Available online at http://scik.org
Adv. Fixed Point Theory, 2025, 15:16
https://doi.org/10.28919/afpt/9238
ISSN: 1927-6303

INNOVATIVE APPROACHES TO $(\mathscr{A},\mathscr{B})$-CONTRACTION AND $\mathbb{T}$-COUPLING FIXED POINTS IN BIPOLAR PARAMETRIC METRIC SPACES

C. USHA BHAVANI[1,*], P. NARESH[2], B. SRINUVASA RAO[3], G. UPENDER REDDY[4]

[1]Department of Humanities & Basic Science, Sumathi Reddy Institute of Technology for women, Anantha Sagar, Warangal-506371, Telangana, India

[2] Department of Mathematics, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad-501301,Telangana, India

[3]Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam- 532410, Andhra Pradesh, India

[4] Department of Mathematics, Mahatma Gandhi University, Nalgonda-508003, Telangana, India

Abstract: This article explores $(\mathscr{A},\mathscr{B})$-contraction type $\mathbb{T}$-coupling and demonstrates the establishment of unique strong common coupled fixed points (USCCFP) within bipolar parametric metric spaces (BPPMS). Our research expands and generalizes several relevant findings from existing literature. Additionally, we provide two examples to support our key results. Finally, we apply our findings to systems of non-linear integral equations and homotopy.

Keywords: bipolar parametric metric space (BPPMS); coupled coincidence point (CCIP); common coupled fixed point (CCFP); strong coupled fixed point (SCFP); SCC-map.

2020 AMS Subject Classification: 54H25, 47H10, 54E50.

1. INTRODUCTION

Research on the theoretical underpinnings of metric fixed point theory has been ongoing. Banach's introduction of the contraction principle is one of its fundamental theorems [1]. There

*Corresponding author
E-mail address: n.ushabhavani@gmail.com
Received March 09, 2025

are numerous uses for this idea in fixed point theory. There is a great deal of interest in and literature on the study of contractive mappings over different spaces ([2]- [5]).

The concept of a coupled fixed point(CFP) of mapping was introduced by Bhaskar *et al.* [6]. Subsequently, Lakshmikantham *et al.* [7] introduced the idea of a coupled coincidence point(CIP). The broader concept of coupling was detailed by authors in [[8], [9]]. Numerous studies have since explored the existence and uniqueness of CFP and CIP, using Kannan type contractions within complete metric spaces ([10]- [14]). These investigations have demonstrated the existence and uniqueness of strong coupled fixed points(SCFP) under these conditions.

The study of fixed points(FP) and associated properties for couplings meeting different kinds of inequalities was presented as an open problem by Choudhury *et al.* [9]. For (ϕ,ψ)-contraction type coupling in complete partial metric spaces, Aydi et al. [10] demonstrated the existence and uniqueness of a strong coupled fixed point (SCFP). By introducing SCC-Map and ϕ-contraction type T-coupling, as well as generalising the ϕ-contraction type coupling provided by Aydi et al. [10] to ϕ-contraction type T-coupling, Rashid and Khan [11] attempted to address this open problem and demonstrated the existence theorem of CCIP for metric spaces that are not complete.Subsequntily, Fuad Abdulkerim *et al.* [15] established CFP results by using (ϕ,ψ)-contraction type T-coupling mappings.

N. Hussain *et al.* recently introduced and researched the idea that parametric metric spaces are a natural generalization of metric spaces ([16], [17]). As a generalization of parametric metric space, Kumar, Ege, Mor, Kumar, and De la Sen [18] established FPT and introduced the idea of binary operation at the place non-negative parameter $\mathfrak{t}$. The concept of BPPMS was introduced and some FPT were proved on this space in 2024 by M. I. Pasha *et al.* [19].

In this paper, we use $(\mathscr{A},\mathscr{B})$-contraction type $\mathbb{T}$-coupling function to provide various CFPT in the context of BPPMS. Additionally, we are able to give appropriate instances that are pertinent to homotopy and integral equations.

First we recall some basic definitions and results.

2. Preliminaries

Definition 2.1:([19]) Suppose $\rho_c : \pounds \times \Im \times (0,\infty) \to \mathbb{R}^+$ is a function defined on two non empty sets $\pounds$ and $\Im$ such that.

(1) If $\rho_c(\mathfrak{x},\mathfrak{y},c) = 0$ for all $c > 0$ then $\mathfrak{x} = \mathfrak{y}$, for all $(\mathfrak{x},\mathfrak{y}) \in \pounds \times \Im$.
(2) If $\mathfrak{x} = \mathfrak{y}$, then $\rho_c(\mathfrak{x},\mathfrak{y},c) = 0$, for all $c > 0$ and $(\mathfrak{x},\mathfrak{y}) \in \pounds \times \Im$
(3) $\rho_c(\mathfrak{x},\mathfrak{y},c) = \rho_c(\mathfrak{y},\mathfrak{x},c)$, for all $c > 0$ and $\mathfrak{x},\mathfrak{y} \in \pounds \cap \Im$
(4) $\rho_c(\mathfrak{x},\mathfrak{y},c) \leq \rho_c(\mathfrak{x},\mathfrak{z},c) + \rho_c(\mathfrak{d},\mathfrak{z},c) + \rho_c(\mathfrak{d},\mathfrak{y},c)$, for all $c > 0$, $\mathfrak{x},\mathfrak{d} \in \pounds$ and $\mathfrak{y},\mathfrak{z} \in \Im$.

The triplet $(\pounds,\Im,\rho_c)$ is called a BPPMS.

Example 2.2:([19]) For all $\mathfrak{x} \in \pounds$, $\mathfrak{y} \in \Im$, and $c > 0$, let $\pounds = [-1,0]$ and $\Im = [0,1]$ be equipped with $\rho_c(\mathfrak{x},\mathfrak{y},c) = c|\mathfrak{x}-\mathfrak{y}|$. Consequently, $(\pounds,\Im,\rho_c)$ is a complete BPPMS.

Definition 2.3:([19]) Let $(\pounds_1,\Im_1,\rho_{c1})$ and $(\pounds_2,\Im_2,\rho_{c2})$ be BPPMSs and $\Omega : \pounds_1 \cup \Im_1 \to \pounds_2 \cup \Im_2$ be a function. If $\Omega(\pounds_1) \subseteq \pounds_2$ and $\Omega(\Im_1) \subseteq \Im_2$, then Ω is termed as a covariant map and this is written as $\Omega : (\pounds_1,\Im_1,\rho_{c1}) \rightrightarrows (\pounds_2,\Im_2,\rho_{c2})$. If Ω is mapping from $(\pounds_1,\Im_1,\rho_{c1})$ to $(\Im_2,\pounds_2,\overline{\rho_{c2}})$, then Ω is termed as a contravariant and this is denoted as $\Omega : (\pounds_1,\Im_1,\rho_{c1}) \rightleftarrows (\pounds_2,\Im_2,\rho_{c2})$.

Definition 2.4: ([19]) Let $(\pounds,\Im,\rho_c)$ be a BPPMS. Then

(z1) The points of the sets $\pounds$, $\Im$, and $\pounds \cap \Im$ are referred to as left, right, and central points, respectively. A sequence on $(\pounds,\Im,\rho_c)$ that consists solely of left, right, or central points is termed a left, right, or central sequence.
(z2) $\rho_c(\mathfrak{x}_a,\mathfrak{y},c) < \wp$ for all $a \geq a_0$ and $c > 0$. This means that a left sequence $\{\mathfrak{x}_a\}$ converges to a right point $\mathfrak{y}$ if and only if for every $\wp > 0$ there exists a $a_0 \in \mathbb{N}$.

Similar to this, a right seqence $\{\mathfrak{y}_a\}$ converges to a left point $\mathfrak{x}$ if and only if we can locate a $a_0 \in \mathbb{N}$ satisfying, whenever $a \geq a_0, c > 0, \rho_c(\mathfrak{x},\mathfrak{y}_a,c) < \wp$.

Definition 2.5:([19]) Let $(\pounds,\Im,\rho_c)$ be a BPPMS.

(i) A sequence $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\}) \subseteq \pounds \times \Im$ is called a bisequence on $(\pounds,\Im,\rho_c)$.
(ii) The bisequence $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\})$ is said to be convergent if both $\{\mathfrak{x}_a\}$ and $\{\mathfrak{y}_a\}$ are convergent. This bisequence is said to be biconvergent if $\{\mathfrak{x}_a\}$ and $\{\mathfrak{y}_a\}$ converge to the same point $u \in \pounds \cap \Im$.

(iii) $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\})$ is a bisequence for $(\pounds,\Im,\rho_c)$ is known as a Cauchy bisequence (CBS) if, for every $\wp > 0$, we can locate a number $a_0 \in \mathbb{N}$, which, for all positive integers $a,b \geq a_0, c > 0, \rho_c(\mathfrak{x}_a,\mathfrak{y}_b,c) < \wp$, is called a CBS.

Definition 2.6: [20] Let $\mathfrak{L} = \{\mathscr{A}/\mathscr{A} : [0,\infty) \to [0,\infty)\}$ be a family of altering distance functions such that

(i) $\mathscr{A}$ is continuous and monotone non-decreasing functions;

(ii) $\mathscr{A}(\mathfrak{e}) = 0 \iff \mathfrak{e} = 0$;

3. Main Results

In this section, we prove CFP theorems on BPPMS.

Definition 3.1: Let $(\pounds,\Im,\rho_c)$ be a BPPMS and a pair $(\wp,\varpi)$ is called

(a) a CFP of mapping $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ if $\Omega(\wp,\varpi) = \wp$, $\mathbb{Q}(\varpi,\wp) = \varpi$ for $(\wp,\varpi) \in \pounds^2 \cup \Im^2$;

(a_i) a SCFP of mapping $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ if $(\wp,\varpi)$ is CFP and $\wp = \varpi$ *i.e* $\mathbb{Q}(\wp,\wp) = \wp$;

(b) a CCIP of $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\mathbb{Q}(\wp,\varpi) = \Lambda\wp$, $\mathbb{Q}(\varpi,\wp) = \Lambda\varpi$;

(b_i) a SCCIP of $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\wp = \varpi$. *i.e* $\mathbb{Q}(\wp,\wp) = \Lambda\wp$;

(c) a CCFP of $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\mathbb{Q}(\wp,\varpi) = \Lambda\wp = \wp$, $\mathbb{Q}(\varpi,\wp) = \Lambda\varpi = \varpi$;

(c_i) a SCCFP of $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ and $\Lambda : (\pounds,\Im) \rightrightarrows (\pounds,\Im)$ if $\wp = \varpi$. *i.e* $\mathbb{Q}(\wp,\wp) = \Lambda\wp = \wp$;

(d) the pair $(\mathbb{Q},\Lambda)$ is weakly compatible (ω-compt) if $\Lambda(\mathbb{Q}(\wp,\varpi)) = \mathbb{Q}(\Lambda\wp,\Lambda\varpi)$ and $\Lambda(\mathbb{Q}(\varpi,\wp)) = \mathbb{Q}(\Lambda\varpi,\Lambda\wp)$ whenever $\mathbb{Q}(\wp,\varpi) = \Lambda\wp$, $\mathbb{Q}(\varpi,\wp) = \Lambda\varpi$.

Definition 3.2: Let $(\pounds,\Im,\rho_c)$ be a BPPMS, $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty subsets of $\pounds$ and $\Im$ respectively. Then a covariant map $\mathbb{Q} : (\pounds^2,\Im^2) \rightrightarrows (\pounds,\Im)$ is said to be a coupling with respect to $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$ if $\mathbb{Q}(\wp,\varpi) \in \mathscr{P}\cup\mathscr{Q}$ and $\mathbb{Q}(\varpi,\wp) \in \mathscr{S}\cup\mathscr{T}$ where $(\wp,\varpi) \in (\mathscr{S}\cup\mathscr{T})^2 \cup (\mathscr{P}\cup\mathscr{Q})^2$.

Definition 3.3: Let $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty subsets of $\pounds$ and $\Im$ respectively. Any function $\Lambda:(\pounds,\Im)\rightrightarrows(\pounds,\Im)$ is said to be

- (i) a cyclic (with respect to $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$) if $\Lambda(\mathscr{S}\cup\mathscr{T})\subseteq\mathscr{P}\cup\mathscr{Q}$ and $\Lambda(\mathscr{P}\cup\mathscr{Q})\subseteq\mathscr{S}\cup\mathscr{T}$.
- (ii) a self-cyclic (with respect to $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$) if $\Lambda(\mathscr{S}\cup\mathscr{T})\subseteq\mathscr{S}\cup\mathscr{T}$ and $\Lambda(\mathscr{P}\cup\mathscr{Q})\subseteq\mathscr{P}\cup\mathscr{Q}$.

Definition 3.4: Let $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ be any nonempty subsets of a BPPMS $(\pounds,\Im,\rho_c)$ and $\mathbb{T}:(\pounds,\Im)\rightrightarrows(\pounds,\Im)$ be a self-covariant map. Then $\mathbb{T}$ is said to be SCC-covariant map with respect to $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$, if

- (i) $\mathbb{T}(\mathscr{S}\cup\mathscr{T})\subseteq\mathscr{S}\cup\mathscr{T}$ and $\mathbb{T}(\mathscr{P}\cup\mathscr{Q})\subseteq\mathscr{P}\cup\mathscr{Q}$;
- (ii) $\mathbb{T}(\mathscr{S}\cup\mathscr{T})$ and $\mathbb{T}(\mathscr{P}\cup\mathscr{Q})$ are closed in $\pounds\cup\Im$.

Definition 3.5: Let $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty subsets of a BPPMS $(\pounds,\Im,\rho_c)$ and $\mathbb{T}:(\pounds,\Im)\rightrightarrows(\pounds,\Im)$ is a SCC-covariant map on $\pounds\cup\Im$ w.r.t $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$. Then a coupling $\mathbb{Q}:\left(\pounds^2,\Im^2\right)\rightrightarrows(\pounds,\Im)$ is said to be $(\mathscr{A},\mathscr{B})$-contraction type $\mathbb{T}$-coupling w.r.t $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$ if there exist two altering distance functions $\mathscr{A},\mathscr{B}\in\mathfrak{L}$ and $\ell\in(0,1)$ such that

$$\mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{z},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{y}),c)\right)\leq\mathscr{A}\left(\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{e})\right)-\mathscr{B}\left(\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{e})\right)\tag{1}$$

where,

$\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{e})=\ell\max\left\{\ \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right),\rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right)\ \right\}$ for all $\mathfrak{x}\in\mathscr{S},\mathfrak{z}\in\mathscr{P}$ and $\mathfrak{y}\in\mathscr{T},\mathfrak{e}\in\mathscr{Q}$.

Theorem 3.6: Let $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ be a nonempty closed subsets of a complete BPPMS $(\pounds,\Im,\rho_c)$, $\mathbb{T}:(\pounds,\Im)\rightrightarrows(\pounds,\Im)$ is a SCC-covariant map on $\pounds\cup\Im$ (w.r.t $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$), and a coupling $\mathbb{Q}:\left(\pounds^2,\Im^2\right)\rightrightarrows(\pounds,\Im)$ be $(\mathscr{A},\mathscr{B})$-contraction type $\mathbb{T}$-coupling (w.r.t $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$), Assume

- (i) $\mathbb{T}(\mathscr{S}\cup\mathscr{T})\cap\mathbb{T}(\mathscr{P}\cup\mathscr{Q})\neq\emptyset$;
- (ii) $\mathbb{Q}$ and $\mathbb{T}$ have a CCIP in $(\mathscr{S}\cup\mathscr{T})^2\cup(\mathscr{P}\cup\mathscr{Q})^2$;
- (iii) If $\mathbb{Q}$ and $\mathbb{T}$ are ω-compt.

Then $\mathbb{Q}$ and $\mathbb{T}$ have a unique SCCFP in $(\mathscr{S}\cup\mathscr{T})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$.

Proof Since $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ are non-empty subsets of $(\pounds,\Im)$ and $\mathbb{Q}$ is $(\mathscr{A},\mathscr{B})$-contraction type-$\mathbb{T}$ coupling w.r.t $(\mathscr{S},\mathscr{P})$ and $(\mathscr{T},\mathscr{Q})$, then for $\mathfrak{a}_0\in\mathscr{S},\mathfrak{b}_0\in\mathscr{T}$ and $\mathfrak{c}_0\in\mathscr{P},\mathfrak{d}_0\in\mathscr{Q}$. For each $\kappa \in \mathbb{N}$, define

$$\mathbb{Q}(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa) = \mathbb{T}\mathfrak{c}_\kappa = \mathfrak{z}_\kappa, \quad \mathbb{Q}(\mathfrak{c}_\kappa, \mathfrak{d}_\kappa) = \mathbb{T}\mathfrak{a}_{\kappa+1} = \mathfrak{x}_{\kappa+1}$$
$$\mathbb{Q}(\mathfrak{b}_\kappa, \mathfrak{a}_\kappa) = \mathbb{T}\mathfrak{d}_\kappa = \mathfrak{e}_\kappa, \quad \mathbb{Q}(\mathfrak{d}_\kappa, \mathfrak{c}_\kappa) = \mathbb{T}\mathfrak{b}_{\kappa+1} = \mathfrak{y}_{\kappa+1}$$

Then $(\{\mathfrak{x}_\kappa\}, \{\mathfrak{z}_\kappa\}), (\{\mathfrak{y}_\kappa\}, \{\mathfrak{e}_\kappa\})$ are bisequences in $(\mathscr{S}, \mathscr{P})$ and $(\mathscr{T}, \mathscr{Q})$.
Say $\mathfrak{D}_\kappa = \frac{\ell^{2\kappa}}{1-\ell} \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_0, \mathfrak{z}_0, c), \\ \rho_c(\mathfrak{y}_0, \mathfrak{e}_0, c) \end{array} \right\}$. Then for all $\kappa, \iota \in \mathbb{Z}^+$ and from (1), we can get

$$\begin{aligned}
\mathscr{A}(\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c)) &= \mathscr{A}(\rho_c(\mathbb{Q}(\mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1}), \mathbb{Q}(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa), c)) \\
&\leq \mathscr{A}(\mathbb{M}(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa, \mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1})) - \mathscr{B}(\mathbb{M}(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa, \mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1})) \\
&\leq \mathscr{A}\left(\ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\}\right) - \mathscr{B}\left(\ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\}\right) \\
&\leq \mathscr{A}\left(\ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\}\right).
\end{aligned}$$

Because of,

$$\begin{aligned}
\mathbb{M}(\mathfrak{a}_\kappa, \mathfrak{b}_\kappa, \mathfrak{c}_{\kappa-1}, \mathfrak{d}_{\kappa-1}) &= \ell \max \left\{ \rho_c(\mathbb{T}\mathfrak{a}_\kappa, \mathbb{T}\mathfrak{c}_{\kappa-1}, c), \rho_c(\mathbb{T}\mathfrak{b}_\kappa, \mathbb{T}\mathfrak{d}_{\kappa-1}, c) \right\} \\
&= \ell \max \left\{ \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \right\}.
\end{aligned}$$

By using property of $\mathscr{A}$, we have

$$\rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c) \leq \ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\} \tag{2}$$

Similarly, we can prove

$$\rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) \leq \ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\} \tag{3}$$

Combining (2) and (3), we have

$$\begin{aligned}
\max \left\{ \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_\kappa, c), \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_\kappa, c) \right\} &\leq \ell \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_\kappa, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_\kappa, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\} \\
&\leq \ell^2 \max \left\{ \begin{array}{c} \rho_c(\mathfrak{x}_{\kappa-1}, \mathfrak{z}_{\kappa-1}, c), \\ \rho_c(\mathfrak{y}_{\kappa-1}, \mathfrak{e}_{\kappa-1}, c) \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\leq \ell^{2\kappa}\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\} \\
&\leq (1-\ell)\mathfrak{D}_\kappa \leq \mathfrak{D}_\kappa.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathscr{A}(\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)) &= \mathscr{A}(\rho_c(\mathbb{Q}(\mathfrak{c}_\kappa,\mathfrak{d}_\kappa),\mathbb{Q}(\mathfrak{a}_\kappa,\mathfrak{b}_\kappa),c)) \\
&\leq \mathscr{A}(\mathbb{M}(\mathfrak{a}_\kappa,\mathfrak{b}_\kappa,\mathfrak{c}_\kappa,\mathfrak{d}_\kappa)) - \mathscr{B}(\mathbb{M}(\mathfrak{a}_\kappa,\mathfrak{b}_\kappa,\mathfrak{c}_\kappa,\mathfrak{d}_\kappa)) \\
&\leq \mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\}\right) - \mathscr{B}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\}\right) \\
&\leq \mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\}\right).
\end{aligned}
$$

Because of,

$$
\begin{aligned}
\mathbb{M}(\mathfrak{a}_\kappa,\mathfrak{b}_\kappa,\mathfrak{c}_\kappa,\mathfrak{d}_\kappa) &= \ell\max\left\{\rho_c(\mathbb{T}\mathfrak{a}_\kappa,\mathbb{T}\mathfrak{c}_\kappa,c),\rho_c(\mathbb{T}\mathfrak{b}_\kappa,\mathbb{T}\mathfrak{d}_\kappa,c)\right\} \\
&= \ell\max\left\{\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\right\}.
\end{aligned}
$$

By using property of $\mathscr{A}$, we have

$$
\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c) \leq \ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\} \tag{4}
$$

Similarly, we can prove

$$
\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c) \leq \ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\} \tag{5}
$$

Combining (4) and (5), we have

$$
\begin{aligned}
\max\left\{\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c),\rho_c(\mathfrak{y}_{\kappa+1},\mathfrak{e}_\kappa,c)\right\} &\leq \ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c),\\ \rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)\end{array}\right\} \\
&\leq \ell^{2\kappa+1}\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_\kappa,c) &\leq \rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)\\
&\leq \rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+2},c)+\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+2},c)+\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+1},c)\\
&\quad +(\ell^{2\kappa+2}+\ell^{2\kappa+1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}\\
&\leq \rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+2},c)+(\ell^{2\kappa+4}+\ell^{2\kappa+3}+\ell^{2\kappa+2}+\ell^{2\kappa+1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}\\
&\ \vdots\\
&\leq \rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+\iota-1},c)+(\ell^{2\kappa+2\iota-2}+\cdots+\ell^{2\kappa+1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}\\
&\leq (\ell^{2\kappa+2\iota-1}+\ell^{2\kappa+2\iota-2}+\ell^{2\kappa+2\iota-3}+\cdots+\ell^{2\kappa+1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}\\
&\leq \ell^{2\kappa+1}\sum_{i=0}^{\infty}\ell^i\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}=\ell\mathfrak{D}_\kappa<\mathfrak{D}_\kappa
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_{\kappa+\iota},c) &\leq \rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{x}_{\kappa+1},\mathfrak{z}_{\kappa+\iota},c)\\
&\leq (\ell^{2\kappa}+\ell^{2\kappa+1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}\\
&\quad +\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+1},c)+\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+\iota},c)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\ell^{2\kappa}+\ell^{2\kappa+1}+\ell^{2\kappa+2}+\ell^{2\kappa+3})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}+\rho_c(\mathfrak{x}_{\kappa+2},\mathfrak{z}_{\kappa+\iota},c)\\
&\ \vdots\\
&\leq (\ell^{2\kappa}+\ell^{2\kappa+1}+\cdots+\ell^{2\kappa+2\iota-1})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}+\rho_c(\mathfrak{x}_{\kappa+\iota},\mathfrak{z}_{\kappa+\iota},c)\\
&\leq (\ell^{2\kappa}+\ell^{2\kappa+1}+\cdots+\ell^{2\kappa+2\iota-1}+\ell^{2\kappa+2\iota})\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}
\end{aligned}
$$

$$\leq \quad \ell^{2\kappa}\sum_{i=0}^{\infty}\ell^{i}\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_0,\mathfrak{z}_0,c),\\ \rho_c(\mathfrak{y}_0,\mathfrak{e}_0,c)\end{array}\right\}=\mathfrak{D}_{\kappa}.$$

Now, since $0<\ell<1$, for any $\varsigma>0$, we can find an integer κ_0 such that $\mathfrak{D}_{\kappa_0}<\frac{\varsigma}{3}$. Hence, $\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\iota,c)\leq\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_{\kappa_0},c)+\rho_c(\mathfrak{x}_{\kappa_0},\mathfrak{z}_{\kappa_0},c)+\rho_c(\mathfrak{x}_{\kappa_0},\mathfrak{z}_\iota,c)\leq 3\mathfrak{D}_{\kappa_0}<\varsigma$ and $(\{\mathfrak{x}_\kappa\},\{\mathfrak{z}_\kappa\})$ is a CBS in $(\mathscr{S},\mathscr{P})$. Similarly, we can prove $(\{\mathfrak{y}_\kappa\},\{\mathfrak{e}_\kappa\})$ is a CBS in $(\mathscr{T},\mathscr{Q})$. Since $\mathbb{T}(\mathscr{S}\cup\mathscr{T})$ and $\mathbb{T}(\mathscr{P}\cup\mathscr{Q})$ are closed subset of a complete BPPMS $(\pounds,\Im,\rho_c)$. Then the sequences $\{\mathfrak{x}_\kappa\}$, $\{\mathfrak{y}_\kappa\}\subseteq\mathbb{T}(\mathscr{S}\cup\mathscr{T})$ and $\{\mathfrak{z}_\kappa\}$, $\{\mathfrak{e}_\kappa\}\subseteq\mathbb{T}(\mathscr{P}\cup\mathscr{Q})$ are convergence in complete BPPMS $(\mathbb{T}(\mathscr{S}\cup\mathscr{T}),\mathbb{T}(\mathscr{P}\cup\mathscr{Q}),\rho_c)$. Therefore, there exist $\mathfrak{a},\mathfrak{b}\in\mathbb{T}(\mathscr{S}\cup\mathscr{T})$ and $\mathfrak{l},\mathfrak{m}\in\mathbb{T}(\mathscr{P}\cup\mathscr{Q})$ such that

$$\begin{aligned}&\lim_{\kappa\to\infty}\mathfrak{x}_\kappa=\lim_{\kappa\to\infty}\mathbb{T}\mathfrak{a}_\kappa=\mathfrak{l},\ \lim_{\kappa\to\infty}\mathfrak{y}_\kappa=\lim_{\kappa\to\infty}\mathbb{T}\mathfrak{b}_\kappa=\mathfrak{m},\\ &\lim_{\kappa\to\infty}\mathfrak{z}_\kappa=\lim_{\kappa\to\infty}\mathbb{T}\mathfrak{c}_\kappa=\mathfrak{a},\ \lim_{\kappa\to\infty}\mathfrak{e}_\kappa=\lim_{\kappa\to\infty}\mathbb{T}\mathfrak{d}_\kappa=\mathfrak{b}.\end{aligned}\tag{6}$$

Then there exists $\kappa_1\in\mathbb{N}$ with $\rho_c(\mathfrak{x}_\kappa,\mathfrak{l},c)<\frac{\varsigma}{3}$, $\rho_c(\mathfrak{y}_\kappa,\mathfrak{m},c)<\frac{\varsigma}{3}$, $\rho_c(\mathfrak{a},\mathfrak{z}_\kappa,c)<\frac{\varsigma}{3}$ and $\rho_c(\mathfrak{b},\mathfrak{e}_\kappa,c)<\frac{\varsigma}{3}$ for all $\kappa\geq\kappa_1$ and every $\varsigma>0$. Since $(\{\mathfrak{x}_\kappa\},\{\mathfrak{z}_\kappa\})$ and $(\{\mathfrak{y}_\kappa\},\{\mathfrak{e}_\kappa\})$ are CBS, we get $\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c)<\frac{\varsigma}{3}$ and $\rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)<\frac{\varsigma}{3}$. Now consider,

$$\rho_c(\mathfrak{a},\mathfrak{l},c)\leq\rho_c(\mathfrak{a},\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{x}_\kappa,\mathfrak{z}_\kappa,c)+\rho_c(\mathfrak{x}_\kappa,\mathfrak{l},c)<\varsigma\tag{7}$$

and

$$\rho_c(\mathfrak{b},\mathfrak{m},c)\leq\rho_c(\mathfrak{b},\mathfrak{e}_\kappa,c)+\rho_c(\mathfrak{y}_\kappa,\mathfrak{e}_\kappa,c)+\rho_c(\mathfrak{y}_\kappa,\mathfrak{m},c)<\varsigma\tag{8}$$

Therefore, from (7) and (8), we have

$$\mathfrak{a}=\mathfrak{l}\text{ and }\mathfrak{b}=\mathfrak{m}\tag{9}$$

it follows that $\mathfrak{a},\mathfrak{b}\in\mathbb{T}(\mathscr{S}\cup\mathscr{T})\cap\mathbb{T}(\mathscr{P}\cup\mathscr{Q})\neq\emptyset$. Now, since $\mathfrak{a},\mathfrak{b}\in\mathbb{T}(\mathscr{S}\cup\mathscr{T})$ and $\mathfrak{l},\mathfrak{m}\in\mathbb{T}(\mathscr{P}\cup\mathscr{Q})$, there exist $\mathfrak{f},\mathfrak{g}\in\mathscr{S}\cup\mathscr{T}$ and $\mathfrak{p},\mathfrak{q}\in\mathscr{P}\cup\mathscr{Q}$ such that $\mathbb{T}\mathfrak{f}=\mathfrak{a},\mathbb{T}\mathfrak{g}=\mathfrak{b}$ and $\mathbb{T}\mathfrak{p}=\mathfrak{l},\mathbb{T}\mathfrak{q}=\mathfrak{m}$.

From (6) and (9), we get

$$\begin{aligned}&\mathbb{T}\mathfrak{a}_\kappa\to\mathbb{T}\mathfrak{p},\ \mathbb{T}\mathfrak{b}_\kappa\to\mathbb{T}\mathfrak{q},\mathbb{T}\mathfrak{c}_\kappa\to\mathbb{T}\mathfrak{f},\ \mathbb{T}\mathfrak{d}_\kappa\to\mathbb{T}\mathfrak{g}\\ &\mathbb{T}\mathfrak{p}=\mathbb{T}\mathfrak{f}\text{ and }\mathbb{T}\mathfrak{q}=\mathbb{T}\mathfrak{g}\end{aligned}\tag{10}$$

Claim that $\mathbb{Q}(\mathfrak{f},\mathfrak{g})=\mathfrak{a},\mathbb{Q}(\mathfrak{g},\mathfrak{f})=\mathfrak{b}$ and $\mathbb{Q}(\mathfrak{p},\mathfrak{q})=\mathfrak{l},\mathbb{Q}(\mathfrak{q},\mathfrak{p})=\mathfrak{m}$.

By using Eq.(1), (10), (d), and the properties of $\mathscr{A}$, $\mathscr{B}$, we have

$$\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c)\leq\rho_c(\mathfrak{a},\mathbb{T}\mathfrak{c}_\kappa,c)+\rho_c(\mathbb{T}\mathfrak{a}_\kappa,\mathbb{T}\mathfrak{c}_\kappa,c)+\rho_c(\mathbb{T}\mathfrak{a}_\kappa,\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c)$$

Letting $\kappa\to\infty$, we get

$$\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c)\leq\lim_{\kappa\to\infty}\rho_c(\mathbb{T}\mathfrak{a}_\kappa,\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c).$$

It follows that

$$\begin{aligned}
&\mathscr{A}(\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c))\leq\lim_{\kappa\to\infty}\mathscr{A}(\rho_c(\mathbb{T}\mathfrak{a}_\kappa,\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c))\\
\leq&\lim_{\kappa\to\infty}\mathscr{A}(\rho_c(\mathbb{Q}(\mathfrak{c}_{\kappa-1},\mathfrak{d}_{\kappa-1}),\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c))\\
\leq&\lim_{\kappa\to\infty}\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{c}_{\kappa-1},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{d}_{\kappa-1},c)\end{array}\right\}\right)-\lim_{\kappa\to\infty}\mathscr{B}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{c}_{\kappa-1},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{d}_{\kappa-1},c)\end{array}\right\}\right)\\
\leq&\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathfrak{a},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathfrak{b},c)\end{array}\right\}\right).
\end{aligned}$$

Similarly, we have

$$\mathscr{A}(\rho_c(\mathfrak{b},\mathbb{Q}(\mathfrak{g},\mathfrak{f}),c))\leq\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathfrak{a},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathfrak{b},c)\end{array}\right\}\right).$$

Since,

$$\begin{aligned}
\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c),\\ \rho_c(\mathfrak{b},\mathbb{Q}(\mathfrak{g},\mathfrak{f}),c)\end{array}\right\}\right)&=\max\left\{\begin{array}{c}\mathscr{A}(\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c)),\\ \mathscr{A}(\rho_c(\mathfrak{b},\mathbb{Q}(\mathfrak{g},\mathfrak{f}),c))\end{array}\right\}\\
&\leq\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathfrak{a},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathfrak{b},c)\end{array}\right\}\right)=0.
\end{aligned}$$

we have

$$\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(\mathfrak{a},\mathbb{Q}(\mathfrak{f},\mathfrak{g}),c),\\ \rho_c(\mathfrak{b},\mathbb{Q}(\mathfrak{g},\mathfrak{f}),c)\end{array}\right\}\right)=0.$$

So that $\mathbb{Q}(\mathfrak{f},\mathfrak{g})=\mathfrak{a}$ and $\mathbb{Q}(\mathfrak{g},\mathfrak{f})=\mathfrak{b}$. Similarly, we can prove $\mathbb{Q}(\mathfrak{p},\mathfrak{q})=\mathfrak{l},\mathbb{Q}(\mathfrak{q},\mathfrak{p})=\mathfrak{m}$. Hence, from (9), $\mathbb{Q}(\mathfrak{f},\mathfrak{g})=\mathbb{T}\mathfrak{f}=\mathfrak{a}=\mathfrak{l}=\mathbb{T}\mathfrak{p}=\mathbb{Q}(\mathfrak{p},\mathfrak{q})$ and $\mathbb{Q}(\mathfrak{g},\mathfrak{f})=\mathbb{T}\mathfrak{g}=\mathfrak{b}=\mathfrak{m}=\mathbb{T}\mathfrak{q}=\mathbb{Q}(\mathfrak{q},\mathfrak{p})$. Therefore,$(\mathfrak{f},\mathfrak{g})\in(\mathscr{S}\cup\mathscr{T})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$ is the CCIP, and $(\mathbb{T}(\mathfrak{f}),\mathbb{T}(\mathfrak{g}))$ is the CCIP of $\mathbb{Q}$ and

$\mathbb{T}$. Let $(\mathfrak{f}^\star,\mathfrak{g}^\star)$ be another CCIP of $\mathbb{Q}$ and $\mathbb{T}$. So, we will prove that $\mathbb{T}(\mathfrak{f})=\mathbb{T}(\mathfrak{f}^\star)$ and $\mathbb{T}(\mathfrak{g})=\mathbb{T}(\mathfrak{g}^\star)$. From (1), we have

$$\begin{aligned}
&\mathscr{A}\left(\rho_c(\mathbb{T}(\mathfrak{f}),\mathbb{T}(\mathfrak{f}^\star),c)\right)\\
=\ &\mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{f},\mathfrak{g}),\mathbb{Q}(\mathfrak{f}^\star,\mathfrak{g}^\star),c)\right)\\
\leq\ &\mathscr{A}\left(\mathbb{M}(\mathfrak{f},\mathfrak{g},\mathfrak{f}^\star,\mathfrak{g}^\star)\right)-\mathscr{B}\left(\mathbb{M}(\mathfrak{f},\mathfrak{g},\mathfrak{f}^\star,\mathfrak{g}^\star)\right)\\
\leq\ &\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c)\end{array}\right\}\right)-\mathscr{B}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c)\end{array}\right\}\right)\\
\leq\ &\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c)\end{array}\right\}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c)\end{array}\right\}\right) &= \max\left\{\begin{array}{c}\mathscr{A}(\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c)),\\ \mathscr{A}(\rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c))\end{array}\right\}\\
&\leq \mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{f}^\star,c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{g}^\star,c)\end{array}\right\}\right).
\end{aligned}$$

Using the property of $\mathscr{A}$ and $\ell\in(0,1)$, we get $\mathbb{T}(\mathfrak{f})=\mathbb{T}(\mathfrak{f}^\star)$ and $\mathbb{T}(\mathfrak{g})=\mathbb{T}(\mathfrak{g}^\star)$. Hence, the CCIP of $\mathbb{Q}$ and $\mathbb{T}$ is unique. Finally, we prove $\mathbb{T}(\mathfrak{f})=\mathbb{T}(\mathfrak{g})$.

$$\begin{aligned}
\mathscr{A}\left(\rho_c(\mathbb{T}(\mathfrak{f}),\mathbb{T}(\mathfrak{g}),c)\right) &= \mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{f},\mathfrak{g}),\mathbb{Q}(\mathfrak{g},\mathfrak{f}),c)\right)\\
&\leq \mathscr{A}\left(\mathbb{M}(\mathfrak{f},\mathfrak{g},\mathfrak{g},\mathfrak{f})\right)-\mathscr{B}\left(\mathbb{M}(\mathfrak{f},\mathfrak{g},\mathfrak{g},\mathfrak{f})\right)
\end{aligned}$$

$$\begin{aligned}
\leq\ &\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{g},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{f},c)\end{array}\right\}\right)-\mathscr{B}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{g},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{f},c)\end{array}\right\}\right)\\
\leq\ &\mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathbb{T}\mathfrak{f},\mathbb{T}\mathfrak{g},c),\\ \rho_c(\mathbb{T}\mathfrak{g},\mathbb{T}\mathfrak{f},c)\end{array}\right\}\right).
\end{aligned}$$

by the property of $\mathscr{A}$, we have

$$\rho_c(\mathbb{T}(\mathfrak{f}),\mathbb{T}(\mathfrak{g}),c)\leq\ell\rho_c(\mathbb{T}(\mathfrak{f}),\mathbb{T}(\mathfrak{g}),c)$$

Since, $\ell \in (0,1)$. Thus, $(\mathbb{T}(\mathfrak{f}), \mathbb{T}(\mathfrak{f}))$ is the UCCIP of the mappings $\mathbb{Q}$ and $\mathbb{T}$ w.r.t $\mathscr{S} \cup \mathscr{T}$ and $\mathscr{P} \cup \mathscr{Q}$. Now, we show that $\mathbb{Q}$ and $\mathbb{T}$ have USCCFP. For this let $\mathbb{T}(\mathfrak{f}) = \mathfrak{h}$, then, we have $\mathfrak{h} = \mathbb{T}(\mathfrak{f}) = \mathbb{Q}(\mathfrak{f}, \mathfrak{f})$ by the ω-compt of $\mathbb{Q}$ and $\mathbb{T}$, we have

$$\mathbb{T}\mathfrak{h} = \mathbb{T}(\mathbb{T}(\mathfrak{f})) = \mathbb{T}\mathbb{Q}(\mathfrak{f}, \mathfrak{f}) = \mathbb{Q}(\mathbb{T}\mathfrak{f}, \mathbb{T}\mathfrak{f}) = \mathbb{Q}(\mathfrak{h}, \mathfrak{h})$$

Thus, $(\mathbb{T}\mathfrak{h}, \mathbb{T}\mathfrak{h})$ is CCIP of $\mathbb{Q}$ and $\mathbb{T}$. By the UCCIP of $\mathbb{Q}$ and $\mathbb{T}$, we have $\mathbb{T}\mathfrak{h} = \mathbb{T}\mathfrak{f}$. Thus, we obtain $\mathfrak{h} = \mathbb{T}\mathfrak{h} = \mathbb{Q}(\mathfrak{h}, \mathfrak{h})$. Therefore, $(\mathfrak{h}, \mathfrak{h})$ is the USCCFP of $\mathbb{Q}$ and $\mathbb{T}$.

Corollary 3.7: Let $\mathscr{S}, \mathscr{T}$ and $\mathscr{P}, \mathscr{Q}$ be a nonempty closed subsets of a complete BPPMS $(\pounds, \Im, \rho_c)$, a coupling covariant map $\mathbb{Q} : (\pounds^2, \Im^2) \rightrightarrows (\pounds, \Im)$ be satisfying $(\mathscr{A}, \mathscr{B})$-contraction type coupling (w.r.t $(\mathscr{S}, \mathscr{P})$ and $(\mathscr{T}, \mathscr{Q})$), then $\mathbb{Q}$ has a USCFP in $(\mathscr{S} \cup \mathscr{T})^2 \cap (\mathscr{P} \cup \mathscr{Q})^2$.

Proof Following the lines of Theorem (3.6), by taking as a $\mathbb{T} = I_{\pounds \cup \Im}$.

Example 3.8: Let $\pounds = \{0, 1, 2, 7\}$ and $\Im = \{0, \frac{1}{4}, \frac{1}{2}, \frac{7}{4}, 3\}$ be equipped with $\rho_c(\sigma, \eta, c) = c|\sigma - \eta|$ for all $\sigma \in \pounds$, $\eta \in \Im$ and $c > 0$. Then, $(\pounds, \Im, \aleph)$ is a complete BPPMS. Let $\mathscr{S} = \{0\}$, $\mathscr{T} = \{0, 1\}$, $\mathscr{P} = \{0\}$ and $\mathscr{Q} = \{0, \frac{1}{2}\}$. Then $\mathscr{S}, \mathscr{T}$ and $\mathscr{P}, \mathscr{Q}$ are closed subsets of $\pounds$ and $\Im$ respectively. Define $\mathbb{Q} : \pounds^2 \cup \Im^2 \rightleftarrows \pounds \cup \Im$ given by $\mathbb{Q}(\mathfrak{x}, \mathfrak{z}) = \min\{\mathfrak{x}, \mathfrak{z}\}$ for all $(\mathfrak{x}, \mathfrak{z}) \in \pounds^2 \cup \Im^2$. Let $\mathbb{T} : \pounds \cup \Im \rightleftarrows \pounds \cup \Im$ as $\mathbb{T}(\mathfrak{x}) = \begin{cases} 0, & \text{if } 0 \leq \mathfrak{x} < 1, \\ 1, & \text{if } 1 \leq \mathfrak{x} \leq 7. \end{cases}$

Also, we define $\mathscr{A}, \mathscr{B} : [0, \infty) \to [0, \infty)$ as $\mathscr{A}(\mathfrak{x}) = \mathfrak{x}^2$ and $\mathscr{B}(\mathfrak{x}) = \mathfrak{x}^3$, then clearly $\mathscr{A}, \mathscr{B}$ are altering distances functions. Observer that $\mathbb{T}(\mathscr{S} \cup \mathscr{T}) = \{0, 1\}$ and $\mathbb{T}(\mathscr{P} \cup \mathscr{Q}) = \{0, \frac{1}{2}\}$ are closed in $\pounds \cup \Im$. Hence $\mathbb{T}$ is a SCC-map. For all $(\mathfrak{x}, \mathfrak{z}) \in (\mathscr{S} \cup \mathscr{T})^2 \cup (\mathscr{P} \cup \mathscr{Q})^2$, we have $\mathbb{Q}(\mathfrak{x}, \mathfrak{z}) = 0 \in \mathscr{P} \cup \mathscr{Q}$ and $\mathbb{Q}(\varpi, \wp) = 0 \in \mathscr{S} \cup \mathscr{T}$. Which show that $\mathbb{Q}$ is $\mathbb{T}$-coupling w.r.t $(\mathscr{S}, \mathscr{P})$ and $(\mathscr{T}, \mathscr{Q})$. Finally, we prove that $\mathbb{Q}$ is $(\mathscr{A}, \mathscr{B})$-Contraction type $\mathbb{T}$-Coupling w.r.t. $(\mathscr{S}, \mathscr{P})$ and $(\mathscr{T}, \mathscr{Q})$. For all $\mathfrak{x} \in \mathscr{S}, \mathfrak{z} \in \mathscr{P}$ and $\mathfrak{y} \in \mathscr{T}, \mathfrak{e} \in \mathscr{Q}$. $\{i.e\}$ $\mathfrak{x} = 0, \mathfrak{y} = 0, 1, \mathfrak{z} = 0, \mathfrak{e} = 0, \frac{1}{2}$ and take $\ell = \frac{1}{2} \in (0, 1)$, $c = 2 > 0$. Four cases will arise for $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{e}$.

(i) $\mathfrak{x} = 0, \mathfrak{y} = 0, \mathfrak{z} = 0, \mathfrak{e} = 0$

(ii) $\mathfrak{x} = 0, \mathfrak{y} = 1, \mathfrak{z} = 0, \mathfrak{e} = 0$

(iii) $\mathfrak{x} = 0, \mathfrak{y} = 0, \mathfrak{z} = 0, \mathfrak{e} = \frac{1}{2}$

(iv) $\mathfrak{x} = 0, \mathfrak{y} = 1, \mathfrak{z} = 0, \mathfrak{e} = \frac{1}{2}$

For case(i), we have $\mathbb{Q}(\mathfrak{z}, \mathfrak{e}) = \mathbb{Q}(0, 0) = 0$, $\mathbb{Q}(\mathfrak{x}, \mathfrak{y}) = \mathbb{Q}(0, 0) = 0$, $\mathbb{T}\mathfrak{x} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{z} = \mathbb{T}0 = 0$,

$\mathbb{T}\mathfrak{y} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{e} = \mathbb{T}0 = 0$ and $\rho_c(0,0,c) = 0$ then

$$\begin{aligned} 0 &= \mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{z},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{y}),c)\right) \\ &\leq \mathscr{A}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) - \mathscr{B}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) \\ &\leq \mathscr{A}\left(\frac{1}{2}\max\left\{\ 0,0\ \right\}\right) - \mathscr{B}\left(\frac{1}{2}\max\left\{\ 0,0\ \right\}\right) \leq \mathscr{A}(0) - \mathscr{B}(0) = 0 \end{aligned}$$

which proves case(i).

For case(ii), we have $\mathbb{Q}(\mathfrak{z},\mathfrak{e}) = \mathbb{Q}(0,0) = 0$, $\mathbb{Q}(\mathfrak{x},\mathfrak{y}) = \mathbb{Q}(0,1) = 0$, $\mathbb{T}\mathfrak{x} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{z} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{y} = \mathbb{T}1 = 1$, $\mathbb{T}\mathfrak{e} = \mathbb{T}0 = 0$ and $\rho_c(0,0,c) = 0$ then

$$\begin{aligned} 0 &= \mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{z},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{y}),c)\right) \\ &\leq \mathscr{A}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) - \mathscr{B}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) \\ &\leq \mathscr{A}\left(\frac{1}{2}\max\left\{\ 0,c\ \right\}\right) - \mathscr{B}\left(\frac{1}{2}\max\left\{\ 0,c\ \right\}\right) \leq \mathscr{A}(1) - \mathscr{B}(1) = 0 \end{aligned}$$

which proves case(ii).

For case(iii), we have $\mathbb{Q}(\mathfrak{z},\mathfrak{e}) = \mathbb{Q}(0,\frac{1}{2}) = 0$, $\mathbb{Q}(\mathfrak{x},\mathfrak{y}) = \mathbb{Q}(0,0) = 0$, $\mathbb{T}\mathfrak{x} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{z} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{y} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{e} = \mathbb{T}\frac{1}{2} = 0$ and $\rho_c(0,0,c) = 0$ then

$$\begin{aligned} 0 &= \mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{z},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{y}),c)\right) \\ &\leq \mathscr{A}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) - \mathscr{B}\left(\ell \max\left\{\begin{array}{c} \rho_c\left(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c\right), \\ \rho_c\left(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c\right) \end{array}\right\}\right) \end{aligned}$$

$$\begin{aligned} &\leq \mathscr{A}\left(\frac{1}{2}\max\left\{\ 0,0\ \right\}\right) - \mathscr{B}\left(\frac{1}{2}\max\left\{\ 0,0\ \right\}\right) \\ &\leq \mathscr{A}(0) - \mathscr{B}(0) = 0 \end{aligned}$$

which proves case(iii).

For case(iv), we have $\mathbb{Q}(\mathfrak{z},\mathfrak{e}) = \mathbb{Q}(0,\frac{1}{2}) = 0$, $\mathbb{Q}(\mathfrak{x},\mathfrak{y}) = \mathbb{Q}(0,1) = 0$, $\mathbb{T}\mathfrak{x} = \mathbb{T}0 = 0$, $\mathbb{T}\mathfrak{z} = \mathbb{T}0 = 0$,

$\mathbb{T}\mathfrak{y} = \mathbb{T}1 = 1$, $\mathbb{T}\mathfrak{e} = \mathbb{T}\frac{1}{2} = 0$ and $\rho_c(0,0,c) = 0$ then

$$\begin{aligned} 0 &= \mathscr{A}\left(\rho_c(\mathbb{Q}(\mathfrak{z},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{y}),c)\right) \\ &\leq \mathscr{A}\left(\ell \max\left\{ \begin{array}{c} \rho_c(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c), \\ \rho_c(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c) \end{array} \right\}\right) - \mathscr{B}\left(\ell \max\left\{ \begin{array}{c} \rho_c(\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{z},c), \\ \rho_c(\mathbb{T}\mathfrak{y},\mathbb{T}\mathfrak{e},c) \end{array} \right\}\right) \\ &\leq \mathscr{A}\left(\frac{1}{2}\max\left\{ 0,c \right\}\right) - \mathscr{B}\left(\frac{1}{2}\max\left\{ 0,c \right\}\right) \\ &\leq \mathscr{A}(1) - \mathscr{B}(1) = 0 \end{aligned}$$

which proves case(iv).

From the cases (i) to (iv) $\mathbb{Q}$ and $\mathbb{T}$ satisfy all the conditions of Theorem 3.6. Thus $\mathbb{Q}$ and $\mathbb{T}$ have a SCFP in $(\mathscr{S}\cup\mathscr{T})\cap(\mathscr{P}\cup\mathscr{Q})$. Obviously, $\mathbb{T}(\mathscr{S}\cup\mathscr{T})\cap\mathbb{T}(\mathscr{P}\cup\mathscr{Q}) = \{0\} \neq \emptyset$. 0 is the USCCIP and $(0,0)$ is the USCCFP of $\mathbb{Q}$ and $\mathbb{T}$ in $(\mathscr{S}\cup\mathscr{T})\cap(\mathscr{P}\cup\mathscr{Q})$ as
$\mathbb{T}(0) = \mathbb{Q}(0,0) = \min\{0,0\} = 0$.

4. APPLICATION

4.1. Application to the existence of solutions of integral equations.

In this section, we present an application of our coupled fixed point results derived in our Corollary (3.7) to establish the existence and uniqueness of a solution of a system of integral equations.

We consider a coupled system of two nonlinear integral equations as follows:

$$\left\{ \begin{array}{l} \mathfrak{x}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))d\mathfrak{u}, \mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2 \\ \mathfrak{y}(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\mathscr{F}(\mathfrak{u},\mathfrak{y}(\mathfrak{u}),\mathfrak{x}(\mathfrak{u}))d\mathfrak{u}, \mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2 \end{array} \right. \tag{11}$$

where

(*i*) $\mathscr{G} : \mathscr{E}_1^2 \cup \mathscr{E}_2^2 \to \mathbb{R}$ and $\mathscr{F} : (\mathscr{E}_1\cup\mathscr{E}_2)\times\mathbb{R}^2 \to \mathbb{R}$ are continuous;

(*ii*) $\mathfrak{f} : \mathscr{E}_1\cup\mathscr{E}_2 \to \mathbb{R}$ is continuous and measurable at $\mathfrak{u}\in\mathscr{E}_1\cup\mathscr{E}_2$, $\forall\ \mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2$;

(*iii*) $\mathscr{G}(\mathfrak{v},\mathfrak{u}) \geq 0\ \forall\ \mathfrak{v},\mathfrak{u}\in\mathscr{E}_1\cup\mathscr{E}_2$ and $\int\limits_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})d\mathfrak{u} \leq 1\ \forall\ \mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2$;

(*iv*) $|\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u})) - \mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))| \leq \frac{\ell}{2c}\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w})$ where $\ell\in(0,1)$, $c>0$ and $\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w}) = \max\left\{ \rho_c(\mathfrak{x},\mathfrak{z},c), \rho_c(\mathfrak{y},\mathfrak{w},c) \right\}$ for all $\mathfrak{x}\in\mathscr{S}, \mathfrak{y}\in\mathscr{T}, \mathfrak{z}\in\mathscr{P}, \mathfrak{w}\in\mathscr{Q}$ where $\mathscr{S},\mathscr{T}$ and $\mathscr{P},\mathscr{Q}$ are closed subsets of $\pounds$ and $\Im$ respectively,

Theorem 4.1 Suppose that (i) and (iv) hold. Then, the existence of a coupled solution for Eq. (11) provides the existence of solution in $(\mathscr{S}\cup\mathscr{T})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$ for the integral Eq. (11).

Proof Let $\pounds = C(\mathscr{L}^\infty(\mathscr{E}_1)), \mathfrak{I} = C(\mathscr{L}^\infty(\mathscr{E}_2))$ be the set of essential bounded measurable continuous functions on $\mathscr{E}_1$ and $\mathscr{E}_2$ where $\mathscr{E}_1, \mathscr{E}_2$ are two Lebesgue measurable sets with $m(\mathscr{E}_1\cup\mathscr{E}_2)<\infty$. Define $\rho_c : \pounds\times\mathfrak{I}\times(0,\infty)\to\mathbb{R}^+$ as $\rho_c(\ell,\sigma,c) = c.\sup_{\mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2}|\ell(\mathfrak{v})-\sigma(\mathfrak{v})|$ for all $\ell\in\pounds, \sigma\in\mathfrak{I}$, $c\in(0,\infty)$. Therefore, $(\pounds,\mathfrak{I},\rho_c)$ is a complete BPPMS. We define $\mathscr{A},\mathscr{B} : [0,\infty)\to[0,\infty)$ as $\mathscr{A}(\mathfrak{x}) = \mathfrak{x}$ and $\mathscr{B}(\mathfrak{x}) = \frac{\mathfrak{x}}{2}$, then clearly $\mathscr{A},\mathscr{B}$ are altering distances functions.

Define $\Gamma : (\pounds^2,\mathfrak{I}^2) \rightrightarrows (\pounds,\mathfrak{I})$ as $\Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_{\mathscr{E}_1\cup\mathscr{E}_2} \mathscr{G}(\mathfrak{v},\mathfrak{u})\mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))d\mathfrak{u}, \mathfrak{u}\in\mathscr{E}_1\cup\mathscr{E}_2$.
Using the inequalities, (i), (ii), (iii), (iv) and for every $\mathfrak{v}\in\mathscr{E}_1\cup\mathscr{E}_2$, we have

$$\begin{aligned}
|\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v})-\Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v})| &= |\int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}(\mathfrak{v},\mathfrak{u})\left(\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u}))-\mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))\right)d\mathfrak{u}| \\
&\leq \int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}(\mathfrak{v},\mathfrak{u})|\mathscr{F}(\mathfrak{u},\mathfrak{z}(\mathfrak{u}),\mathfrak{w}(\mathfrak{u}))-\mathscr{F}(\mathfrak{u},\mathfrak{x}(\mathfrak{u}),\mathfrak{y}(\mathfrak{u}))|d\mathfrak{u} \\
&\leq \int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}(\mathfrak{v},\mathfrak{u})\frac{\ell}{2c}\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w})d\mathfrak{u} \\
&\leq \frac{\ell}{2c}\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w})\int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}(\mathfrak{v},\mathfrak{u})d\mathfrak{u} \leq \frac{\ell}{2c}\mathbb{M}(\mathfrak{x},\mathfrak{y},\mathfrak{z},\mathfrak{w})
\end{aligned}$$

which implies that

$$\rho_c(\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}),\Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v}),c) \leq \frac{\ell}{2}\max\left\{\ \rho_c(\mathfrak{x},\mathfrak{z},c), \rho_c(\mathfrak{y},\mathfrak{w},c)\ \right\}$$

taking $\mathscr{A}(\mathfrak{x}) = \mathfrak{x}$ and $\mathscr{B}(\mathfrak{x}) = \frac{\mathfrak{x}}{2}$, then we get

$$\mathscr{A}\left(\rho_c(\Gamma(\mathfrak{z},\mathfrak{w})(\mathfrak{v}),\Gamma(\mathfrak{x},\mathfrak{y})(\mathfrak{v}),c)\right) \leq \mathscr{A}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x},\mathfrak{z},c),\\ \rho_c(\mathfrak{y},\mathfrak{w},c)\end{array}\right\}\right) - \mathscr{B}\left(\ell\max\left\{\begin{array}{c}\rho_c(\mathfrak{x},\mathfrak{z},c),\\ \rho_c(\mathfrak{y},\mathfrak{w},c)\end{array}\right\}\right)$$

Hence, all the conditions of Corollary (3.7) hold, we conclude that Γ has a unique coupled solution in $(\mathscr{S}\cup\mathscr{T})^2\cap(\mathscr{P}\cup\mathscr{Q})^2$ to the integral equation (11).

4.2. Application to the existence of solutions of Homotopy.

In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem 4.2 Let $(\pounds,\mathfrak{I},\rho_c)$ be a complete BPPMS, $((\mathscr{S},\mathscr{G}),(\mathscr{I},\mathscr{L}))$ and $\left((\overline{\mathscr{S}},\overline{\mathscr{G}}),(\overline{\mathscr{I}},\overline{\mathscr{L}})\right)$ be an open and closed subsets of $(\pounds,\mathfrak{I})$ such that $((\mathscr{S},\mathscr{G}),(\mathscr{I},\mathscr{L}))\subseteq\left((\overline{\mathscr{S}},\overline{\mathscr{G}}),(\overline{\mathscr{I}},\overline{\mathscr{L}})\right)$. Suppose $\mathscr{H}_c:\left((\overline{\mathscr{S}}\cup\overline{\mathscr{G}})\times(\overline{\mathscr{I}}\cup\overline{\mathscr{L}})\right)\cup\left((\overline{\mathscr{I}}\cup\overline{\mathscr{L}})\times(\overline{\mathscr{S}}\cup\overline{\mathscr{G}})\right)\times[0,1]\to\pounds\cup\mathfrak{I}$ be an operator with following conditions are satisfying,

$\star_1$) $\mathfrak{x}\neq\mathscr{H}_c(\mathfrak{x},\mathfrak{e},s)$, $\mathfrak{e}\neq\mathscr{H}_c(\mathfrak{e},\mathfrak{x},s)$, for each $\mathfrak{x}\in\partial(\mathscr{S}\cup\mathscr{G}),\mathfrak{e}\in\partial(\mathscr{I}\cup\mathscr{L})$ and $s\in[0,1]$ (Here $\partial(\mathscr{S}\cup\mathscr{G})\cup\partial(\mathscr{I}\cup\mathscr{L})$ is boundary of $(\mathscr{S}\cup\mathscr{G})\cup(\mathscr{I}\cup\mathscr{L})$ in $\pounds\cup\mathfrak{I}$);

$\star_2$) for all $\mathfrak{x}\in\overline{\mathscr{S}}$, $\mathfrak{e}\in\overline{\mathscr{G}}$, $\iota\in\overline{\mathscr{I}}$, $\varsigma\in\overline{\mathscr{L}}$, $s\in[0,1]$ and $\mathscr{A},\mathscr{B}\in\mathfrak{L}$ such that

$$\mathscr{A}\left(\rho_c\left(\mathscr{H}_c(\mathfrak{x},\iota,s),\mathscr{H}_c(\varsigma,\mathfrak{e},s),c\right)\right)\leq\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(\mathfrak{x},\varsigma,c),\\ \rho_c(\mathfrak{e},\iota,c)\end{array}\right\}\right)-\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c(\mathfrak{x},\varsigma,c),\\ \rho_c(\mathfrak{e},\iota,c)\end{array}\right\}\right)$$

$\star_3$) $\exists\, M\geq0\ni\rho_c(\mathscr{H}_c(\mathfrak{x},\iota,s),\mathscr{H}_c(\varsigma,\mathfrak{e},t),c)\leq Mc|s-t|$ for every $\mathfrak{x}\in\overline{\mathscr{S}},\mathfrak{e}\in\overline{\mathscr{G}},\iota\in\overline{\mathscr{I}},\varsigma\in\overline{\mathscr{L}}$ and $s,t\in[0,1]$.

Then $\mathscr{H}_c(.,0)$ has a CFP $\iff$ $\mathscr{H}_c(.,1)$ has a CFP.

Proof Let the set

$$\Theta=\left\{\ s\in[0,1]:\mathscr{H}_c(\mathfrak{x},\iota,s)=\mathfrak{x},\mathscr{H}_c(\iota,\mathfrak{x},s)=\iota\text{ for some }\mathfrak{x}\in\mathscr{S},\iota\in\mathscr{I}\ \right\}.$$

$$\Upsilon=\left\{\ t\in[0,1]:\mathscr{H}_c(\varsigma,\mathfrak{e},t)=\varsigma,\mathscr{H}_c(\mathfrak{e},\varsigma,t)=\mathfrak{e}\text{ for some }\mathfrak{e}\in\mathscr{G},\varsigma\in\mathscr{L}\ \right\}.$$

Suppose that $\mathscr{H}_c(.,0)$ has a CFP in $((\mathscr{S}\cup\mathscr{G})\times(\mathscr{I}\cup\mathscr{L}))\cup((\mathscr{I}\cup\mathscr{L})\times(\mathscr{S}\cup\mathscr{G}))$, we have that $(0,0)\in(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$. Now we show that $(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$ is both closed and open in $[0,1]$ and hence by the connectedness $\Theta=\Upsilon=[0,1]$. As a result, $\mathscr{H}_c(.,1)$ has a CFP in $(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$. First we show that $(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$ closed in $[0,1]$. To see this, Let $(\{a_p\}_{p=1}^{\infty},\{x_p\}_{p=1}^{\infty})\subseteq(\Theta,\Upsilon)$ and $(\{y_p\}_{p=1}^{\infty},\{b_p\}_{p=1}^{\infty})\subseteq(\Upsilon,\Theta)$ with $(a_p,x_p)\to(\alpha,\alpha)$, $(y_p,b_p)\to(\alpha,\alpha)\in[0,1]$ as $p\to\infty$. We must show that $(\alpha,\alpha)\in(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$. Since $(a_p,x_p)\in(\Theta,\Upsilon),(y_p,b_p)\in(\Upsilon,\Theta)$ for $p=0,1,2,3,\cdots$, there exists sequences $\left(\{\mathfrak{x}_p\},\{\iota_p\}\right)$ and $\left(\{\mathfrak{e}_p\},\{\varsigma_p\}\right)$ with $\mathfrak{x}_p=\mathscr{H}_c(\mathfrak{x}_p,\iota_p,a_p)$, $\iota_p=\mathscr{H}_c(\iota_p,\mathfrak{x}_p,x_p)$ and $\mathfrak{e}_p=\mathscr{H}_c(\mathfrak{e}_p,\varsigma_p,y_p)$, $\varsigma_p=\mathscr{H}_c(\varsigma_p,\mathfrak{e}_p,b_p)$.

Consider

$$\begin{aligned}\rho_c(\mathfrak{x}_p,\varsigma_{p+1},c)&=\rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\iota_p,a_p),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},b_{p+1}),c\right)\\&\leq\rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\iota_p,a_p),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},a_p),c\right)\\&\quad+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p+1},\iota_{p+1},a_{p+1}),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},a_p),c\right)\end{aligned}$$

$$
\begin{aligned}
&+\rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p+1},\iota_{p+1},a_{p+1}),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},b_{p+1}),c\right)\\
\leq\ &\rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\iota_p,a_p),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},a_p),c\right)\\
&+Mc|a_{p+1}-a_p|+Mc|a_{p+1}-b_{p+1}|.
\end{aligned}
$$

Letting $p\to\infty$ and using properties of $\mathscr{A}$, we get

$$
\begin{aligned}
&\lim_{p\to\infty}\mathscr{A}\left(\rho_c(\mathfrak{x}_p,\varsigma_{p+1},c)\right)\leq\lim_{p\to\infty}\mathscr{A}\left(\rho_c\left(\mathscr{H}_c(\mathfrak{x}_p,\iota_p,a_p),\mathscr{H}_c(\varsigma_{p+1},\mathfrak{e}_{p+1},a_p),c\right)\right)\\
\leq\ &\lim_{p\to\infty}\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right)-\lim_{p\to\infty}\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right).
\end{aligned}
$$

Similarly, we can prove

$$
\begin{aligned}
\lim_{p\to\infty}\mathscr{A}\left(\rho_c(\mathfrak{e}_{p+1},\iota_p,c)\right)\ \leq\ &\lim_{p\to\infty}\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right)\\
&-\lim_{p\to\infty}\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{p\to\infty}\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right)\ \leq\ &\lim_{p\to\infty}\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right)\\
&-\lim_{p\to\infty}\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right).
\end{aligned}
$$

By the definition of $\mathscr{B}$, it follows that $\lim_{p\to\infty}\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right),\\ \rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)\end{array}\right\}\right)=0$.

So that $\lim_{p\to\infty}\rho_c\left(\mathfrak{x}_p,\varsigma_{p+1},c\right)=0$ and $\lim_{p\to\infty}\rho_c\left(\mathfrak{e}_{p+1},\iota_p,c\right)=0$.

We will prove $\left(\{\mathfrak{x}_p\},\{\iota_p\}\right)$ and $\left(\{\mathfrak{e}_p\},\{\varsigma_p\}\right)$ are a CBS. Assume there are $\varepsilon>0$ and $\{q_k\},\{p_k\}$ so that for $p_k>q_k>k$,

$$
\rho_c(\mathfrak{x}_{p_k},\varsigma_{q_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{x}_{p_{k-1}},\varsigma_{q_k},c)<\varepsilon,\rho_c(\mathfrak{e}_{p_k},\iota_{q_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{e}_{p_{k-1}},\iota_{q_k},c)<\varepsilon \tag{12}
$$

and

$$
\rho_c(\mathfrak{x}_{q_k},\varsigma_{p_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{x}_{q_k},\varsigma_{p_{k-1}},c)<\varepsilon,\rho_c(\mathfrak{e}_{q_k},\iota_{p_k},c)\geq\varepsilon,\ \ \rho_c(\mathfrak{e}_{q_k},\iota_{p_{k-1}},c)<\varepsilon \tag{13}
$$

By view of (12) and triangle inequality, we get

$$\begin{aligned}
\varepsilon &\leq \rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c) \\
&\leq \rho_c(\mathfrak{x}_{p_k}, \varsigma_{p_{k-1}}, c) + \rho_c(\mathfrak{x}_{p_{k-1}}, \varsigma_{p_{k-1}}, c) + \rho_c(\mathfrak{x}_{p_{k-1}}, \varsigma_{q_k}, c) \\
&< \rho_c(\mathfrak{x}_{p_k}, \varsigma_{p_{k-1}}, c) + \rho_c\left(\mathscr{H}_c(\mathfrak{x}_{p_{k-1}}, \iota_{p_{k-1}}, a_{p_{k-1}}), \mathscr{H}_c(\varsigma_{p_{k-1}}, \mathfrak{e}_{p_{k-1}}, b_{p_{k-1}}), c\right) + \varepsilon \\
&< \rho_c(\mathfrak{x}_{p_k}, \varsigma_{p_{k-1}}, c) + Mc|a_{p_{k-1}} - b_{p_{k-1}}| + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c) = \varepsilon \tag{14}$$

Using (13), one can prove

$$\lim_{k\to\infty} \rho_c(\mathfrak{x}_{q_k}, \varsigma_{p_k}, c) = \varepsilon \tag{15}$$

Similarly, we can prove

$$\lim_{k\to\infty} \rho_c(\mathfrak{e}_{p_k}, \iota_{q_k}, c) = \varepsilon, \ \lim_{k\to\infty} \rho_c(\mathfrak{e}_{q_k}, \iota_{p_k}, c) = \varepsilon.$$

For all $k \in \mathbb{N}$, by $(\star_2)$ we have

$$\begin{aligned}
\lim_{k\to\infty} \mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c), \\ \rho_c\left(\mathfrak{e}_{p_k}, \iota_{q_k}, c\right)\end{array}\right\}\right) &\leq \lim_{k\to\infty} \mathscr{A}\left(max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c), \\ \rho_c\left(\mathfrak{e}_{p_k}, \iota_{q_k}, c\right)\end{array}\right\}\right) \\
&\quad - \lim_{k\to\infty} \mathscr{B}\left(max\left\{\begin{array}{c}\rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c), \\ \rho_c\left(\mathfrak{e}_{p_k}, \iota_{q_k}, c\right)\end{array}\right\}\right).
\end{aligned}$$

By the definition of $\mathscr{B}$, it follows that $\mathscr{B}\left(max\left\{\begin{array}{c}\lim_{k\to\infty} \rho_c(\mathfrak{x}_{p_k}, \varsigma_{q_k}, c), \\ \lim_{k\to\infty} \rho_c\left(\mathfrak{e}_{p_k}, \iota_{q_k}, c\right)\end{array}\right\}\right) = 0$ implies that $\mathscr{B}(\varepsilon) = 0$ then $\varepsilon = 0$ which is contradictory, by applying (14) and (15).

Therefore, $\left(\{\mathfrak{x}_p\}, \{\iota_p\}\right) \subseteq (\mathscr{S}, \mathscr{I})$ and $\left(\{\mathfrak{e}_p\}, \{\varsigma_p\}\right) \subseteq (\mathscr{G}, \mathscr{L})$ are CBS. By completeness, there exist $(a, x) \in \mathscr{S} \times \mathscr{I}$ and $(y, b) \in \mathscr{G} \times \mathscr{L}$ with

$$\lim_{p\to\infty} \mathfrak{x}_{p+1} = x \quad \lim_{p\to\infty} \iota_{p+1} = a \quad \lim_{p\to\infty} \mathfrak{e}_{p+1} = b \quad \lim_{p\to\infty} \varsigma_{p+1} = y$$

we have

$$\begin{aligned}\rho_c(\mathscr{H}_c(y,b,\alpha),x,c) &\leq \rho_c(\mathscr{H}_c(y,b,\alpha),\varsigma_p,c)+\rho_c(\mathfrak{x}_p,\varsigma_p,c)+\rho_c(\mathfrak{x}_p,x,c)\\ &\leq \rho_c(\mathscr{H}_c(y,b,\alpha),\mathscr{H}_c(\varsigma_p,\mathfrak{e}_p,b_p),c)+Mc|a_p-b_p|+\rho_c(\mathfrak{x}_{p+},x,c).\end{aligned}$$

Letting $p\to\infty$ in the above inequality and using conditions of $\mathscr{A}$, we have

$$\begin{aligned}&\mathscr{A}(\rho_c(\mathscr{H}_c(y,b,\alpha),x,c))\leq \lim_{k\to\infty}\mathscr{A}(\rho_c(\mathscr{H}_c(y,b,\alpha),\mathscr{H}_c(\varsigma_p,\mathfrak{e}_p,\alpha),c))\\ \leq\;& \lim_{k\to\infty}\mathscr{A}\left(\max\left\{\begin{array}{c}\rho_c(y,\varsigma_p,c),\\ \rho_c(\mathfrak{e}_p,b,c)\end{array}\right\}\right)-\lim_{k\to\infty}\mathscr{B}\left(\max\left\{\begin{array}{c}\rho_c(y,\varsigma_p,c),\\ \rho_c(\mathfrak{e}_p,b,c)\end{array}\right\}\right)\\ \leq\;& \mathscr{A}(0)-\mathscr{B}(0)=0\end{aligned}$$

it follows that $\mathscr{A}(\rho_c(\mathscr{H}_c(y,b,\alpha),x,c))=0$ implies that $\mathscr{H}_c(y,b,\alpha)=x$. Similarly, we can prove that $\mathscr{H}_c(b,y,\alpha)=a$ and $\mathscr{H}_c(x,a,\alpha)=y$, $\mathscr{H}_c(a,x,\alpha)=b$.

On the other hand,

$$\rho_c(y,x,c)=\rho_c\left(\lim_{p\to\infty}\varsigma_p,\lim_{p\to\infty}\mathfrak{x}_p,c\right)=\lim_{p\to\infty}\rho_c(\mathfrak{x}_p,\varsigma_p,c)=0$$

and

$$\rho_c(a,b,c)=\rho_c\left(\lim_{p\to\infty}\iota_p,\lim_{p\to\infty}\mathfrak{e}_p,c\right)=\lim_{p\to\infty}\rho_c(\mathfrak{e}_p,\iota_p,c)=0$$

Therefore, $x=y$, $a=b$ and consequently, $a=x$. Hence $(\alpha,\alpha)\in(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$.

Clearly $(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$ is closed in $[0,1]$. Let $(\alpha_0,\beta_0)\in\Theta\times\Upsilon$, there exists bisequences $(\mathfrak{x}_0,\iota_0)$ and $(\mathfrak{e}_0,\varsigma_0)$ with $\mathfrak{x}_0=\mathscr{H}_c(\mathfrak{x}_0,\iota_0,\alpha_0)$, $\iota_0=\mathscr{H}_c(\iota_0,\mathfrak{x}_0,\beta_0)$ and
$\mathfrak{e}_0=\mathscr{H}_c(\mathfrak{e}_0,\varsigma_0,\alpha_0)$, $\varsigma_0=\mathscr{H}_c(\varsigma_0,\mathfrak{e}_0,\beta_0)$. Since $((\mathscr{S}\cup\mathscr{G})\times(\mathscr{I}\cup\mathscr{L}))\cup((\mathscr{I}\cup\mathscr{L})\times(\mathscr{S}\cup\mathscr{G}))$ is open, then there exist $\delta>0$ such that $B_{\rho_c}(\mathfrak{x}_0,\delta)$, $B_{\rho_c}(\mathfrak{e}_0,\delta)$, $B_{\rho_c}(\iota_0,\delta)$ and
$B_{\rho_c}(\varsigma_0,\delta)\subseteq((\mathscr{S}\cup\mathscr{G})\times(\mathscr{I}\cup\mathscr{L}))\cup((\mathscr{I}\cup\mathscr{L})\times(\mathscr{S}\cup\mathscr{G}))$.
Choose $\alpha\in(\alpha_0-\varepsilon,\alpha_0+\varepsilon)$ such that $|\alpha-\alpha_0|\leq\frac{1}{M^p}<\frac{\varepsilon}{2}$, $|\alpha_0-\beta_0|\leq\frac{1}{M^p}<\frac{\varepsilon}{2}$.
Then for, $\varsigma\in\overline{B}_{\pounds\cup\mathfrak{I}}(\mathfrak{x}_0,\delta)=\{\varsigma,\varsigma_0\in\mathfrak{I}/\rho_c(\mathfrak{x}_0,\varsigma,c)\leq\rho_c(\mathfrak{x}_0,\varsigma_0,c)+\delta\}$,
$\iota\in\overline{B}_{\pounds\cup\mathfrak{I}}(\delta,\mathfrak{e}_0)=\{\iota,\iota_0\in\mathfrak{I}/\rho_c(\mathfrak{e}_0,\iota,c)\leq\rho_c(\mathfrak{e}_0,\iota_0,c)+\delta\}$
$\mathfrak{x}\in\overline{B}_{\pounds\cup\mathfrak{I}}(\delta,\varsigma_0)=\{\mathfrak{x},\mathfrak{x}_0\in\pounds/\rho_c(\mathfrak{x},\varsigma_0,c)\leq\rho_c(\mathfrak{x}_0,\varsigma_0,c)+\delta\}$

$\mathfrak{e} \in \overline{B}_{\pounds\cup\mathfrak{I}}(\imath_0, \delta) = \{\mathfrak{e}, \mathfrak{e}_0 \in \pounds / \rho_c(\mathfrak{e}, \imath_0, c) \leq \mathfrak{e}(\mathfrak{e}_0, \imath_0, c) + \delta\}$.

Now consider

$$\begin{aligned}
\rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \varsigma_0, c\right) &= \rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \mathscr{H}_c(\varsigma_0, \mathfrak{e}_0, \beta_0), c\right) \\
&\leq \rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \mathscr{H}_c(\varsigma, \mathfrak{e}, \alpha_0), c\right) + \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma, \mathfrak{e}, \alpha_0), c\right) \\
&\qquad \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma_0, \mathfrak{e}_0, \beta_0), c\right) \\
&\leq Mc|\alpha - \alpha_0| + \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma, \mathfrak{e}, \alpha_0), c\right) + Mc|\alpha_0 - \beta_0| \\
&\leq \frac{2c}{M^{p-1}} + \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma, \mathfrak{e}, \alpha_0), c\right)
\end{aligned}$$

Letting $p \to \infty$ and using $\mathscr{A}$ property, then we have

$$\begin{aligned}
\mathscr{A}\left(\rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \varsigma_0, c\right)\right) &\leq \mathscr{A}\left(\rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma, \mathfrak{e}, \alpha_0), c\right)\right) \\
&\leq \mathscr{A}\left(\max\left\{\begin{array}{c} \rho_c(\mathfrak{x}_0, \varsigma, c), \\ \rho_c(\mathfrak{e}, \imath_0, c) \end{array}\right\}\right) - \mathscr{B}\left(\max\left\{\begin{array}{c} \rho_c(\mathfrak{x}_0, \varsigma, c), \\ \rho_c(\mathfrak{e}, \imath_0, c) \end{array}\right\}\right) \\
&\leq \mathscr{A}\left(\max\left\{\rho_c(\mathfrak{x}_0, \varsigma, c), \rho_c(\mathfrak{e}, \imath_0, c)\right\}\right)
\end{aligned}$$

It follows that

$$\rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \varsigma_0, c\right) \leq \max\left\{\rho_c(\mathfrak{x}_0, \varsigma, c), \rho_c(\mathfrak{e}, \imath_0, c)\right\}$$

Similarly we can prove

$$\rho_c\left(\mathfrak{x}_0, \mathscr{H}_c(\varsigma, \mathfrak{e}, \beta), c\right) \leq \max\left\{\rho_c(\mathfrak{x}_0, \varsigma, c), \rho_c(\mathfrak{e}, \imath_0, c)\right\}.$$

Therefore,

$$\max\left\{\begin{array}{c} \rho_c\left(\mathscr{H}_c(\mathfrak{x}, \imath, \alpha), \varsigma_0, c\right), \\ \rho_c\left(\mathfrak{x}_0, \mathscr{H}_c(\varsigma, \mathfrak{e}, \beta), c\right) \end{array}\right\} \leq \max\left\{\begin{array}{c} \rho_c(\mathfrak{x}_0, \varsigma, c), \\ \rho_c(\mathfrak{e}, \imath_0, c) \end{array}\right\} \leq \max\left\{\begin{array}{c} \rho_c(\mathfrak{x}_0, \varsigma_0, c) + \delta, \\ \rho_c(\mathfrak{e}_0, \imath_0, c) + \delta \end{array}\right\}$$

Similarly, we have

$$\max\left\{\rho_c\left(\mathscr{H}_c(\mathfrak{e}, \varsigma, \alpha), \imath_0, c\right), \rho_c\left(\mathfrak{e}_0, \mathscr{H}_c(\imath, \mathfrak{x}, \beta), c\right)\right\} \leq \max\left\{\rho_c(\mathfrak{x}_0, \varsigma_0, c) + \delta, \rho_c(\mathfrak{e}_0, \imath_0, c) + \delta\right\}$$

On the other hand,

$$\rho_c(\mathfrak{x}_0, \varsigma_0, c) = \rho_c\left(\mathscr{H}_c(\mathfrak{x}_0, \imath_0, \alpha_0), \mathscr{H}_c(\varsigma_0, \mathfrak{e}_0, \beta_0)\right) \leq Mc|\alpha_0 - \beta_0| < \frac{c}{M^{p-1}} \to 0 \text{ as } p \to \infty$$

$$\rho_c(\mathfrak{e}_0, \imath_0, c) = \rho_c\left(\mathscr{H}_c(\mathfrak{e}_0, \varsigma_0, \alpha_0), \mathscr{H}_c(\imath_0, \mathfrak{x}_0, \beta_0)\right) \leq Mc|\alpha_0 - \beta_0| < \frac{c}{M^{p-1}} \to 0 \text{ as } p \to \infty.$$

So $\mathfrak{x}_0 = \varsigma_0$ and $\imath_0 = \mathfrak{e}_0$ and hence $\alpha = \beta$. Thus for each fixed $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, $\mathscr{H}_c(.,\alpha) : \overline{B}_{\Theta\cup\Upsilon}(\mathfrak{x}_0,\delta) \to \overline{B}_{\Theta\cup\Upsilon}(\mathfrak{x}_0,\delta)$ and $\mathscr{H}_c(.,\alpha) : \overline{B}_{\Theta\cup\Upsilon}(\imath_0,\delta) \to \overline{B}_{\Theta\cup\Upsilon}(\imath_0,\delta)$. Thus, we conclude that $\mathscr{H}_c(.,\alpha)$ has a coupled fixed point in $\big((\overline{\mathscr{S}}\cup\overline{\mathscr{G}})\times(\overline{\mathscr{I}}\cup\overline{\mathscr{L}})\big)\cup\big((\overline{\mathscr{I}}\cup\overline{\mathscr{L}})\times(\overline{\mathscr{S}}\cup\overline{\mathscr{G}})\big)$. But this must be in $((\mathscr{S}\cup\mathscr{G})\times(\mathscr{I}\cup\mathscr{L}))\cup((\mathscr{I}\cup\mathscr{L})\times(\mathscr{S}\cup\mathscr{G}))$. Therefore, $(\alpha,\alpha) \in (\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$ for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$.

Hence, $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \subseteq (\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$. Clearly, $(\Theta\times\Upsilon)\cap(\Upsilon\times\Theta)$ is open in $[0,1]$. For the reverse implication, we use the same strategy.

5. Conclusions

This paper utilizes covariant mappings of $(\mathscr{A},\mathscr{B})$-contraction type $\mathbb{T}$-coupling functions to establish certain strong common fixed point theorems (SCFPT) within the framework of complete bipolar parametric metric spaces (BPPMS). It provides pertinent examples to highlight the primary findings. Additionally, applications to integral equations and homotopy are presented.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] S. Banach, Sur Les Operations Dans Les Ensembles Abstraits et Leur Application Aux Equations Integrales, Fund. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[2] A. Mutlu, U. Gurdal, Bipolar Metric Spaces and Some Fixed Point Theorems, J. Nonlinear Sci. Appl. 09 (2016), 5362–5373. https://doi.org/10.22436/jnsa.009.09.05.

[3] A. Mutlu, K. *Ö*zkan, U. G*ü*rdal, Coupled Fixed Point Theorems on Bipolar Metric Spaces, Eur. J. Pure Appl. Math. 10 (2017), 655–667.

[4] S.H. Cho, Fixed Point Theorem for (α,β)-Z-Contractions in Metric Spaces, Int. J. Math. Ana. 13 (2019), 161–174.

[5] M. Simkhah, D. Turkoglu, S. Sedghi, N. Shobe, Suzuki Type Fixed Point Results in P-Metric Spaces, Commun. Optim. Theory, 2019 (2019), 13.

[6] T.G. Bhaskar, V. Lakshmikantham, Fixed Point Theorems in Partially Ordered Metric Spaces and Applications, Nonlinear Anal.: Theory Methods Appl. 65 (2006), 1379–1393. https://doi.org/10.1016/j.na.2005.10.017.

[7] V. Lakshmikantham, L. Ciric, Coupled Fixed Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces, Nonlinear Anal.: Theory Methods Appl. 70 (2009), 4341–4349. https://doi.org/10.1016/j.na.2008.09.020.

[8] B.S. Choudhury, P. Maity, Cyclic Coupled Fixed Point Result Using Kannan Type Contractions, J. Oper. 2014 (2014), 876749. https://doi.org/10.1155/2014/876749.

[9] B.S. Choudhury, P. Maity, P. Konar, Fixed Point Results for Couplings on Metric Spaces, U.P.B. Sci. Bull., Ser. A, 79 (2017), 77–88.

[10] H. Aydi, M. Barakat, A. Felhi, H. Işik, On ϕ-Contraction Type Couplings in Partial Metric Spaces, J. Math. Anal. 8 (2017), 78–89.

[11] T. Rashid, Q.H. Khan, Coupled Coincidence Point of ϕ-Contraction Type T-Coupling and (ψ,ϕ)-Contraction Type Coupling in Metric Spaces, arXiv:1710.10054 [math.FA] (2017). https://doi.org/10.48550/arXiv.1710.10054.

[12] D. Eshi, B. Hazarika, N. Saikia, S. Aljohani, S.K. Panda, A. Aloqaily, N. Mlaiki, Some Strong Coupled Coincidence and Coupled Best Proximity Results on Complete Metric Spaces, J. Inequal. Appl. 2024 (2024), 151. https://doi.org/10.1186/s13660-024-03234-x.

[13] S.M. Anushia, S.N. Leena Nelson, Fixed Points From Kannan Type S-Coupled Cyclic Mapping In Complete S-Metric Space, Nanotechnol. Percept. 20 (2024), 335–356. https://doi.org/10.62441/nano-ntp.vi.2779.

[14] G.V.R. Babu, P.D. Sailaja, G. Srichandana, Strong Coupled Fixed Points of Chatterjea Type (ϕ,ψ)-Weakly Cyclic Coupled Mappings in S-Metric Spaces, Proc. Int. Math. Sci. 2 (2020), 60–78.

[15] F. Abdulkerim, K. Koyas, S. Gebregiorgis, Coupled Coincidence and Coupled Common Fixed Points of (ϕ,ψ)-Contraction Type T-Coupling in Metric Spaces, Iran. J. Math. Sci. Inform. 19 (2024), 61–75.

[16] N. Hussain, S. Khaleghizadeh, P. Salimi, A.A.N. Abdou, A New Approach to Fixed Point Results in Triangular Intuitionistic Fuzzy Metric Spaces, Abstr. Appl. Anal. 2014 (2014), 690139. https://doi.org/10.1155/2014/690139.

[17] N. Hussain, P. Salimi, V. Parvaneh, Fixed Point Results for Various Contractions in Parametric and Fuzzy b-Metric Spaces, J. Nonlinear Sci. Appl. 8 (2015), 719–739.

[18] M. Kumar, O. Ege, V. Mor, P. Kumar, M. De La Sen, Boyd-Wong Type Contractions in Generalized Parametric Bipolar Metric Space, Heliyon 10 (2024), e23998. https://doi.org/10.1016/j.heliyon.2024.e23998.

[19] M.I. Pasha, K.R.K. Rao, G. Mani, et al. Solving a Fractional Differential Equation via the Bipolar Parametric Metric Space, J. Math. 2024 (2024), 5533347. https://doi.org/10.1155/2024/5533347.

[20] M.S. Khan, M. Swaleh, S. Sessa, Fixed Point Theorems by Altering Distances between the Points, Bull. Aust. Math. Soc. 30 (1984), 1–9. https://doi.org/10.1017/S0004972700001659.