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STRICT UNIFORM STABILITY ANALYSIS IN TERMS OF TWO MEASURES OF CAPUTO FRACTIONAL DYNAMIC SYSTEMS ON TIME SCALE

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Abstract. This work investigates the $(m, m_0)-$strict uniform stability of Caputo fractional dynamic systems on time scales, leveraging the Caputo fractional derivative's ability to model memory and hereditary effects for a more accurate representation of real-world dynamics. Traditional stability concepts, such as Lyapunov and asymptotic stability, often lack the granularity to fully capture complex system behaviors. To address this, we focus on $(m, m_0)-$strict uniform stability, which provides a stringent and comprehensive framework for analyzing system robustness and convergence rates. Using vector Lyapunov functions, we enable component-wise stability analysis, offering a detailed understanding of multi-dimensional dynamics, particularly in high-dimensional systems with interdependent variables. We also demonstrate the practical relevance of our approach through a comprehensive example.

Keywords: strict stability; vector Lyapunov functions; fractional dynamic equations; time scale.

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1. INTRODUCTION

The analysis of stability in dynamic systems [5] has long been a foundational aspect of mathematical research, with profound implications across diverse fields such as engineering and biology. This encompasses both integer-order [27, 28, 30, 31] and fractional (non-integer) [26, 29, 33] stability concepts. Traditional stability concepts, such as Lyapunov stability and asymptotic stability, have provided valuable tools for understanding system behavior [12, 16, 17, 18]. However, these notions often fall short in capturing the intricate dynamics of complex systems, particularly those that exhibit both continuous and discrete behaviors or possess memory and hereditary properties. This limitation has spurred the development of more refined stability concepts, such as strict stability and strict uniform stability, which offer a deeper and more comprehensive understanding of system dynamics.

In this work, we delve into the strict stability of Caputo fractional dynamic equations on time scales, a framework that unifies the analysis of continuous and discrete systems. The Caputo fractional derivative, known for its ability to model systems with memory and hereditary effects, provides a more accurate representation of real-world dynamics compared to integer-order derivatives [6, 8]. This is particularly advantageous in systems where past states significantly influence present and future behavior, a feature that integer-order models often fail to capture. By leveraging the fractional derivative, we aim to provide a more robust and versatile framework for analyzing dynamic systems [19, 20, 22].

Our focus is on (m, m_0)-strict uniform stability, a concept that offers significant advantages over other stability notions. While Lyapunov stability and asymptotic stability provide useful insights, they often lack the granularity needed to fully capture the rate of convergence or the robustness of the system under perturbations. Strict uniform stability addresses these limitations by providing a more stringent and comprehensive framework for analyzing system behavior. This is particularly important in applications where predictability and robustness are critical, such as in control systems, where even small deviations from equilibrium can have significant consequences.

To achieve our objective, we employ vector Lyapunov functions (LFs), a powerful tool that allows for a more granular and flexible analysis of system stability [7, 9, 10, 11, 12]. Unlike scalar LFs, which provide a general view of stability, vector LFs enable component-wise analysis, offering a more detailed understanding of multi-dimensional dynamics. This is especially beneficial in high-dimensional systems, where the interactions between variables can be complex and interdependent. By using vector LFs, we can capture the individual behaviors of system components and their contributions to overall stability, providing a more comprehensive understanding of system dynamics. The use of vector LFs in our examination of $(m,m_0)-$strict uniform stability offers several advantages including allowing us to analyze the stability of each component of the system independently, providing insights into how individual variables contribute to the overall system behavior. This is particularly useful in systems with interdependent variables, where the stability of one component can significantly influence the stability of others.

One of the key motivations for this work is the need to address the limitations of existing stability concepts in the literature. While previous studies have explored various forms of stability, they often rely on comparison theorems or focus on uniform stability, which may not fully capture the intricacies of system behavior. By focusing exclusively on $(m,m_0)-$strict uniform stability and employing vector LFs, we aim to provide a more refined and rigorous framework for analyzing dynamic systems. This approach not only enhances our understanding of system behavior but also provides a foundation for future research in areas such as variational Lyapunov stability and other related fields.

Consider the Caputo fractional dynamic system of order α with $0<\alpha<1$,

$$\begin{aligned} {}^{C}\Delta^{\alpha}\upsilon &= \Xi(t,\upsilon),\ t\in\mathbb{T}, \\ \upsilon(t_0) &= \upsilon_0,\ t_0\geq 0, \end{aligned} \tag{1}$$

where $\Xi\in C_{rd}[\mathbb{T}\times\mathbb{R}^N,\mathbb{R}^N]$, $\Xi(t,0)\equiv 0$ and ${}^{C}\Delta^{\alpha}\upsilon$ is the Caputo FrΔD of $\upsilon\in\mathbb{R}^N$ of order α with respect to $t\in\mathbb{T}$. Let $\upsilon(t)=\upsilon(t,t_0,\upsilon_0)\in C^{\alpha}_{rd}[\mathbb{T},\mathbb{R}^N]$ (the fractional derivative of order alpha of $\upsilon(t)$ exist and it is rd-continuous) be a solution of (1) and assume the solution exists and is unique (results on existence and uniqueness of (1) are contained in [4, 14, 15, 19, 22, 23, 24, 25, 34], this work aims to investigate the (m,m_0)-strict uniform stability of the system (1).

The study begins in the next section by outlining foundational definitions. Then, in Section 3, we develop the (m, m_0)-strict uniform stability criteria for the Caputo fractional dynamic system (1). Next, in Section 4, we provide a comprehensive example demonstrating the significance and practical relevance of our results and finally in Section 6, we give a concluding remark.

2. PRELIMINARIES, DEFINITIONS, AND NOTATIONS

Definition 2.1 ([2])**.** *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

(i) if $\sigma(t) > t$, t is right scattered,
(ii) if $\rho(t) < t$, t is left scattered,
(iii) if $t < max\mathbb{T}$ and $\sigma(t) = t$, then t is called right dense,
(iv) if $t > min\mathbb{T}$ and $\rho(t) = t$, then t is called left dense.

Definition 2.2 ([2])**.** *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Definition 2.3 ([2])**.** *A function $\psi : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous if it is continuous at all right-dense points of $\mathbb{T}$, and if it has finite left-sided limits at left-dense points of $\mathbb{T}$. The set of all such right-dense continuous functions is denoted by*

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Definition 2.4 ([2])**.** *A function $\phi : [0, r] \to [0, \infty)$ is of class $\mathscr{K}$ if it is continuous, and strictly increasing on $[0, r]$ with $\phi(0) = 0$.*

Definition 2.5. [8] *We define the Caputo FrΔDiD of the Lyapunov function, $\mathscr{L}(t, \upsilon) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}^N_+]$ (which is locally Lipschitz with respect to its second argument and* *satisfies* $\mathscr{L}(t,0) \equiv 0$*) along the trajectories of solutions of the system (1) as:*

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}(t,\upsilon) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg[\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\left({}^{\alpha}C_{r}\right)\left[\mathscr{L}(\sigma(t)-r\mu,\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon(t))\right. \\ &\qquad \left.-\mathscr{L}(t_0,\upsilon_0)\right]\Bigg], \end{aligned} \tag{2}$$

and can be expanded as

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}(t,\upsilon) = \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg\{&\mathscr{L}(\sigma(t),\upsilon(\sigma(t)) - \mathscr{L}(t_0,\upsilon_0) \\ &-\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r+1}\left({}^{\alpha}C_{r}\right)\left[\mathscr{L}(\sigma(t)-r\mu,\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon(t))-\mathscr{L}(t_0,\upsilon_0)\right]\Bigg\}, \end{aligned} \tag{3}$$

where $t\in\mathbb{T}$*,* $\upsilon,\upsilon_0\in\mathbb{R}^{N}$*,* $\mu=\sigma(t)-t$*, and* $\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon)\in\mathbb{R}^{N}$*.*

Applying (9) *to* (3)*, we obtain*

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}(t,\upsilon) &= \limsup_{\mu\to 0^{+}} \frac{1}{\mu^{\alpha}}\Bigg\{\mathscr{L}(\sigma(t),\upsilon(\sigma(t)) \\ &\qquad +\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\left({}^{\alpha}C_{r}\right)\left[\mathscr{L}(\sigma(t)-r\mu,\upsilon(\sigma(t))-\mu^{\alpha}\Xi(t,\upsilon(t))\right]\Bigg\} \end{aligned} \tag{4}$$

$$-\frac{\mathscr{L}(t_0,\upsilon_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5}$$

If $\mathbb{T}$ *is discrete and* $\mathscr{L}(t,\upsilon(t))$ *is continuous at* t*, the Caputo Fr∆DiD of the LF in discrete times is given by:*

$${}^{C}\Delta_{+}^{\alpha}\mathscr{L}(t,\upsilon) = \frac{1}{\mu^{\alpha}}\left[\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^{r}\left({}^{\alpha}C_{r}\right)\left(\mathscr{L}(\sigma(t),\upsilon(\sigma(t)))-\mathscr{L}(t_0,\upsilon_0)\right)\right] \tag{6}$$

and if $\mathbb{T}$ *is continuous, i.e.,* $\mathbb{T}=\mathbb{R}$*, and* $\mathscr{L}(t,\upsilon(t))$ *is continuous at* t*, we have that*

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}(t,\upsilon) = \limsup_{d\to 0^{+}} \frac{1}{d^{\alpha}}\Bigg\{&\mathscr{L}(t,\upsilon(t)) - \mathscr{L}(t_0,\upsilon_0) \\ &-\sum_{r=1}^{\left[\frac{t-t_0}{d}\right]} (-1)^{r+1}\left({}^{\alpha}C_{r}\right)\left[\mathscr{L}(t-rd,\upsilon(t))-d^{\alpha}\Xi(t,\upsilon(t))-\mathscr{L}(t_0,\upsilon_0)\right]\Bigg\}, \end{aligned} \tag{7}$$

for $d>0$*.*

From [13], we state the following:

$$\lim_{\mu\to 0^+} \sum_{r=1}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r = -1, \tag{8}$$

and

$$^{C\mathbb{T}}D_+^{\alpha}\psi^{\Delta}(t) = \limsup_{\mu\to 0^+} \frac{1}{\mu^{\alpha}} \sum_{r=0}^{[\frac{(t-t_0)}{\mu}]} (-1)^{r\alpha} C_r = ^{RL\mathbb{T}} D^{\alpha}(1) = \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \geq t_0. \tag{9}$$

Definition 2.6. *The zero solution of* (1) *is said to be* $(m_0, m)-$*strictly uniformly stable if given* $\varepsilon_1 > 0$ *and* $t_0 \in \mathbb{T}$*, there exists a* $\delta_1 = \delta_1(\varepsilon_1) > 0$ *such that for any* $\upsilon_0 \in \mathbb{R}^N$*, the inequality* $|m(t_0, \upsilon(t_0))| < \delta_1$ *implies* $|m(t\upsilon(t)\| < \varepsilon_1$*,* $t \geq t_0$ *and given* $\delta_2 \in (0, \delta_1]$*, we can also find* $\varepsilon_2 \in (0, \delta_2)$ *such that* $\delta_2 < |m(t_0, x(t_0))|$ *implies* $\varepsilon_2 < |m(t, \upsilon(t))|$*,* $t \geq t_0$*.*

3. MAIN RESULTS

In this section, we will obtain sufficient conditions for $(m_0, m)-$strict uniform stability of the fractional dynamic system (1) for $\alpha = (0,1)$. Also, inequalities between vectors are taken to be component-wise inequalities.

Theorem 3.1 ($(m_0, m)-$Strict Uniform Stability)**.** *Let* $\mathscr{L}(t, \upsilon(t)) \in C_{rd}[\mathbb{T} \times \mathbb{R}^N, \mathbb{R}_+^N]$ *and* $m \in \Lambda$ *be such that*

(i) $\mathscr{L}$ *is locally Lipschitzian in* υ *with* $\mathscr{L}(t, 0) \equiv 0$*;*

(ii) for positive numbers ψ, ζ *were* $\psi \in (0, \zeta)$*, we have that when* $|m(t, \upsilon(t)) \geq \psi$ *and*

$$b_1(|m(t, \upsilon(t))|) \leq \mathscr{L}_{0_{\psi}}(t, \upsilon) \leq a_1(|m(t, \upsilon(t))|), \tag{10}$$

then the inequality

$$^{C}\Delta_+^{\alpha} \mathscr{L}_{0\psi}(t, \upsilon(t)) \leq 0, \tag{11}$$

holds, $a_1, b_1 \in \mathscr{K}$*,* $\mathscr{L}_0(t, \upsilon) = \sum_{i=1}^N \mathscr{L}_i(t, \upsilon(t))$*;*

(iii) for any points $t, t_0 \geq 0$ *and positive numbers* ϕ, ζ*, were* $\phi \in (0, \zeta)$*, we have that when* $|h(t, \upsilon(t))| \leq \phi$ *and*

$$b_2(|m(t, \upsilon(t))|) \leq \mathscr{L}_{0\phi}(t, \upsilon) \leq a_2(|m(t, \upsilon(t))|), \tag{12}$$

the inequality

$$ {}^{C}\Delta_{+}^{\alpha}\mathscr{L}_{0\phi}(t,\upsilon(t)) \geq 0, \tag{13}$$

holds, were $a_2, b_2 \in \mathscr{K}$ *and* $\upsilon \in \mathbb{R}^N$*.*

Then the zero solution of the FrDE (1) *is* $(m_0,m)-$*strictly uniformly stable.*

Proof. We shall make this proof in two phases.

Phase 1

Let $\varepsilon \in (0,\zeta)$ and $t_0 \in \mathbb{T}$ be given. Set $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that

$$a_1(\delta_1) < b_1(\varepsilon_1), \tag{14}$$

then we assert that,

$$|m_0(t_0,\upsilon(t_0))| < \delta_1 \implies |m(t,\upsilon(t))| < \varepsilon_1 \text{ for } t \geq t_0 \tag{15}$$

If the assertion is false, then there would exist some time $t_1 > t_0$ were for any solution $\upsilon(t)$, $h_0(t_0,\upsilon_0) < \delta_1$ would imply

$$|m(t,\upsilon(t_1))| = \varepsilon_1 \tag{16}$$

and

$$|m(t,\upsilon(t))| < \varepsilon_1$$

for $t \in [t_0,t_1)$.

Combining (10), (14), and (16) at $t = t_1$, we obtain

$$b_1(\varepsilon_1) = b_1(|m(t_1,\upsilon(t_1))|) \leq \mathscr{L}_{0_\psi}(t_1,\upsilon) \leq a_1(|m(t_1,\upsilon(t_1))|) \leq a_1(\delta_1) < b_1(\varepsilon),$$

which is clearly a contradiction, implying that (15) is true.

Phase 2

Let $\varepsilon_2 > 0$ be given, we can pick $\delta_2 \in (0,\delta_1]$ and if we set $|m_0(t_0,\upsilon(t_0))| < \delta_2 < \delta_1$ such that

$$a_2(\varepsilon_2) < b_2(\delta_2), \tag{17}$$

we could now make the assertion that

$$\delta_2 < |m_0(t_0,\upsilon_0(t_0))| < \delta_1 \implies \varepsilon_2 < |m(t,\upsilon(t))| < \varepsilon_1,\ t = t_0. \tag{18}$$

If this assertion is false, then by the validity of (15), there exist a solution $\upsilon(t) = \upsilon(t;t_0,x_0)$ of (1) and a time $t_1 > t_2 > t_0$ such that $\delta_2 < |m_0(t_0,\upsilon(t_0))| < \delta_1$ implies

$$|m(t_1,\upsilon(t_1))| = \varepsilon_2, \text{ and } m(t,\upsilon(t)) \leq \delta_2, \text{ for } t \in [t_2,t_1]. \tag{19}$$

Set $\phi = \delta_2$, then from (12), we obtain

$$a_2(\varepsilon_2) = a_2(|m(t_1,\upsilon(t_1))|) \geq \mathscr{L}_{0\phi}(t_1,\upsilon(t_1)) \geq \mathscr{L}_{0\phi}(t_2,\upsilon(t_2)) \geq b_2(|m(t_2,\upsilon(t_2))|),$$

contradicting (17), implying (18) holds. Phase 1 and Phase 2 satisfies Definition 2.6 so that we conclude that the zero solution of (1) is $(m_0,m)-$strictly uniformly stable. □

4. APPLICATION

Consider the Caputo fractional dynamic system

$$\begin{aligned} {}^{C}\Delta^{\alpha}\upsilon_1(t) &= 3\upsilon_1 - 6\frac{\upsilon_3^2}{\upsilon_1} \\ {}^{C}\Delta^{\alpha}\upsilon_2(t) &= -2\frac{\upsilon_1^2+}{\upsilon_2} + 4\upsilon_2 - 5\frac{\upsilon_3^2}{\upsilon_2} \\ {}^{C}\Delta^{\alpha}\upsilon_3(t) &= -\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3} + 5\upsilon_3, \end{aligned} \tag{20}$$

for $t \geq t_0$, with initial conditions

$$\upsilon_1(t_0) = \upsilon_{10}, \quad \upsilon_2(t_0) = \upsilon_{20}, \quad \text{and} \quad \upsilon_3(t_0) = \upsilon_{30},$$

where $\upsilon = (\upsilon_1,\upsilon_2,\upsilon_3)$, and $\Xi = (\Xi_1,\Xi_2,\Xi_3)$.

Consider a vector $V = (\mathscr{L}_1,\mathscr{L}_2,\mathscr{L}_3)^T$, where $\mathscr{L}_1 = \upsilon_1^2$, $\mathscr{L}_2 = \upsilon_2^2$ and $\mathscr{L}_3 = \upsilon_3^2$, for $t \in \mathbb{T}$ and $(\upsilon_1,\upsilon_2,\upsilon_3) \in \mathbb{R}^3$. Then condition 2(ii) of Theorem 3.1 is satisfied, for $\phi = \frac{1}{2}r$, and $\theta = r^2$ where $\phi,\theta \in \mathscr{K}$, so that the associated norm $\|\upsilon\| = \sqrt{\upsilon_1^2+\upsilon_2^2+\upsilon_3^2}$.
and

$$\mathscr{L}_0(\upsilon_1,\upsilon_2,\upsilon_3) = \sum_{i=1}^{2}\mathscr{L}_i(\upsilon_1,\upsilon_2,\upsilon_3) = \upsilon_1^2+\upsilon_2^2+\upsilon_3^2,$$

then $\phi(\|\psi\|) \leq \mathscr{L}(\psi_1,\psi_2,\psi_3) \leq \theta(\|\upsilon\|)$. From (3), we compute the Caputo FrΔDiD for $\mathscr{L}_1 = \upsilon_1^2$ as follows:

$${}^{C}\Delta^{\alpha}_{+}\mathscr{L}_1$$

$$
\begin{aligned}
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ \left[(\upsilon_1(\sigma(t)))^2\right] - \left[(\upsilon_{10})^2\right] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\,({}^{\alpha}C_r)\,[(\upsilon_1(\sigma(t)) - \mu^\alpha \Xi_1(t,\upsilon))^2] - [((\upsilon_{10})^2]\Bigg\} \\
&= \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\Bigg\{ \left[(\upsilon_1(\sigma(t)))^2\right] - \left[(\upsilon_{10})^2\right] \\
&\quad + \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\,({}^{\alpha}C_r)\,[(\upsilon_1(\sigma(t)))^2 - 2\upsilon_1(\sigma(t))\mu^\alpha \Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3) \\
&\quad + \mu^{2\alpha}(\Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3))^2] - \left[(\upsilon_{10})^2\right]\Bigg\} \\
&= -\limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{ \sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\,({}^{\alpha}C_r)\left[(\upsilon_{10})^2\right]\right\} \\
&\quad + \limsup_{\mu\to 0^+} \frac{1}{\mu^\alpha}\left\{ \sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\,({}^{\alpha}C_r)\left[(\upsilon_1(\sigma(t)))^2\right]\right\} \\
&\quad - \limsup_{\mu\to 0^+} \left\{ \sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]} (-1)^r\,({}^{\alpha}C_r)\,[2\upsilon_1(\sigma(t))\mu^\alpha \Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3)]\right\}.
\end{aligned}
$$

Applying (8) and (9) we obtain

$$
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_1 \;\le\; \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2\right] - [2\upsilon_1(\sigma(t))\Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3)].
$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_1(\sigma(t)))^2\right] \to 0$, which is

$$
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_1 \;\le\; -2[\upsilon_1(\sigma(t))\Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3)].
$$

applying $\upsilon(\sigma(t)) \le \mu^{C}\Delta^{\alpha}\upsilon(t) + \upsilon(t)$

$$
\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_1 &= -2\Big[\mu(t)\Xi_1^2(t,\upsilon_1,\upsilon_2,\upsilon_3) + \upsilon_1(t)\Xi_1(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big] \\
&= -2\left[\mu(t)\left(3\upsilon_1 - 6\frac{\upsilon_3^2}{\upsilon_1}\right)^2 + \upsilon_1\left(3\upsilon_1 - 6\frac{\upsilon_3^2}{\upsilon_1}\right)\right] \\
&= -2\mu(t)\left[\left(3\upsilon_1 - 6\frac{\upsilon_3^2}{\upsilon_1}\right)^2\right] - 2\upsilon_1\left[3\upsilon_1 - 6\frac{\upsilon_3^2}{\upsilon_1}\right]. \qquad (21)
\end{aligned}
$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (21) becomes:

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_1(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\upsilon_1\left[3\upsilon_1-6\frac{\upsilon_3^2}{\upsilon_1}\right]\\
&= -6\upsilon_1^2+0\upsilon_2^2+12\upsilon_3^2\\
&= (-6 \quad 0 \quad 12)\cdot(\mathscr{L}_1 \quad \mathscr{L}_2 \quad \mathscr{L}_3)^T. \qquad (22)
\end{aligned}$$

If $\mathbb{T}=\mathbb{N}_0$, we have that $\mu=1$, so that (21) becomes:

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_1(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\left[\left(3\upsilon_1-6\frac{\upsilon_3^2}{\upsilon_1}\right)^2\right]-2\upsilon_1\left[3\upsilon_1-\frac{6\upsilon_3^2}{\upsilon_1}\right]\\
&\leq -2\upsilon_1\left[3\upsilon_1-6\frac{\upsilon_3^2}{\upsilon_1}\right],
\end{aligned}$$

leading to the same conclusion as (22). Clearly, this also works for any other discrete time.

Similarly, compute the Caputo FrΔDiD for $\mathscr{L}_2(\upsilon)=\upsilon_2^2$ as follows:

$$\begin{aligned}
&{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon)\\
= \;& \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_2(\sigma(t)))^2\right]-\left[(\upsilon_{20})^2\right]\\
&+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_2(\sigma(t))-\mu^\alpha\Xi_2(t,\upsilon))^2\right]-\left[((\upsilon_{20})^2\right]\Bigg\}\\
= \;& \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_2(\sigma(t)))^2\right]-\left[(\upsilon_{20})^2\right]\\
&+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_2(\sigma(t)))^2-2\upsilon_2(\sigma(t))\mu^\alpha\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.\\
&\left.+\mu^{2\alpha}(\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3))^2\right]-\left[(\upsilon_{20})^2\right]\Bigg\}\\
= \;& -\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_{20})^2\right]\right\}\\
&+\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[(\upsilon_2(\sigma(t)))^2\right]\right\}
\end{aligned}$$

$$-\limsup_{\mu\to 0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^{\alpha}C_r\right)\left[2\upsilon_2(\sigma(t))\mu^{\alpha}\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\right]\right\}.$$

Applying (8) and (9) we obtain

$${}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon) \;\leq\; \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_2(\sigma(t)))^2\right]-\left[2\upsilon_2(\sigma(t))\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\right].$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_2(\sigma(t)))^2\right]\to 0$, which is

$${}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon) \;\leq\; -2[\upsilon_2(\sigma(t))\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3)],$$

applying $\upsilon(\sigma(t))\leq\mu^{C}\Delta^{\alpha}\upsilon(t)+\upsilon(t)$

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon) &= -2\Big[\mu(t)\Xi_2^2(t,\upsilon_1,\upsilon_2,\upsilon_3)+\upsilon_2(t)\Xi_2(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\
&= -2\left[\mu(t)\left(-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right)^2+\upsilon_2\left(-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right)\right]\\
&= -2\mu(t)\left[\left(-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right)^2\right]-2\upsilon_1\left[-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right]. \qquad (23)
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (23) becomes;

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon_1,\upsilon_2) &= -2\upsilon_2\left[-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right]\\
&= 4\upsilon_1^2-8\upsilon_2^2+10\upsilon_3^2\\
&= (4 \quad -8 \quad 10)\cdot(\mathscr{L}_1 \quad \mathscr{L}_2 \quad \mathscr{L}_3)^T. \qquad (24)
\end{aligned}$$

If $\mathbb{T}=\mathbb{N}_0$, we have that $\mu=1$, so that (21) becomes:

$$\begin{aligned}
{}^{C}\Delta_{+}^{\alpha}\mathscr{L}_2(\upsilon_1,\upsilon_2) &= -2\left[\left(-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right)^2\right]-2\upsilon_1\left[-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right]\\
&\leq -2\upsilon_1\left[-2\frac{\upsilon_1^2+}{\upsilon_2}+4\upsilon_2-5\frac{\upsilon_3^2}{\upsilon_2}\right],
\end{aligned}$$

this also leads to the same conclusion as (24). Clearly, this also works for any other discrete time.

Similarly, compute the Caputo FrΔDiD for $\mathscr{V}_3(\upsilon_1,\upsilon_2,\upsilon_3)=\upsilon_3^2$ as follows:

$${}^{C}\Delta_{+}^{\alpha}\mathscr{L}_3$$

$$\begin{aligned}
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_3(\sigma(t)))^2\right]-\left[(\upsilon_{30})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_3(\sigma(t))-\mu^\alpha\Xi_3(t,\upsilon))^2\right]-\left[((\upsilon_{30})^2\right]\Bigg\}\\
&= \limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\Bigg\{\left[(\upsilon_3(\sigma(t)))^2\right]-\left[(\upsilon_{30})^2\right]\\
&\quad+\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_3(\sigma(t)))^2-2\upsilon_3(\sigma(t))\mu^\alpha\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right.\\
&\quad\left.+\mu^{2\alpha}(\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3))^2\right]-\left[(\upsilon_{30})^2\right]\Bigg\}\\
&= -\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_{30})^2\right]\right\}\\
&\quad+\limsup_{\mu\to 0^+}\frac{1}{\mu^\alpha}\left\{\sum_{r=0}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[(\upsilon_3(\sigma(t)))^2\right]\right\}\\
&\quad-\limsup_{\mu\to 0^+}\left\{\sum_{r=1}^{\left[\frac{t-t_0}{\mu}\right]}(-1)^r\left({}^\alpha C_r\right)\left[2\upsilon_3(\sigma(t))\mu^\alpha\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right]\right\}.
\end{aligned}$$

Applying (8) and (9) we obtain

$$ {}^C\Delta^\alpha_+\mathscr{L}_3 \;\le\; \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_3(\sigma(t)))^2\right]-\left[2\upsilon_1(\sigma(t))\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right].$$

As $t\to\infty$, $\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\left[(\upsilon_3(\sigma(t)))^2\right]\to 0$, then

$$ {}^C\Delta^\alpha_+\mathscr{L}_3 \;\le\; -2\left[\upsilon_3(\sigma(t))\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\right].$$

applying $\upsilon(\sigma(t))\le\mu\,{}^C\Delta^\alpha\upsilon(t)+\upsilon(t)$

$$\begin{aligned}
{}^C\Delta^\alpha_+\mathscr{L}_3 &= -2\Big[\mu(t)\Xi_3^2(t,\upsilon_3,\upsilon_2,\upsilon_3)+\upsilon_3(t)\Xi_3(t,\upsilon_1,\upsilon_2,\upsilon_3)\Big]\\
&= -2\left[\mu(t)\left(-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right)^2+\upsilon_3\left(-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right)\right]\\
&= -2\mu(t)\left[\left(-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right)^2\right]-2\upsilon_3\left[-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right]. \qquad (25)
\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$ we have that $\mu=0$, so that (25) becomes;

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}_3(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\upsilon_3\left[-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right] \\ &= 2\upsilon_1^2+2\upsilon_2-10\upsilon_3^2 \\ &= (2 \quad 2 \quad -10)\cdot(\mathscr{L}_1 \quad \mathscr{L}_2 \quad \mathscr{L}_3)^T. \end{aligned} \tag{26}$$

If $\mathbb{T}=\mathbb{N}_0$, we have that $\mu=1$, so that (21) becomes:

$$\begin{aligned} {}^{C}\Delta_{+}^{\alpha}\mathscr{L}_3(\upsilon_1,\upsilon_2,\upsilon_3) &= -2\left[\left(-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right)^2\right]-2\upsilon_1\left[-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right] \\ &\leq -2\upsilon_1\left[-\frac{\upsilon_1^2+\upsilon_2}{\upsilon_3}+5\upsilon_3\right], \end{aligned}$$

this also leads to the same conclusion as (26). Clearly, this also works for any other discrete time.

Combining (22), (24) and (26), we have that

$$ {}^{C}\Delta_{+}^{\alpha}\mathscr{L} \leq \begin{pmatrix} -6 & 0 & 12 \\ 4 & -8 & 10 \\ 2 & 2 & -10 \end{pmatrix}\begin{pmatrix} \mathscr{L}_1 \\ \mathscr{L}_2 \\ \mathscr{L}_3 \end{pmatrix} \tag{27}$$

$$\text{If } A=\begin{pmatrix} -6 & 0 & 12 \\ 4 & -8 & 10 \\ 2 & 2 & -10 \end{pmatrix}.$$

The vectorial inequality (27) and all other conditions of Theorem 3.1 are satisfied if A has eigen values with negative real parts, since the eigen values of A are $\lambda_1=-14.248, \quad \lambda_2=-9.20293, \quad \lambda_3=-0.549103$, then (20) is uniformly stable. Therefore, we conclude that the zero solution υ_0 of the system (20) is $(m_0,m)-$strictly uniformly stable..

5. CONCLUSION

In this work, we have explored the $(m_0,m)-$strict uniform stability of Caputo fractional dynamic systems on time scales, leveraging the unique properties of the Caputo fractional derivative to model systems with memory and hereditary effects. By focusing on $(m_0,m)-$strict uniform stability, we have provided a more refined and rigorous framework for analyzing dynamic systems, addressing the limitations of traditional stability concepts such as Lyapunov stability and asymptotic stability. The use of vector LFs has been pivotal in our analysis, allowing for a component-wise examination of system stability. This approach has enabled us to capture the individual behaviors of system components and their contributions to overall stability, offering a more detailed and comprehensive understanding of multi-dimensional dynamics. The practical relevance of our findings has been demonstrated through a comprehensive example, highlighting the applicability of our results in real-world scenarios. The insights gained from this work not only enhance our understanding of system behavior but also pave the way for future research in areas such as variational Lyapunov stability and other related fields. As the demand for more accurate and reliable models of dynamic systems continues to grow, the concepts and methodologies developed in this work will serve as valuable tools for researchers and practitioners alike.

AUTHORS' CONTRIBUTIONS

All authors contributed equally to the manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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