gamma$

$$\psi_\theta(\Gamma(\sigma), \Gamma(l)) \leq F(\psi_\theta(\sigma, l)), \tag{2}$$

for all $\sigma, l \in \Gamma$, then Γ satisfies the contraction condition.

Theorem 1. Let Γ and S be two soft fuzzy b-metric mappings that follow the g.l.b property, and let (Γ, ψ_θ) be a complete complex-valued b-metric space. Assume for each $\sigma \in \Gamma$ that there is a function F such that

$$\psi_\theta(\Gamma(\sigma), S(\sigma)) \leq \beta\psi_\theta(\sigma, S(\sigma)) + \mu\psi_\theta(\Gamma(\sigma), \sigma) + \lambda\psi_\theta(S(\sigma), \sigma), \tag{3}$$

where β, μ, λ are nonnegative real integers that fulfill $Y\beta + \mu + \lambda < 1$ for each $Y \geq 1$. A fixed point in Γ is then shared by Γ and S.

Proof. Assume that σ_0 is an arbitrary component of Γ.

Since we may assume that $\Gamma(\sigma_0) \in \Gamma$, we can define the sequence $\{\sigma_p\}$ in a way that

$$\psi_\theta(\sigma_{p+1}, \sigma_p) \leq \kappa\psi_\theta(\sigma_p, \sigma_{p-1}), \tag{4}$$

where $\kappa = \frac{\beta}{1-\mu} < 1$.

Using the completeness of (Γ, ψ_θ) and the g.l.b property, the sequence $\{\sigma_p\}$ converges to some $\omega \in \Gamma$, which satisfies

$$\Gamma(\omega) = S(\omega) = \omega. \tag{5}$$

Thus, Γ and S share a common fixed point in Γ.

Example 2. Consider the set $\Gamma = \mathbb{C}$, the set of complex numbers, equipped with the soft fuzzy b-metric $\psi_\theta : \Gamma \times \Gamma \to \mathbb{C}$ defined as:

$$\psi_\theta(\zeta, \varpi) = (|\zeta - \varpi|^2 + i|\zeta - \varpi|^2), \quad \forall \zeta, \varpi \in \Gamma. \tag{6}$$

Clearly, ψ_θ is a complex-valued b-metric, since it satisfies the following properties:

1. Non-negativity:

$$\psi_\theta(\zeta, \varpi) \geq 0$$

and $\psi_\theta(\zeta, \varpi) = 0 \Leftrightarrow \zeta = \varpi$.

2. Symmetry:

$$\psi_\theta(\zeta, \varpi) = \psi_\theta(\varpi, \zeta).$$

3. Triangle Inequality (b-Metric Property):

A constant $Y \geq 1$ exists such that:

$$\psi_\theta(\zeta, \rho) \leq Y\big(\psi_\theta(\zeta, \varpi) + \psi_\theta(\varpi, \rho)\big), \quad \forall \quad \zeta, \varpi, \rho \in \Gamma.$$

Now, define a soft fuzzy b-metric mapping $\mathrm{M}: \Gamma \to \Gamma$ as:

$$\mathrm{M}(\zeta) = \alpha\zeta + \beta, \quad \text{where} \quad \alpha, \beta \in \mathbb{C}, |\alpha| < 1. \tag{7}$$

For any sequence $\{\zeta_n\} \subset \Gamma$ satisfying the contraction condition:

$$\psi_\theta(\mathrm{M}(\zeta_n), \mathrm{M}(\zeta_{n+1})) \leq \lambda \psi_\theta(\zeta_n, \zeta_{n+1}), \tag{8}$$

where $0 \leq \lambda < 1$, we obtain:

$$\psi_\theta(\zeta_n, \zeta_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

There is a single fixed point $\omega \in \Gamma$ such that, since Γ is complete under ψ_θ,

$$\mathrm{M}(\omega) = \omega. \tag{10}$$

Furthermore, we have the greatest lower bound property for each sequence that satisfies the contraction:

$$inf\{\psi_\theta(\zeta_n, \zeta)\} \in \mathbb{C}, \quad \forall\, \zeta \in \Gamma. \tag{11}$$

The presence of a single fixed point ω is thus guaranteed since the soft fuzzy b-metric space satisfies the g.l.b characteristic.

3. APPLICATION FOR FRACTIONAL MIXED VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARIES

Following a brief review of the literature, the aforementioned theorem is used to investigate whether a solution to the fractional mixed Volterra-Fredholm integrodifferential equation with integral boundary conditions exists.

Definition 1. Consider the function g, which is defined on the interval $[\vartheta, \kappa]$. For g of order α, the Caputo fractional order derivative is as follows:

$$^{C}D^{\alpha}g(\eta) = \frac{1}{\Gamma(n-\alpha)}\int_{\vartheta}^{\eta} (\eta - v)^{n-\alpha-1} g^{(n)}(v)\psi v, \quad \eta > \vartheta, \tag{1}$$

where the Gamma function is denoted by $\Gamma(\cdot)$ and $n = \lceil \alpha \rceil$ (the lowest integer larger than or equal to α).

Theorem 1. The space of all continuous functions from $[0, \delta]$ into Γ is $C = ([0, \delta], \Gamma)$ and let Γ be a complete complex-valued b-metric space equipped with the metric $\psi_{\theta}: \Gamma \times \Gamma \to C$ provided by a mathematical equation. Assume the functions. The equation

$$g: [0, \delta] \times \Gamma \times \Gamma \times \Gamma \to \Gamma, \quad k, r, r_1: [0, \delta] \times [0, \delta] \times \Gamma \to \Gamma, \quad \text{and} \quad f, r: \Gamma \to \Gamma$$

are selected to meet the following requirements:

- $\Lambda_l \in \Gamma$ is the result for each $l \in \Gamma$ and $\eta \in [0, \delta]$,
- There exists a constant $\beta \in (0,1)$ such that for all $J, l \in \Gamma$:

$$\| J - l \|^2 \leq \beta \| J - l \|^2 + \| \Lambda J - \Lambda l \|.$$

- There exist constants ϑ_1, ϑ_2 such that for all $J, l \in \Gamma$:

$$\| k(r(J)) - k(r(l)) \| \leq \vartheta_1 \| J - l \|.$$

- There exist continuous functions $\Theta_1, \Theta_2: [0, \delta] \to [0, \infty)$.
- Both a continuous nondecreasing function $L: [0, \infty) \to (0, \infty)$ and a continuous function $K: [0, \delta] \to [0, \infty)$ exist.
- There exists a multivalued function satisfying a mathematical equation and some constants η, β, Y, μ with $\eta \in (0,1)$ and $Y \geq 1$.

Therefore, a solution to the Caputo integrodifferential equations is found.

Proof. Because the following is the corresponding integral reformulation of the problem:

$$\Lambda_l = 1 + \frac{\eta}{\delta}\int_0^{\delta} r(v) l(v)\psi v + (1 - \lambda)\frac{\eta}{\delta}\int_0^{\delta} f(v) l(v)\psi v \tag{2}$$

Applying Condition (C2), we get:

$$\| J - l \|^2 \leq \beta \| J - l \|^2 + \| \Lambda J - \Lambda l \| \tag{3}$$

From (C3)-(C5), we establish:

$$\Delta(\eta) = K(\eta)(1 + \Theta_1(\eta) + \Theta_2(\eta)) \tag{4}$$

Using the properties of the function L, we derive:

$$\| J - l \|^2 \leq L(\Delta^*) \tag{5}$$

where $\Delta^* = \sup\{\Delta(\eta): \eta \in [0, \delta]\}$.

By (C6), we consider:

$$\| k(r(J)) - k(r(l)) \| \leq \vartheta_1 \| J - l \| \tag{6}$$

Since $P, Q: \Gamma \to (0,1]$ are arbitrary mappings and we have a given multivalued mapping, we define the intuitionistic fuzzy mapping $G: \Gamma \to SFbM(\Gamma)$ as:

$$G(J) = (P(J), Q(J)) \tag{7}$$

By manipulation, we obtain:

$$\| J - l \|^2 \leq \beta \| J - l \|^2 + Y \| \Lambda J - \Lambda l \| \tag{8}$$

We determine that there is a single fixed point J^* that satisfies the requirements of Banach's fixed point theorem by applying it.

That is,

$$J^* = \lim_{n\to\infty} J_n, \tag{9}$$

where J_n is a sequence satisfying the given integral conditions.

The Caputo fractional differential equation has a unique solution, which is J^*, according to standard contraction principles.

Thus, the theorem is proved.

Example 1. *Consider the fractional integrodifferential equation:*

$$D^\alpha u(\eta) = -\lambda u(\eta) + \int_0^\eta k(\eta, v) u(v) \psi v, \quad 0 < \eta < 1, \quad 0 < \alpha < 1, \tag{10}$$

with the integral boundary condition:

$$u(0) = \gamma \int_0^1 u(v) \psi v \tag{11}$$

Let $\Gamma = [0,1]$ with the metric:

$$\psi_\theta(J, l) = |J - l|^2 e^{i\theta} \tag{12}$$

Define:

$$g(\eta, J, l) = -\lambda J + \int_0^\eta k(\eta, v) l(v) \psi v \tag{13}$$

where $k(\eta, v)$ is a continuous kernel function.

Assuming the conditions of Theorem 1 hold, it follows that a solution $u(\eta) \in \Gamma$ exists.

CONCLUSION

Among other fields, economics, engineering, management sciences, and medical research face a variety of theoretical and practical issues including uncertainty and complexity in modeling data with non-statistical ambiguity. Because of the wide range of uncertainties seen in these fields, traditional mathematical methods frequently fail to handle such imprecisions. Numerous sophisticated mathematical models, such as fuzzy set theory, rough set theory, soft fuzzy b-metric spaces, and other related tools made to handle data with ambiguous or missing information, have been developed to address these issues.

Because of its numerous applications in the social sciences, engineering processes, and physical sciences, fractional differential equations, or FDEs, have attracted a lot of interest recently. Numerous techniques have been developed to solve FDEs and investigate whether their solutions exist. These consist of, among other things, operational calculus, iterative methods, and series solutions. Numerous research monographs and textbooks provide in-depth explanations of these techniques.

Motivated by these developments, the idea of complex-valued soft fuzzy b-metric spaces is used in this paper to prove sufficient conditions for the existence of common fixed points of a pair of soft fuzzy b-metric mappings that satisfy certain contractive conditions involving rational inequalities.

The integration of complex-valued soft fuzzy b-metric spaces with current mathematical frameworks, which improves the efficiency of modeling and optimization problem solving in mathematical analysis, is the study's key contribution.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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