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CONVERGENCE RESULTS OF MODIFIED ISHIKAWA ITERATES FOR ASYMPTOTICALLY NONEXPANSIVE NONSELF MAPS (ANNMS) IN CAT(0) SPACES WITH NEW CONTROL CONDITIONS

P. JOYAL ROY*, A. ANTHONY ELDRED

Department of Mathematics, St. Joseph's College (Autonomous) (Affiliated to Bharathidasan University),

Trichy-620 002, Tamil Nadu, India

Abstract. We establish the convergence of modified Ishikawa iteration to a fixed point of asymptotically nonexpansive nonself-mapping on a nonempty convex closed subset with relaxed conditions on the control sequences that generate the iterative process.

Keywords: asymptotically nonexpansive nonself-mapping; CAT(0) spaces; modified Ishikawa iterative process.

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1. Introduction

One of the main features in recent developments of fixed point theory is the treatment of computational aspects of convergence of fixed points of nonexpansive and asymptotically nonexpansive mappings in the frame work of CAT(0) spaces. In 1965 D. Gohde and Kirk showed that a nonexpansive self mappings of closed and bounded convex subset of a uniformly convex Banach spaces has a fixed point.

*Corresponding author

E-mail address: joyinyou11@gmail.com

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Approximating fixed point theorems of the convergence of some sequences defined by iteration techniques for general nonexpansive mappings was shown by Browder and Pertyshin (1966-1967)[18]. The most important generalization of the class of nonexpansive mappings is the class of asymptotically nonexpansive mappings established by Goebel and Kirk [12]. In uniformly convex Banach spaces there are several papers in terms of approximating fixed points of aysmptotically nonexpansive on closed bounded convex subsets using Modified Mann and Ishikawa iterative procedures. (see [3, 7, 8, 9, 11, 12, 15, 16, 17, 19, 20, 22])

A nonempty closed convex subset $\mathbb{E}$ of Banach space $\mathbb{X}$ is said to be a retract of $\mathbb{X}$ if there exists a continuous map $\tilde{\mathbb{P}} : \mathbb{X} \to \mathbb{E}$ such that $\tilde{\mathbb{P}}x = x$ for all $x \in \mathbb{E}$. All convex closed subsets of a uniformly convex Banach space are retracts. A retraction $\tilde{\mathbb{P}} : \mathbb{X} \to \mathbb{E}$ is nonexpansive if

$$\|\tilde{\mathbb{P}}x - \tilde{\mathbb{P}}y\| \leq \|x - y\|, \forall\, x, y \in \mathbb{X}. \tag{1}$$

Chidume et al.[9] defined a nonself asymptotically nonexpansive mappings as: Let $\tilde{\mathbb{P}} : \mathbb{X} \to \mathbb{E}$ be a nonexpansive retraction of $\mathbb{X}$ into $\mathbb{E}$. $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ a nonself map is asymtotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ satisfying

$$\left\|\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x - \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y\right\| \leq k_n \, \|x - y\|, \forall\, x, y \in \mathbb{X},\ n \geq 1 \tag{2}$$

For some $\mathbb{L} > 0$ if

$$\left\|\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x - \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y\right\| \leq \mathbb{L} \, \|x - y\|, \forall\, x, y \in \mathbb{X},\ n \geq 1 \tag{3}$$

$\mathbb{T}$ is uniformly $\mathbb{L}$-Lipschitzian.

Let $\mathbb{E}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathbb{X}$. For $x_1 \in \mathbb{E}$, we have the Ishikawa iteration process as

$$\begin{cases} x_{n+1} = \tilde{\mathbb{P}}\left(\alpha_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n + (1 - \alpha_n) x_n\right) \\ y_n = \tilde{\mathbb{P}}\left(\beta_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n + (1 - \beta_n) x_n\right) \end{cases} \tag{4}$$

A metric space $\mathbb{X}$ is called a CAT(0) if it is geodesically connected and if every geodesic triangle in $\mathbb{X}$ is at least as thin as its comparison triangle in the Euclidean plane. Let $(\mathbb{X}, d)$ be a metric space. A geodesic path joining $x \in \mathbb{X}$ to $y \in \mathbb{Y}$ is a mapping from $[0, l]$ in $\mathbb{R}$ to $\mathbb{X}$ such that $c(0) = x, c(l) = y$ and $d\left(c(t), c\left(t'\right)\right) = \mid t - t' \mid$ for all $t, t' \in [0, l]$. Specifically c is isometry and $d(x,y)=l$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. The space $(\mathbb{X},d)$ is called a geodesic space if every two points of $\mathbb{X}$ are joined by geodesic and $\mathbb{X}$ is said to be uniquely geodesic if there is exactly one geodesic joining x and y for all $x,y\in\mathbb{X}$. A subset $\mathbb{Y}$ of $\mathbb{X}$ is convex if $\mathbb{Y}$ includes every geodsic segment joining any two of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ is a geodesic metric space $(\mathbb{X},d)$ consisting of three points $x_1,x_2,x_3\in\mathbb{X}$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in $(\mathbb{X},d)$ is a triangle $\triangle(x_1,x_2,x_3):=\triangle(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Eucldean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}\left(\bar{x_i},\bar{x_j}\right)=d\left(x_i,x_j\right)$ for $i,j\in\{1,2,3\}$. A geodesic space is said to be a CAT(0) space if all the geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in $\mathbb{X}$ and $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y\in\triangle$ and all comparison points $\bar{x},\bar{y}\in\bar{\triangle}$, $d(x,y)=d_{\mathbb{E}^2}(\bar{x},\bar{y})$[10].

The existence of fixed points of nonexpansive mappings in CAT(0) spaces was proved by Kirk. Using Mann and Ishikawa iterations Dhompongsa and Panyanak established some results for single map and Abbas and Khan has done a study on common fixed points approximation for two mappings by Ishikawa-type iteration. (See[1, 2, 12, 10, 14]).The classical hyperbolic spaces, Euclidean buildings (see [6]) and the complex Hilbert ball with a hyperbolic metric (see [13]) are CAT(0) spaces.

Lemma 1.1. [10] Let $(\mathbb{X},d)$ be a CAT(0) space. Then

(i) $(\mathbb{X},d)$ is uniquely geodesic.

(ii) Let p,x,y be points of $\mathbb{X}$, and $\alpha\in[0,1]$ and let m_1 and m_2 respectively denote the points of $[p,x]$ and $[p,y]$ satisfying $d(p,m_1)=\alpha\, d(p,x)$ and $d(p,m_2)=\alpha\, d(p,y)$. Then $d(m_1,m_2)\le\alpha\, d(x,y)$.

(iii) Let $x,y\in\mathbb{X},x\neq y$ and $z,w\in[x,y]$ such that $d(x,z)=d(x,w)$. Then $z=w$.

(iv) Let $x,y\in\mathbb{X}$. For each $t\in[0,1]$, there exists a unique point $Z\in[x,y]$ such that

$$d(x,z)=t\,d(x,y)\,\text{and}\;d(y,z)=(1-t)\,d(x,y)\,. \tag{5}$$

For convenience from now on we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (5).

Let $\mathbb{E}$ be a nonempty closed convex subset of CAT(0) space $\mathbb{X}$. For $x_1 \in \mathbb{E}$, we have Ishikawa iteration process for a fixed point where $\{\alpha_n\}$ and $\{\beta_n\} \in [0,1]$

$$\begin{cases} x_{n+1} = \alpha_n \mathbb{T}^n y_n \oplus (1-\alpha_n) x_n \\ y_n = \beta_n \mathbb{T}^n x_n \oplus (1-\beta_n) x_n \end{cases} \tag{6}$$

Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ be a ANNM where $\mathbb{E}$ is a nonempty convex subset of a CAT(0) space $\mathbb{X}$ and $\{\alpha_n\}, \{\beta_n\} \in [0,1]$. Define sequence $\{x_n\}_{n=1}^{\infty}$ as

$$\begin{cases} x_0 \in \mathbb{E} \\ x_{n+1} = \tilde{\mathbb{P}} \left(\alpha_n \mathbb{T} \left(\tilde{\mathbb{P}} \mathbb{T} \right)^{n-1} y_n \oplus (1-\alpha_n) x_n \right) \\ y_n = \tilde{\mathbb{P}} \left(\beta_n \mathbb{T} \left(\tilde{\mathbb{P}} \mathbb{T} \right)^{n-1} x_n \oplus (1-\beta_n) x_n \right) \end{cases} \tag{7}$$

is called modified Ishikawa iterative process.

Control sequence range affects substantially on convergence of iteration process $\{\alpha_n\}$ and $\{\beta_n\}$. The uniform convexity of the Banach space requires $\{\alpha_n\}$ and $\{\beta_n\}$ must be contained in $(\varepsilon, 1-\varepsilon)$ where $\varepsilon \in [0, 1/2)$. Since a nonlinear version of Hilbert Banach spaces is a CAT(0) space, we show that it is possible to relax these conditions and still discuss convergence of modified Ishikawa and Mann iterations ensuring a faster rate of convergence.

We establish that if $\mathbb{E}$ is a nonempty covex closed bounded subset of a CAT(0) space $\mathbb{X}$, $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ is a ANNM and a sequence defined as Mann iterative procedure

$$x_{n+1} = \frac{1}{n} \mathbb{T}^n x_n \oplus \left(1 - \frac{1}{n}\right) x_n$$

or the Ishikawa iteration

$$x_{n+1} = \left(1 - \frac{1}{n}\right) \mathbb{T}^n y_n \oplus \frac{1}{n} x_n$$

$$y_n = \frac{1}{n} \mathbb{T}^n x_n \oplus \left(1 - \frac{1}{n}\right) x_n$$

then $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

2. PRELIMINARIES

Definition 2.1. [2] Let $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ be a mapping where $\mathbb{E}$ is a nonempty subset of a metric space $(\mathbb{X},d)$. A sequence $\{x_n\}$ in $\mathbb{E}$ is called approximating fixed point sequence of $\mathbb{T}$ if $\lim_{n\to\infty} d(x_n,\mathbb{T}x_n)=0$.

Definition 2.2. [2] Let $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ be a mapping where $\mathbb{E}$ is a nonempty subset of a metric space $(\mathbb{X},d)$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty} k_n=1$ satisfying

$$d(\mathbb{T}^n x_n,\mathbb{T}^n y_n)\le k_n d(x_n,y_n),\forall x,y\in\mathbb{E},n\in\mathbb{N} \tag{8}$$

Note 2.3. [2] In a CAT(0) space $(\mathbb{X},d)$ we observe that if $x,y,z\in\mathbb{X}$ then CAT(0) inequality gives

$$d^2\left(x,\frac{y\oplus z}{2}\right)\le\frac{1}{2}d^2(x,y)+\frac{1}{2}d^2(x,z)-\frac{1}{4}d^2(y,z) \tag{CN}$$

is called CN inequality due to Bruhat and Tits [4].

Lemma 2.4. [10] Let $\mathbb{X}$ be a CAT(0) space. Then for all $x,y,z\in\mathbb{X}$ and $t\in[0,1]$ $d((1-t)x\oplus ty,z)\le(1-t)d(x,z)+td(y,z)$.

Lemma 2.5. [10] Let $\mathbb{X}$ be a CAT(0) space. Then for all $x,y,z\in\mathbb{X}$ and $t\in[0,1]$ $d^2((1-t)x\oplus ty,z)\le(1-t)d^2(x,z)+td^2(y,z)-t(1-t)d^2(x,y)$.

Lemma 2.6. [22] Suppose $g:[0,\infty)\to[0,\infty)$ is a strictly increasing function with $g(0)=0$. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies $\lim_{n\to\infty} g(x_n)=0$. Then $\lim_{n\to\infty} x_n=0$.

Lemma 2.7. [19] Let $\mathbb{E}$ be a convex subset of a normed space $\mathbb{X}$. Let $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ be uniformly $\mathbb{L}$ lipschitzian and $\{\alpha_n\}$ and $\{\beta_n\}\in[0,1]$. Suppose $\{x_n\}$ is defined as in equation (6) and set $c_n=\|\mathbb{T}^n(x_n)-x_n\|,\ \forall n\in\mathbb{N}$. Then $\|x_n-\mathbb{T}(x_n)\|\le c_n+c_{n-1}L\left(1+3L+2L^2\right),\ \forall n\in\mathbb{N}$.

Lemma 2.8. [8] Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences with $\sum_{n=1}^{\infty} b_n<\infty$. Then $\lim_{n\to\infty} a_n$ exists if one of the conditions is satisfied:

$$a_n+b_n\ge a_{n+1},\quad n\ge1$$

$$a_n(1+b_n)\ge a_{n+1},\quad n\ge1$$

Lemma 2.9. [11] Let positive real sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy:

(i) a_n is decreasing,

(ii) $\sum a_n = \infty$,

(iii) $\sum a_n b_n < \infty$.

Then there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that sequence $\{b_{n_k},\ b_{n_k+1}\} = b_{n_1}, b_{n_1+1}, b_{n_2}, b_{n_2+1}, \ldots, b_{n_k}, b_{n_k+1}, \ldots$ converges to zero.

3. Main Results

Lemma 3.1. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ be a ANNMs with at least one fixed point and sequence $\{k_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $\mathbb{E}$ is a nonempty closed convex subset of a CAT(0) space $\mathbb{X}$ and $\tilde{\mathbb{P}} : \mathbb{X} \to \mathbb{E}$ be a nonexpansive retraction of $\mathbb{X}$ into $\mathbb{E}$. Then $\lim_{n\to\infty} d(x_n,t)$ exists for every $t \in F(\mathbb{T})$ when $\{x_n\}$ is defined as in (7).

Proof. Suppose $t \in F(\mathbb{T})$. We have

$$\begin{aligned}
d(x_{n+1},t) &= d\left(\tilde{\mathbb{P}}\left(\alpha_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n \oplus (1-\alpha_n) x_n\right), t\right) \\
&\le \alpha_n d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n, t\right) + (1-\alpha_n)\, d(x_n,t) \text{ [By Lemma 2.4]} \\
&\le \alpha_n k_n d(y_n,t) + (1-\alpha_n)\, d(x_n,t) \\
&= \alpha_n k_n d\left(\tilde{\mathbb{P}}\left(\beta_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n \oplus (1-\beta_n) x_n\right), t\right) + (1-\alpha_n)\, d(x_n,t) \\
&\le \alpha_n k_n \beta_n d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n, t\right) + \alpha_n k_n\, (1-\beta_n)\, d(x_n,t) + (1-\alpha_n)\, d(x_n,t) \\
&\le \alpha_n k_n^2 \beta_n d(x_n,t) + \alpha_n k_n\, (1-\beta_n)\, d(x_n,t) + (1-\alpha_n)\, d(x_n,t) \\
&= d(x_n,t)\left[\alpha_n k_n^2 \beta_n + \alpha_n k_n - \alpha_n k_n \beta_n + 1 - \alpha_n\right] \\
&= d(x_n,t)\left[\alpha_n k_n \beta_n (k_n - 1) + \alpha_n\, (k_n - 1) + 1\right] \\
&= d(x_n,t)\left[1 + \zeta_n\right] \text{ where } \zeta_n = \alpha_n k_n \beta_n (k_n - 1) + \alpha_n\, (k_n - 1)
\end{aligned}$$

Thus

$$d(x_{n+1},t) \le d(x_n,t)\left[1 + \zeta_n\right] \tag{9}$$

Consequently,

$$d\left(x_{n+1},t\right) \leq d\left(x_1,t\right)\left[1+\zeta_1\right]\left[1+\zeta_2\right]\ldots\left[1+\zeta_n\right] \leq d\left(x_1,t\right)e^{\sum\limits_{n=1}^{\infty}\zeta_n}$$

Hence $d\left(x_n,t\right)$ is bounded. Since $\sum\limits_{n=1}^{\infty}\left(k_n-1\right)<\infty$ and applying (9) in Lemma 2.8 we obtain that $\lim\limits_{n\to\infty} d\left(x_n,t\right)$ exists. □

Lemma 3.2. Let $\mathbb{T}:\mathbb{E}\to\mathbb{X}$ be continuous and $\{x_n\}$ be defined in $\mathbb{E}$ such that $\lim\limits_{n\to\infty} d\left(x_n,t\right)$ exists for each $t\in F(\mathbb{T})$, where $\mathbb{E}$ is a nonempty closed convex subset of a CAT(0) snace $\mathbb{X}$. If there exists a convergent subsequence $\{x_{n_k}\}$ such that $d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_{n_k},x_{n_k}\right)\to 0$, then $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to some $x\in\mathbb{E}$. Since $\tilde{\mathbb{P}}$ and $\mathbb{T}$ are continuous and as $d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_{n_k},x_{n_k}\right)\to 0$ we get $\mathbb{T}x=x$. Thus $x_n\to x$ as $\lim\limits_{n\to\infty} d\left(x_n,x\right)$ exists. □

Theorem 3.3. Let $\mathbb{T}:\mathbb{E}\to\mathbb{X}$ be ANNMs with at least one fixed point and sequence $\{k_n\}\geq 1$ be decreasing satisfying $\sum\limits_{n=1}^{\infty}\left(k_n^2-1\right)<\infty$, where $\mathbb{E}$ is a nonempty convex closed subset of a CAT(0) space $\mathbb{X}$. Suppose $\{x_n\}$ is defined as in (7). Let sequences $\{\alpha_n\}$ and $\{\beta_n\}\in[0,1]$ satisfy one of the conditions:

(I) $\sum\limits_{n=1}^{\infty}\alpha_n\left(1-\alpha_n\right)=\infty$ and $\beta_n\downarrow\beta<1$

(II) $\sum\limits_{n=1}^{\infty}\beta_n\left(1-\beta_n\right)=\infty$ and $\alpha_n\downarrow\alpha>0$ and $\beta_n\downarrow\beta<1$

(III) $0\leq\alpha_n\leq b<1$, $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$ and $\beta_n\to 0$ as $n\to\infty$

Then we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_{n_k},x_{n_k}\right)\to 0$.

Proof. Assume $t\in F(\mathbb{T})$. From Lemma 2.5, we have

$$\begin{aligned}
d^2\left(x_{n+1},t\right) &= d^2\left(\tilde{\mathbb{P}}\left(\alpha_n\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\oplus\left(1-\alpha_n\right)x_n\right),t\right)\\
&\leq \alpha_n d^2\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n,t\right)+\left(1-\alpha_n\right)d^2\left(x_n,t\right)\\
&\quad -\alpha_n\left(1-\alpha_n\right)d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)\\
&\leq \alpha_n k_n^2 d^2\left(y_n,t\right)+\left(1-\alpha_n\right)d^2\left(x_n,t\right)-\alpha_n\left(1-\alpha_n\right)d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n k_n^2 d^2\left(\tilde{\mathbb{P}}\left(\beta_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n \oplus (1-\beta_n) x_n\right), t\right) + (1-\alpha_n)\, d^2(x_n,t) \\
&\quad - \alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right) \\
&\leq \alpha_n \beta_n k_n^2\, d^2\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n, t\right) + \alpha_n (1-\beta_n)\, k_n^2\, d^2(x_n,t) \\
&\quad - \alpha_n \beta_n (1-\beta_n)\, k_n^2\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right) + (1-\alpha_n)\, d^2(x_n,t) \\
&\quad - \alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right) \\
&\leq \alpha_n \beta_n k_n^4\, d^2(x_n,t) + \alpha_n (1-\beta_n)\, k_n^2\, d^2(x_n,t) \\
&\quad - \alpha_n \beta_n (1-\beta_n)\, k_n^2\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right) + (1-\alpha_n)\, d^2(x_n,t) \\
&\quad - \alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right) \\
&= \alpha_n \beta_n k_n^4\, d^2(x_n,t) + \alpha_n (1-\beta_n)\, k_n^2\, d^2(x_n,t) \\
&\quad - \alpha_n \beta_n (1-\beta_n)\, k_n^2\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right) + d^2(x_n,t) - \alpha_n d^2(x_n,t) \\
&\quad - \alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right) + \alpha_n \beta_n (1-\beta_n)\, k_n^2\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right) \\
&\qquad \leq d^2(x_n,t) - d^2(x_{n+1},t) + \left[\alpha_n\, \beta_n\, k_n^4 + \alpha_n\, k_n^2 - \alpha_n\, \beta_n\, k_n^2 - \alpha_n\right] d^2(x_n,t)
\end{aligned}
$$

From the above inequality we get

$$
\begin{aligned}
&\alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right) \\
&\leq d^2(x_n,t) - d^2(x_{n+1},t) + \left[\alpha_n\, \beta_n\, k_n^2\left(k_n^2-1\right) + \alpha_n\left(k_n^2-1\right)\right] d^2(x_n,t)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\alpha_n\, \beta_n (1-\beta_n)\, k_n^2\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right) \\
&\leq d^2(x_n,t) - d^2(x_{n+1},t) + \left[\alpha_n\, \beta_n\, k_n^2\left(k_n^2-1\right) + \alpha_n\left(k_n^2-1\right)\right] d^2(x_n,t)
\end{aligned} \tag{11}
$$

Case (i): $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (I).

Let $m \geq 1$, then from (10)

$$
\sum_{n=1}^{m} \alpha_n (1-\alpha_n)\, d^2\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right)
$$

$$\leq \sum_{n=1}^{m}\left(d^2(x_n,t)-d^2(x_{n+1},t)\right)+\sum_{n=1}^{m} d^2(x_n,t)\left(\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\alpha_n\left(k_n^2-1\right)\right)$$
$$=\sum_{n=1}^{m}\left(d^2(x_n,t)-d^2(x_{n+1},t)\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\left(k_n^2-1\right)$$

Hence

$$\sum_{n=1}^{m}\alpha_n(1-\alpha_n)\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)$$
$$\leq d^2(x_1,t)-d^2(x_{m+1},t)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\left(k_n^2-1\right)$$

Since $d(x_n,t)$ is bounded and $\sum_{n=1}^{\infty}\left(k_n^2-1\right)<\infty$, as $m\to\infty$,

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)<\infty.$$

Let $u_n=d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)$ and $v_n=d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)$. Take $c_n=\alpha_n(1-\alpha_n)$ and $d_n=g\left(d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)\right)$, then using Lemma 2.9 there exists a subsequence $\left\{g(v_{n_k}),g\left(v_{n_k+1}\right)\right\}$ converges to zero. From the Lemma 2.6 $\left\{v_{n_k},v_{n_k+1}\right\}$ converges to zero. Since

$$\begin{aligned}
d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right) &\leq d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)+d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)\\
&\leq d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)+k_n\,d(y_n,x_n)\\
&= d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)\\
&\quad +k_n d\left(\tilde{\mathbb{P}}\left(\beta_n\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\oplus(1-\beta_n)x_n\right),x_n\right)\\
&\leq d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)+\beta_n\,k_n\,d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)\\
&\quad +(1-\beta_n)\;k_n\,d(x_n,x_n)
\end{aligned}$$

Thus we obtain $(1-\beta_n k_n)\,u_n\leq v_n$ for all n.

As $\beta_n\downarrow\beta<1$ and $k_n\downarrow 1$, also the sequence u_n converges to zero whenever v_n converges to zero. Therefore $\left\{u_{n_k},u_{n_k+1}\right\}$ converges to 0.

Case (ii): $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (II). Then from (11)

$$\sum_{n=1}^{m}\alpha_n\,\beta_n(1-\beta_n)\,k_n^2\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)$$

$$\leq \sum_{n=1}^{m}\left(d^2(x_n,t)-d^2(x_{n+1},t)\right)+\sum_{n=1}^{m} d^2(x_n,t)\left(\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\alpha_n\left(k_n^2-1\right)\right)$$
$$=\sum_{n=1}^{m}\left(d^2(x_n,t)-d^2(x_{n+1},t)\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\left(k_n^2-1\right)$$

Thus

$$\sum_{n=1}^{m}\alpha_n\,\beta_n(1-\beta_n)\,k_n^2\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)$$
$$\leq d^2(x_1,t)-d^2(x_{m+1},t)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\sum_{n=1}^{m} d^2(x_n,t)\,\alpha_n\left(k_n^2-1\right)$$

Letting $m\to\infty$, we see that as in Case(i)

$$\sum_{n=1}^{\infty}\alpha_n\,\beta_n(1-\beta_n)\,k_n^2\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)<\infty$$

As $\alpha_n\downarrow\alpha>0$ and $\beta_n\downarrow\beta<1$. From condition (II)

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\;k_n^2=\infty$$

Let $u_n=d\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}x_n\right)$ then from Lemma 2.9 $\left\{u_{n_k},u_{n_k+1}\right\}$ converges to 0.

Case (iii): $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (III).

Applying $0\leq\alpha\leq b<1$, in (10)

$$\alpha_n(1-b)\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)$$
$$\leq d^2(x_n,t)-d^2(x_{n+1},t)+\left[\alpha_n\;\beta_n\;k_n^2\left(k_n^2-1\right)+\alpha_n\left(k_n^2-1\right)\right]d^2(x_n,t)\,.$$

Thus summing first m terms,

$$\sum_{n=1}^{m}\alpha_n(1-b)\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)$$
$$\leq\sum_{n=1}^{m}\left(d^2(x_n,t)-d^2(x_{n+1},t)\right)+\sum_{n=1}^{m} d^2(x_n,t)\left(\alpha_n\beta_n k_n^2\left(k_n^2-1\right)+\alpha_n\left(k_n^2-1\right)\right)$$
$$=d^2(x_1,t)-d^2(x_{n+1},t)+\alpha_n\beta_n k_n^2\sum_{n=1}^{m} d^2(x_n,t)\left(k_n^2-1\right)+\alpha_n\sum_{n=1}^{m} d^2(x_n,t)\left(k_n^2-1\right)$$

As $\{d(x_n,t)\}$ is bounded and taking $m\to\infty$ thus we obtain

$$\sum_{n=1}^{\infty}\alpha_n(1-b)\,d^2\left(x_n,\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1}y_n\right)<\infty$$

Let $u_n = d\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n\right)$ and $v_n = d\left(x_n, \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n\right)$. We arrive that $\{u_{n_k},\, u_{n_k+1}\}$ converges to zero as done in Case(i).

Hence in all three cases $\{u_{n_k}, u_{n_k+1}\}$ converges to 0. Applying in Lemma 2.7, we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d\left(\mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_{n_k}, x_{n_k}\right) \to 0$. Thus we say that x_n converges to a fixed point t of $\mathbb{T}$. Hence the theorem. □

Theorem 3.4. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ where $\mathbb{E} \neq \phi$ is a bounded closed convex subset of a CAT(0) space $\mathbb{X}$ and sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} \left(k_n^2 - 1\right) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying one of the following conditions (I), (II) and (III) in Theorem 3.3. Define

$$x_{n+1} = \tilde{\mathbb{P}}\left(\alpha_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n \oplus (1-\alpha_n) x_n\right)$$
$$y_n = \tilde{\mathbb{P}}\left(\beta_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n \oplus (1-\beta_n) x_n\right)$$

then sequence $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Define $\{x_n\}$ as above then by Theorem 3.3 and we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ and $x = Tx$ since $\mathbb{T}$ completely continuous. $\lim_{n\to\infty} d(x_n, t)$ exists since t is a fixed point of $\mathbb{T}$. Thus $x_n \to x$. □

For specific sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying any one condition (I), condition (II) and condition (III) of Theorem 3.3 we get the following corollaries.

Corollary 3.5. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ where $\mathbb{E} \neq \phi$ is a bounded closed convex subset of a CAT(0) space $\mathbb{X}$ and sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} \left(k_n^2 - 1\right) < \infty$ and $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$. Suppose

$$x_{n+1} = \tilde{\mathbb{P}}\left(\alpha_n \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{p-1} x_n \oplus (1-\alpha_n) x_n\right),$$

then sequence $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Take $\beta_n = 0$ in Condition I. □

Corollary 3.6. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ where $\mathbb{E} \neq \phi$ is a bounded closed convex subset of a CAT(0) space $\mathbb{X}$ and sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} \left(k_n^2 - 1\right) < \infty$. Suppose

$$x_{n+1} = \tilde{\mathbb{P}}\left(\frac{1}{n} \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n \oplus \left(1 - \frac{1}{n}\right) x_n\right),$$

then sequence $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Take $\alpha_n = \frac{1}{n}$ and $\beta_n = 0$ in Condition I. $\square$

Corollary 3.7. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ where $\mathbb{E} \neq \phi$ is a bounded closed convex subset of a CAT(0) space $\mathbb{X}$ and sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty} \left(k_n^2 - 1\right) < \infty$. Suppose

$$x_{n+1} = \tilde{\mathbb{P}}\left(\left(1 - \frac{1}{n}\right) \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n \oplus \frac{1}{n} x_n\right),$$
$$y_n = \tilde{\mathbb{P}}\left(\frac{1}{n} \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n \oplus \left(1 - \frac{1}{n}\right) x_n\right)$$

then sequence $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Take $\alpha_n = 1 - \frac{1}{n}$ and $\beta_n = \frac{1}{n}$ in either condition (I) or (II) or (III). $\square$

Corollary 3.8. Let $\mathbb{T} : \mathbb{E} \to \mathbb{X}$ where $\mathbb{E} \neq \phi$ is a bounded closed convex subset of a CAT(0) space $\mathbb{X}$ and sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum\limits_{n=1}^{\infty} \left(k_n^2 - 1\right) < \infty$. Suppose

$$x_{n+1} = \tilde{\mathbb{P}}\left(\frac{1}{n} \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} y_n \oplus \left(1 - \frac{1}{n}\right) x_n\right),$$
$$y_n = \tilde{\mathbb{P}}\left(\frac{1}{n} \mathbb{T}\left(\tilde{\mathbb{P}}\mathbb{T}\right)^{n-1} x_n \oplus \left(1 - \frac{1}{n}\right) x_n\right),$$

then sequence $\{x_n\}$ converges to a fixed point of $\mathbb{T}$.

Proof. Take $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{n}$ in condition (I). $\square$

Conflict of Interests

The authors declare that there is no conflict of interests.

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